On Left Integro-Differential Splines and Cauchy Problem

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***Abstract*—In the case of integro-differential splines we use the values of integrals over net intervals. Integro-differential polynomial splines were first used in the works of Kireev V.I. Integro-differential nonpolynomial splines were used by the author of the paper. The error of approximation and results of approximation by the left integro-differential splines are represented in the paper. We construct implicit numerical methods for solving Cauchy problems by using polynomial and nonpolynomial left integro-differential splines. Here we compare the quality of approximation of different methods used for solving differential equations.**

***Index Terms*—Splines, Interpolation, Integro-differential splines, Cauchy problem**

I. INTRODUCTION

A large part of scientific computing is concerned with the solution of differential equations ([1]–[3]). Polynomial interpolation is quite useful for constructing numerical methods for both ordinary and partial differential equations, especially boundary-value problems ([4]–[6]).

As is well known, approximation by splines, in many cases.is preferable to polynomial approximation. Nowadays, there are many different splines for solving different problems ([7]–[16]).

Minimal interpolation polynomial splines were investigated in detail in ([17], [18], [20]). The distinctive feature of these splines is the existence of interpolation basis. The support of the basis spline contains one or several net intervals. These splines are convenient for approximation functions and their derivatives with given error of approximation. Minimal interpolation splines are suitable for solving the interpolation problems of Lagrange, Hermit, and Hermit-Birkhoff. The solution is constructed as the sum of products of the values of the function in the points of interpolation and the values of basic functions (and maybe the values of their derivatives) on every net interval separately.

Integro-differential polynomial splines were suggested in the works of Kireev V. I. These case splines contain the values of integrals over net intervals. Integro-differential nonpolynomial splines were used by the author of the paper. Some results were presented in ([19], [21]).

Here we construct numerical methods for solving Cauchy problem by using polynomial and polynomial integro-differential splines. Numerical methods for solving Cauchy problems by using minimal splines without values of integrals were suggested by the author in ([18]).

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II. ON NON-POLYNOMIAL INTEGRO-DIFFERENTIAL SPLINE CONSTRUCTION

Let $\alpha, m, m_\alpha, l_\alpha, s_\alpha, n, p, q$ — be integer nonnegative numbers, $l_\alpha \geq 1, s_\alpha \geq 1$, $m_\alpha = s_\alpha + l_\alpha$, $m_0 + \ldots + m_q + p = m$, $\{x_k\}$ be a net of ordered nodes, $a < \ldots < x_{k-1} < x_k < x_{k+1} \ldots < b$. Further it will be considered the grid of equidistant points $x_k = a + kh$, $h > 0$. Let function u be such that $u \in C^m([a, b])$.

Suppose that φ_j, $j = 1, \ldots, m$, is a Chebyshev system on $[a, b]$, in which case the functions $\varphi_j \in C^m([a, b])$, $j = 1, \ldots, m$, are nonzero on $[a, b]$. We construct:

$$\widetilde{u}(x) = \sum_{\alpha=0}^{q} \sum_{j=k-l_\alpha+1}^{k+s_\alpha} u^{(\alpha)}(x_j)\ \omega_{j,\alpha}(x) +$$

$$+\delta \sum_{i=1}^{p} \left(\int_{x_{k-i}}^{x_k} u(t)dt \right) \omega_k^{<-i>}(x),$$

for approximating the function $u(x)$ on the interval $[x_k, x_{k+1}]$. Here $\delta = 1$ or $\delta = 0$. If $\delta = 0$ then we put $p = 0$. Functions $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$ are such that $supp\, \omega_{k,\alpha} = [x_{k-s_\alpha}, x_{k+l_\alpha}]$, $\alpha = 0, 1, \ldots, q$, $supp\, \omega_{k,\alpha} \subseteq supp\, \omega_{k,\beta}$, $\beta < \alpha$, $supp\, \omega_k^{<-i>} = [x_k, x_{k+1}]$. Functions $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$, $x \in [x_k, x_{k+1}]$, are determined from the system of equations, which are called the approximation identities:

$$\widetilde{u}(x) = u(x), \quad \text{for} \quad u(x) = \varphi_\nu(x), \quad \nu = 1, \ldots, m.$$

We introduce the notations:

$$\Phi(x) = (\varphi_1(x), \ldots, \varphi_m(x))^T,$$

$$\Phi_\alpha(x) = (\varphi_1^{(\alpha)}(x), \ldots, \varphi_m^{(\alpha)}(x))^T,$$

$$\Psi_{k,\alpha} = (\Phi_\alpha(x_{k-l_\alpha+1}), \ldots, \Phi_\alpha(x_{k+s_\alpha})),$$

$$S\Phi_p = \left(\int_{x_{k-1}}^{x_k} \Phi(t)dt, \ldots, \int_{x_{k-p}}^{x_k} \Phi(t)dt \right).$$

Then the system determinant takes the form:

$$\Delta = det(\Psi_{k,0}, \ldots, \Psi_{k,q}, S\Phi_p).$$

Sometimes a numerical quantity of determinant (if $h \neq const$) may be approximately equal to 0. Suppose that for the chosen values of parameters, the determinant is nonzero. Then the basis functions $\omega_{j,\alpha}(x)$, $\omega_k^{<-i>}(x)$ can be determined by Cramer's formulas. In particular, for finding the basis function $\omega_{k,\alpha}(x)$ on the interval $[x_k, x_{k+1}]$ the following relation can be used:

$$\omega_{k,\alpha}(x) = det(\Psi_{k,0}, \ldots, \Phi_\alpha(x_{k-l_\alpha+1}), \ldots,$$

$$\Phi_\alpha(x_{k-1}), \Phi(x), \Phi_\alpha(x_{k+1}), \ldots$$
$$\ldots, \Phi_\alpha(x_{k+s_\alpha}) \ldots, \Psi_{k,q}, S\Phi_p)/\Delta.$$

Then the constructed splines $\omega_{k,\alpha}(x)$, $\omega_k^{<-i>}(x)$ and the approximation $\widetilde{u}(x)$ have the following properties:

1) at the ends of each interval $[x_k, x_{k+1}]$ we have $u^{(\alpha)}(x_k) = \widetilde{u}^{(\alpha)}(x_k)$, $u^{(\alpha)}(x_{k+1}) = \widetilde{u}^{(\alpha)}(x_{k+1})$, $\alpha = 0, 1, \ldots, q$, $\widetilde{u} \in C^q([a, b])$;

2) $\int_{x_{k-i}}^{x_k} u(t)dt = \int_{x_{k-i}}^{x_k} \widetilde{u}(t)dt$, $i = 1, \ldots, p$;

3) for polynomial and trigonometrical system $\{\varphi_i\}$ on equidistant set of nodes with step h, we have $|\omega_k^{<-i>}(x)| \le \widetilde{K}_i/h$, $|\omega_{k,\alpha}(x)| \le \widetilde{C}_\alpha h^\alpha$, here $\widetilde{K}_i > 0, \widetilde{C}_\alpha > 0$ are certain constants.

In general we assume that a nonpolynomial system of functions $\{\varphi_i\}$ is chosen in such a way that property 3 is fulfilled.

Let system equations matrix consists of the units: $(\mathcal{X}_j, \mathcal{X}_j^{(1)}, \ldots, \mathcal{X}_j^{(s)})$, $j = 1, \ldots, m$, where
$\mathcal{X} = (1, \varphi(t), \ldots, \varphi^m(t))^T$,
$\mathcal{X}_j^{(i)} = (0, (\varphi(t))^{(i)}|_{t=t_j}, \ldots, (\varphi^m(t))^{(i)}|_{t=t_j})^T$.

Lemma. *Let P_s be such that $P_s = 2P_{s-1} - P_{s-2} + 1$, $P_0 = 0$, $P_1 = 1$. Then the following assertion is true:*

$$\det(\mathcal{X}_1, \mathcal{X}_1^{(1)}, \ldots, \mathcal{X}_1^{(s)}, \ldots, \mathcal{X}_m, \mathcal{X}_m^{(1)}, \ldots, \mathcal{X}_m^{(s)}) =$$
$$= (1!\,2! \ldots s!)^m \prod_{1 \le j < i \le m} (\varphi(x_i) - \varphi(x_j))^{(s+1)^2} \left(\prod_{i=1}^m \varphi'(x_i)\right)^{P_s}.$$

Proof. Let us differentiate the Vandermonde determinant:

$$\det(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{s+1}, \ldots, \mathcal{X}_{m(s+1)})$$

as follows: once on x_2; twice on x_3; …; s times on x_{s+1}. After that we put $x_1 = x_2 = \ldots = x_{s+1}$, etc.

Example 1. Let us take $\mathcal{X} = (1, e^t, e^{2t}, e^{3t})^T$. Now $\mathcal{X}_k = (1, e^{t_k}, e^{2t_k}, e^{3t_k})^T$, $\mathcal{X}'_k = (0, e^{t_k}, 2e^{2t_k}, 3e^{3t_k})^T$, $k = j, j+1$. So we have: $\det(\mathcal{X}_j, \mathcal{X}'_j, \mathcal{X}_{j+1}, \mathcal{X}'_{j+1}) = e^{t_j} e^{t_{j+1}} (e^{t_j} - e^{t_{j+1}})^4$.

III. The error of approximation

First we find the relation for $u(x)$ for computing the approximation error. Construct a homogeneous linear equation, which has a fundamental system of solutions $\varphi_1(x), \ldots, \varphi_m(x)$. Let us find the function $u(x)$ in the form convenient for obtaining error estimation. First construct a homogeneous linear equation Lu, which has a fundamental system of functions φ_i. Let us construct the next equation for $x \in [x_k, x_{k+1}] \subset [a, b]$:

$$Lu = \begin{vmatrix} \varphi_1(x), & \varphi_2(x), & \ldots & \varphi_m(x), & u(x) \\ \varphi'_1(x), & \varphi'_2(x), & \ldots & \varphi'_m(x), & u'(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_1^{(m)}(x), & \varphi_2^{(m)}(x), & \ldots & \varphi_m^{(m)}(x), & u^{(m)}(x) \end{vmatrix} = 0.$$

Here the Wronskian

$$W(x) = \begin{vmatrix} \varphi_1(x), & \varphi_2(x), & \ldots & \varphi_m(x) \\ \varphi'_1(x), & \varphi'_2(x), & \ldots & \varphi'_m(x) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_1^{(m-1)}(x), & \varphi_2^{(m-1)}(x), & \ldots & \varphi_m^{(m-1)}(x) \end{vmatrix}$$

does not equal zero. Expanding the determinant according to the elements of the last column and dividing all terms of the obtained equation by $W(x)$, one can obtain the desired equation $Lu = u^{(m)}(x) + Q_1(x)u^{(m-1)}(x) + \ldots + Q_m(x)u(x) = 0$. Now construct a general solution of nonhomogeneous equation $Lu = F$ by the method of variation of the constants. Suppose:

$$u(x) = \sum_{i=1}^m C_i(x)\varphi_i(x).$$

Then:

$$C_i(x) = \int_{x_k}^x \frac{W_{mi}(t)F(t)}{W(t)}dt + c_i,$$

where c_i are arbitrary constants. Since $F = Lu$, one has:

$$u(x) = \sum_{i=1}^m \varphi_i(x) \int_{x_k}^x \frac{W_{mi}(t)\ Lu(t)}{W(t)}dt + \sum_{i=1}^m c_i\varphi_i(x).$$

where $W_{mi}(x)$ are algebraic complements (signed minor) of the element of i-th column of m-th row of determinant $W(x)$. Let us estimate $|r| = |\widetilde{u}(x) - u(x)|$.

It is easy to show that the following relations

$$\sum_{i=1}^m c_i \left(\sum_{\alpha=0}^q \sum_{j=k-l_\alpha+1}^{k+s_\alpha} \varphi^{(\alpha)}(x_j)\ \omega_{j,\alpha}(x) + \right.$$
$$\left. +\delta \sum_{i=1}^p \left(\int_{x_{k-i}}^{x_k} \varphi(t)dt \right) \omega_k^{<-i>}(x) \right) = \sum_{i=1}^m c_i\varphi_i(x),$$
$$\sum_{i=1}^m \varphi_i^{(\alpha)}(x)W_{mi}(x) = 0, \quad \alpha = 0, \ldots, m-2,$$

are valid. One has:

$$u^{(\alpha)}(x) = \sum_{i=1}^m \varphi_i^{(\alpha)}(x) \int_{x_k}^x \frac{W_{mi}(t)\ Lu(t)}{W(t)}dt + \sum_{i=1}^m c_i\varphi_i^{(\alpha)}(x).$$

Representing $\varphi_i(x)$ by the Taylor formula

$$\varphi_i(x) = \sum_{l=1}^{m-1} \frac{\varphi_i^{(l-1)}(t)}{(l-1)!}(x-t)^{l-1} + (x-t)^{m-1}\frac{\varphi_i^{(m-1)}(\tau_i)}{(m-1)!},$$

where τ_i is between x and t, taking into account the similar relations for the derivatives $\varphi_i(x)$ and the previous identities, one obtains:

$$\widetilde{u}(x) - u(x) = \sum_{\alpha=0}^q \sum_{j=k-l_\alpha+1}^{k+s_\alpha} u^{(\alpha)}(x_j)\ \omega_{j,\alpha}(x) +$$
$$+ \sum_{\gamma=1}^p \int_{x_{k-\gamma}}^{x_k} u(z)dz\ \omega_k^{<-\gamma>}(x) - u(x) =$$
$$= \sum_{\alpha=0}^q \sum_{j=k-l_\alpha+1}^{k+s_\alpha} \sum_{i=1}^m \int_{x_k}^{x_j} \frac{\varphi_i^{(m-1)}(\xi_j)}{(m-1-\alpha)!}(x_j - t)^{m-\alpha-1} \times$$
$$\times \frac{W_{mi}(t)\ Lu(t)}{W(t)}dt\ \omega_{j,\alpha}(x) +$$
$$+ \sum_{i=1}^m \sum_{\gamma=1}^p \int_{x_{k-\gamma}}^{x_k} \int_{x_k}^z \frac{\varphi_i^{(m-1)}(\zeta_j)}{(m-1)!}(z-t)^{m-1} \times$$

$$\times \frac{W_{mi}(t)\ Lu(t)}{W(t)} dtdz\ \omega_k^{<-\gamma>}(x)-$$

$$-\sum_{i=1}^{m} \int_{x_k}^{x} \frac{\varphi_i^{(m-1)}(\xi)}{(m-1)!}(x-t)^{m-1} \frac{W_{mi}(t)\ Lu(t)}{W(t)} dt\ ,$$

ξ_j is between x_k and x_j, ξ is between x_k and x, ζ_i is between x_k and z. We use the theorem of the mean of integrals. In view of $|\omega_k^{<\gamma>}(x)| \le K_0/h$, $|\omega_{j,\alpha}(x)| \le K_{1,\alpha}h^\alpha$, $K_0 > 0, K_{1,\alpha} > 0$, and with the notation:

$$K = \sum_{\alpha=0}^{q} \sum_{j=k-l_\alpha+1}^{k+s_\alpha} \sum_{i=1}^{m} \frac{K_{1,\alpha}|k-j||\varphi_i^{(m-1)}(\eta_j)|}{(m-1-\alpha)!} \times$$

$$\times \max_{t\in[x_k,x_{k+1}]} \left|\frac{W_{mi}(t)}{W(t)}\right| +$$

$$+\sum_{i=1}^{m}\sum_{\gamma=1}^{p} \frac{|\varphi_i^{(m-1)}(\mu_\gamma)|}{(m-1)!} K_0(\gamma+1) \max_{t\in[x_k,x_{k+1}]} \left|\frac{W_{mi}(t)}{W(t)}\right| +$$

$$+\sum_{i=1}^{m} \frac{|\varphi_i^{(m-1)}(\eta)|}{(m-1)!} \max_{t\in[x_k,x_{k+1}]} \left|\frac{W_{mi}(t)}{W(t)}\right|,$$

η_j is between x_k and x_j, η is between x_k and x_{k+1}, μ_γ is between $x_{k-\gamma}$ and x_{k+1}, we have:

$$|\widetilde{u}(x) - u(x)| \le h^m K \|\ Lu\|, \quad x \in [x_k, x_{k+1}].$$

Here $\|\ f\| = \max_{x\in[x_{k-l_0+1},x_{k+s_0}]} |f(x)|$.

So, if $\varphi_i = x^{i-1}$, $i = 1, 2, \dots m$, then:

$$|\widetilde{u}(x) - u(x)| \le h^m K_1 \|u^{(m)}\|,\ K_1 > 0,\ x \in [x_k, x_{k+1}].$$

A. *Examples*

Example 2

Here there are some applications of the results representing above.

a) If $\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_3 = \cos(x)$, then we have:
$Lu = u'(x) + u'''(x)$,
$u(x) = 2\int_{x_k}^{x}(u'(\tau)+ u'''(\tau))\sin^2(x/2 - \tau/2)d\tau + c_1 + c_2\sin(x) + c_3\cos(x)$, where c_i are arbitrary constants.

b) If $\varphi_i = x^{i-1}$, $i = 1, 2, 3$, then we have:
$Lu = u'''(x)$,
$u(x) = \frac{1}{2}\int_{x_k}^{x}(u'''(\tau))(x-\tau)^2 d\tau + \sum_{i=1}^{3} c_i x^i$, where c_i are arbitrary constants.

c) If $\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_3 = \cos(x)$, $\varphi_4 = \sin(2x)$, $\varphi_5 = \cos(2x)$, then we have:
$Lu = 4u'(\tau) + 5u'''(\tau) + u^V(\tau)$,
$u(x) = \frac{2}{3}\int_{x_k}^{x}(4u'(\tau) + 5u'''(\tau) + u^V(\tau))\sin^4(x/2 - \tau/2)d\tau + c_1 + c_2\sin(x) + c_3\cos(x) + c_4\sin(2x) + c_5\cos(2x)$, where c_i are arbitrary constants.

d) If $\varphi_i = x^{i-1}$, $i = 1, 2, 3, 4, 5$, then we have:
$Lu = u^V(x)$,
$u(x) = \frac{1}{24}\int_{x_k}^{x}(u^V(\tau))(x-\tau)^4 d\tau + \sum_{i=1}^{5} c_i x^i$, where c_i are arbitrary constants.

e) If $\varphi_1 = 1$, $\varphi_2 = exp(x/2)$, $\varphi_3 = exp(-x/2)$, then we have:
$Lu = 4u'''(x) - u'(x)$,
$u(x) = 2\int_{x_k}^{x}(4u'''(\tau) - u'(\tau))((exp(x/2 - \tau/2) - 1) + (exp(t/2 - x/2) - 1))d\tau + c_1 + c_2 exp(x/2) + c_3 exp(-x/2)$, where c_i are arbitrary constants.

f) If $\varphi_1 = 1$, $\varphi_1 = x$, $\varphi_2 = exp(x)$, $\varphi_3 = exp(-x)$, then we have:
$Lu = u'' - u''''$,
$u(x) = \int_{x_k}^{x}(u''(\tau) - u''''(\tau))(2\tau - 2x + exp(x-\tau) - exp(-x+\tau))d\tau + c_1 + c_2 x + c_3 exp(x) + c_4 exp(-x)$, where c_i are arbitrary constants.

Example 3

Let us take $p = 1$, $q = 0$, $l_0 = s_0 = 1$. Integro-differential left spline has the form:

$$\widetilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + \\ + \int_{x_{j-1}}^{x_j} u(t)dt\ \omega_j^{<-1>}(x),\ x \in [x_j, x_{j+1}], \qquad (1)$$

while in polynomial case ($\varphi_i = x^{i-1}$) we have:

$$\omega_j(x) = \frac{A_1}{(x_j - x_{j+1})(x_j - 3x_{j+1} + 2x_{j-1})(x_j - x_{j-1})},$$

$A_1 = (-x_{j+1} + x)(3x_{j-1}x + 3xx_j - 6x_{j+1}x - 2x_j^2 - 2x_{j-1}^2 - 2x_j x_{j-1} + 3x_{j+1}x_{j-1} + 3x_{j+1}x_j)$,

$$\omega_{j+1}(x) = \frac{(-x_j + x)(3x - x_j - 2x_{j-1})}{(-x_{j+1} + x_j)(x_j - 3x_{j+1} + 2x_{j-1})},$$

$$\omega_j^{<-1>}(x) = -6\frac{(-x_{j+1} + x)(-x_j + x)}{(x_j - x_{j-1})^2(x_j - 3x_{j+1} + 2x_{j-1})}.$$

$$\int_{x_j}^{x_{j+1}} \omega_j(t)dt = \frac{A_2}{B_2(x_j - x_{j-1})},$$

$A_2 = (x_j x_{j-1}^2 + x_{j+1}^2 x_j - 2x_{j+1}x_j x_{j-1} - x_{j+1}^3 - x_{j+1}x_{j-1}^2 + 2x_{j+1}^2 x_{j-1})$,
$B_2 = (x_j - 3x_{j+1} + 2x_{j-1})$,
$\int_{x_j}^{x_{j+1}} \omega_{j+1}(t)dt = -(x_j x_{j-1} - x_{j+1}x_j - x_{j+1}x_{j-1} + x_{j+1}^2)/(x_j - 3x_{j+1} + 2x_{j-1})$,

$$\int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(t)dt = -\frac{A_3}{(x_j - x_{j-1})^2 B_3}.$$

$A_3 = (-x_{j+1}^3 + x_j^3 - 3x_{j+1}x_j^2 + 3x_{j+1}^2 x_j)$,
$B_3 = (x_j - 3x_{j+1} + 2x_{j-1})$,

If $x_{j+1} - x_j = x_j - x_{j-1} = h > 0$, $x = x_j + th$, $t \in [0, 1]$, then:

$$\omega_j(x_j + th) = -(9t+5)(t-1)/5, \quad \int_{x_j}^{x_{j+1}} \omega_j(x)dx = 4h/5,$$

$$\omega_{j+1}(x_j + th) = (3t+2)t/5, \quad \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = 2h/5,$$

$$\omega_j^{<-1>}(x_j + th) = \frac{6t}{5h}(t-1), \quad \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -1/5.$$

Representing $u(x)$, $u(x_{j+1})$, $u(x_{j-1})$ using the Taylor formula in x_j, we obtain in polynomial case:

$$|\widetilde{u}(x) - u(x)| \le K h^3 \|u'''\|_{[x_j,x_{j+2}]}, \quad K > 0,\ x \in [x_j, x_{j+1}].$$

In trigonometrical case ($\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_3 = \cos(x)$,) we have:

$\omega_j(x) = (-\cos(-x_{j+1} + x_{j-1}) + \cos(-x_{j+1} + x_j) + x_j \sin(x - x_{j+1}) - x_{j-1} \sin(x - x_{j+1}) + \cos(x - x_{j-1}) - \cos(x - x_j))/(-\cos(-x_{j+1} + x_{j-1}) + \cos(-x_{j+1} + x_j) + x_j \sin(-x_{j+1}+x_j) - x_{j-1} \sin(-x_{j+1}+x_j) + \cos(x_j - x_{j-1}) - 1)$,

$\int_{x_j}^{x_{j+1}} \omega_j(t)dt = (-x_j + x_{j-1} - \cos(-x_{j+1} + x_{j-1})x_{j+1} + \cos(-x_{j+1} + x_j)x_{j+1} + \cos(-x_{j+1} + x_{j-1})x_j - x_{j-1} \cos(-x_{j+1} + x_j) + \sin(-x_{j+1} + x_j) - \sin(x_j - x_{j-1}) - \sin(-x_{j+1} + x_{j-1}))/(-\cos(-x_{j+1} + x_{j-1}) + \cos(-x_{j+1} + x_j) + x_j \sin(-x_{j+1} + x_j) - x_{j-1} \sin(-x_{j+1} + x_j) + \cos(x_j - x_{j-1}) - 1)$,

$\omega_{j+1}(x) = (\cos(x - x_{j-1}) - \cos(x - x_j) + x_j \sin(x - x_j) - x_{j-1} \sin(x - x_j) - \cos(x_j - x_{j-1}) + 1)/(\cos(-x_{j+1} + x_{j-1}) - \cos(-x_{j+1} + x_j) - x_j \sin(-x_{j+1} + x_j) + x_{j-1} \sin(-x_{j+1} + x_j) - \cos(x_j - x_{j-1}) + 1)$,

$\int_{x_j}^{x_{j+1}} \omega_{j+1}(t)dt = (-x_{j+1} + x_{j-1} + \cos(x_j - x_{j-1})x_{j+1} - \cos(x_j - x_{j-1})x_j + \cos(-x_{j+1} + x_j)x_j - x_{j-1} \cos(-x_{j+1} + x_j) - \sin(-x_{j+1} + x_j) + \sin(x_j - x_{j-1}) + \sin(-x_{j+1} + x_{j-1}))/(-\cos(-x_{j+1} + x_{j-1}) + \cos(-x_{j+1} + x_j) + x_j \sin(-x_{j+1}+x_j) - x_{j-1} \sin(-x_{j+1}+x_j) + \cos(x_j - x_{j-1}) - 1)$,

$\omega_j^{<-1>}(x) = (\sin(x - x_{j+1}) - \sin(-x_{j+1} + x_j) - \sin(x - x_j))/(\cos(-x_{j+1}+x_{j-1}) - \cos(-x_{j+1}+x_j) - x_j \sin(-x_{j+1} + x_j) + x_{j-1} \sin(-x_{j+1} + x_j) - \cos(x_j - x_{j-1}) + 1)$,

$\int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(t)dt = (2\cos(-x_{j+1}+x_j) + x_j \sin(-x_{j+1} + x_j) - 2 - \sin(-x_{j+1} + x_j)x_{j+1})/(\cos(-x_{j+1} + x_{j-1}) - \cos(-x_{j+1} + x_j) - x_j \sin(-x_{j+1} + x_j) + x_{j-1} \sin(-x_{j+1} + x_j) - \cos(x_j - x_{j-1}) + 1)$.

If $x_{j+1} - x_j = x_j - x_{j-1} = h > 0$, $x = x_j + th$, $t \in [0, 1]$, then:

$\omega_j(x_j + th) = (\cos(h + th) - \cos(2h) + h \sin(th - h) + \cos(h) - \cos(th))/(-1 - h \sin(h) - \cos(2h) + 2 \cos(h))$,

$\int\limits_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = 2((\cos(h))^2 + h \sin(h) \cos(h) - 1)/(2 \cos(h) \sin(h) - h \cos(h) - h)$,

$\omega_{j+1}(x_j + th) = (-\cos(h + th) + \cos(th) + \cos(h) - 1 - \sin(th)h)/(-1 - h \sin(h) - \cos(2h) + 2 \cos(h))$,

$\int\limits_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = -2((\cos(h))^2 + h \sin(h) - 1)/(2 \cos(h) \sin(h) - h \cos(h) - h)$,

$\omega_j^{<-1>}(x_j+th) = (-\sin(th-h)+\sin(th)-\sin(h))/(-1-h \sin(h) - \cos(2h) + 2 \cos(h))$.

$\int\limits_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -(h \cos(h) - 2 \sin(h) + h)/(2 \cos(h) \sin(h) - h \cos(h) - h)$,

It can be shown, that next relations are fullfilled:

$\omega_j(x_j + th) = -(9t + 5)(t - 1)/5 + O(h)$,
$\omega_{j+1}(x_j + th) = t(2 + 3t)/5 + O(h)$,
$\omega_j^{<-1>}(x_j + th) = (6/5)t(t - 1)/h + O(1)$.

$\int\limits_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -1/5 + O(h^2)$,

$\int\limits_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = (2/5)h + O(h^3)$,

$\int\limits_{x_j}^{x_{j+1}} \omega_j(x)dx = (4/5)h + O(h^3)$,

The relations above determine correlations between the trigonometrical and the polynomial splines.

We have in the trigonometrical case:

$$|\widetilde{u}(x) - u(x)| \leq Kh^3 \|u' + u'''\|_{[x_j, x_{j+2}]}, K > 0, x \in [x_j, x_{j+1}].$$

In exponential case ($\varphi_1 = 1$, $\varphi_2 = exp(x/2)$, $\varphi_3 = exp(-x/2)$), if $x_{j+1} - x_j = x_j - x_{j-1} = h > 0$, $x = x_j + th$, $t \in [0, 1]$, then we have:

$\omega_j(x_j + th) = (2\ exp(2h) - 2\ exp(3h/2) + 2 - 2\ exp(h/2) + h\ exp(h/2 + th/2) - 2\ exp(3h/2 + th/2) + 2\ exp(h + th/2) - 2\ exp(h/2 - th/2) + 2exp(h - th/2) - h\ exp(3h/2 - th/2))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + h\ exp(h/2) + 4exp(h) - h\ exp(3h/2))$,

$\omega_{j+1}(x_j + th) = (2exp(3h/2 + th/2) - 2exp(h + th/2) + 2exp(h/2 - th/2) - 2exp(h - th/2) + hexp(h - th/2) - 2exp(3h/2) + 4exp(h) - 2exp(h/2) - h\ exp(h + th/2))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + h\ exp(h/2) + 4exp(h) - h\ exp(3h/2))$,

$\omega_j^{<-1>}(x_j + th) = -(-exp(3h/2 - th/2) + exp(h/2 + th/2) - exp(h/2) + exp(h - th/2) - exp(h + th/2) + exp(3h/2))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + h\ exp(h/2) + 4exp(h) - h\ exp(3h/2))$.

It can be shown, that:

$\omega_j(x_j + th) = -(9t + 5)(t - 1)/5 + O(h)$;
$\omega_{j+1}(x_j + th) = t(2 + 3t)/5 + O(h)$,
$\omega_j^{<-1>}(x_j + th) = (6/5)t(t - 1)/h + O(h)$,

We have in the exponential case:

$$|\widetilde{u}(x) - u(x)| \leq Kh^3 \|u' - 4u'''\|_{[x_j, x_{j+2}]}, K > 0, x \in [x_j, x_{j+1}].$$

Let us take $u(x) = 1/(1+25x^2)$. We calculate: $\widetilde{u}(x) - u(x)$ on $[-1, 1]$, $n = 10$, $h = 0.5$. The error of the approximation by the polynomial integro-differential splines (1) is represented on Figure 1a. The error of the approximation of $1/(1 + 25x^2)$ by the trigonometric integro-differential splines (1) is represented on Figure 1b.

(a) (b)

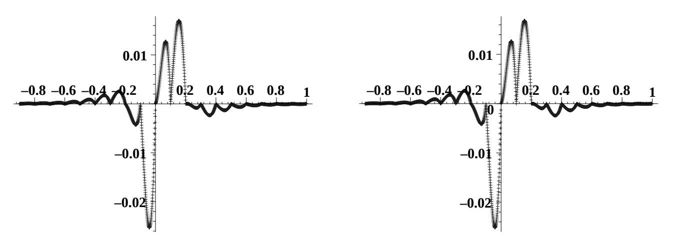

Fig. 1. Plots of the error of approximation $1/(1 + 25x^2)$ by polynomial integro-differential splines (a), and trigonometric integro-differential splines (b)

Let us take $u(x) = \sin(x) + \sin(3x)$. We calculate: $\widetilde{u}(x) - u(x)$ on $[-1, 1]$, $n = 10$, $h = 0.5$. The error of approximation by the polynomial integro-differential splines (1) is represented on Figure 2a. The error of approximation of $\sin(x) \sin(3x)$ by the trigonometric integro-differential splines (1) is represented on Figure 2b.

Table 1 shows the actual errors $R = \max(\tilde{u} - u)$ of approximation of the functions. Here R^P is the actual error of approximation by the polynomial splines (1); R^T is the actual

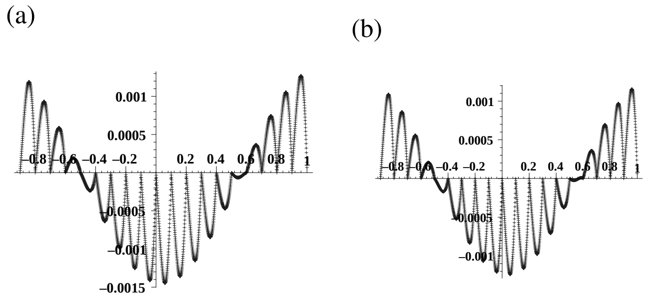

Fig. 2. Plots of the error of approximation $\sin(x)\sin(3x)$ by polynomial integro-differential splines (a), and trigonometric integro-differential splines (b)

error of the approximation by the trigonometrical splines (1), R^E is the actual error of the approximation by the exponential splines (1), when $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 1. Actual error of the approximation by the trigonometrical spline (1), the polynomial spline (1) and the exponential spline (1).

$u(x)$	R^P	R^T	R^E
$1/(1+25x^2)$	0.0253	0.0251	0.0253
$\sin(x)+\sin(3x)$	0.00145	0.00124	0.00150
$\exp(x/2)+\exp(3x)$	0.0224	0.0250	0.0222

IV. Solution of a Cauchy Problem for one Equation

We shall solve a Cauchy problem:

$$y' = f\,(x, y(x)),\ y(x_0) = y_0,\ x \in [x_0, X].$$

Consider the integral identity:

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} y'(x)dx.$$

We replace $y'(x)$ by the integro-differential spline $\tilde{u}(x)$. Now we have:

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} \tilde{u}(x)dx + R,$$

where $R = \int_{x_j}^{x_{j+1}}(u(x) - \widetilde{u}(x))dx$, taking into account the error of approximation by the integro-differential spline, we have:

$|R| \le h^{m+1}K_3\|\ Lu\|,\ K_3 > 0.$

V. Numerical methods for $q = 0$

We have for $q = 0$ and $x \in [x_j, x_{j+1}]$ (here $\omega_k(x) = \omega_{k,0}(x)$):

$$\widetilde{u}(x) = \sum_{k=j-l+1}^{j+s} u(x_k)\,\omega_k(x) + \sum_{i=1}^{p}\left(\int_{x_{j-i}}^{x_j} u(t)dt\right)\omega_j^{<-i>}(x).$$

A. Numerical method 1

Let us take $p = 1$. We put $u(x) = y'(x)$. We replace the integrand in

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} y'(x)dx$$

with $\tilde{u}(x)$, $x \in [x_j, x_{j+1}]$:

$$\tilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) +$$
$$+\left(\int_{x_{j-1}}^{x_j} u(x)dx\right)\omega_j^{<-1>}(x).$$

We have:

$$y(x_{j+1}) = y(x_j) + \int_{x_j}^{x_{j+1}} \tilde{u}(x)dx + R.$$

We obtain:

$$y(x_{j+1}) = y(x_j) + u(x_j)\int_{x_j}^{x_{j+1}} \omega_j(x)dx +$$
$$u(x_{j+1})\int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx +$$
$$+\left(\int_{x_{j-1}}^{x_j} u(x)dx\right)\int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx + R.$$

Now we have the next implicit method:

$$y_{j+1} = y_j(1 + I^{<-1>}) - y_{j-1}(I^{<-1>}) +$$
$$+f(x_j, y_j)I_0 + f(x_{j+1}, y_{j+1})I_1,$$

where

$$I^{<-1>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx,\quad I_0 = \int_{x_j}^{x_{j+1}} \omega_j(x)dx,$$
$$I_1 = \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx.$$

Now we construct $I^{<-1>}$, I_0, I_1 for polynomial and non-polynomial cases.

a) Let us take $\varphi_1(x) = 1, \varphi_2(x) = e^{(x/2)}, \varphi_3(x) = e^{(-x/2)}$, $h = const$.

We easily receive: $I_0 = -2(2hexp(h/2) + 4exp(h/2) + 2hexp(3h/2) - 4exp(3h/2) - h - exp(2h)h - 2 - 2hexp(h) + 2exp(2h))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + hexp(h/2) + 4exp(h) - hexp(3h/2))$,

$I_1 = -4(2exp(3h/2) - 2hexp(h) - 2exp(h/2) - exp(2h) + 1 + hexp(h/2) + hexp(3h/2))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + hexp(h/2) + 4exp(h) - hexp(3h/2))$,

$I^{<-1>} = (4exp(3h/2) + 4exp(h/2) - 8exp(h) - hexp(3h/2) + hexp(h/2))/(2exp(2h) - 4exp(3h/2) + 2 - 4exp(h/2) + hexp(h/2) + 4exp(h) - hexp(3h/2))$,

We have:

$$|R| \le\ Kh^4\|4y^{\mathrm{IV}} - y^{\mathrm{II}}\|_{[x_{j-1},x_{j+1}]},\ K > 0.$$

b) In case $\varphi_1(x) = 1, \varphi_2(x) = x, \varphi_3(x) = x^2$, we have:

$$y_{j+1} = \frac{4}{5}y_j + \frac{1}{5}y_{j-1} + f(x_j, y_j)\frac{4h}{5} + \frac{2h}{5}f(x_{j+1}, y_{j+1}),$$
$$|R| \le\ Kh^4\|y^{\mathrm{IV}}\|_{[x_{j-1},x_{j+1}]},\ K > 0.$$

c) In case $\varphi_1(x) = 1, \varphi_2(x) = \sin(x), \varphi_3(x) = \cos(x)$, we have:

$$I_0 = 2\frac{(\cos^2(h) + h\sin(h)\cos(h) - 1)}{(2\sin(h)\cos(h) - h\cos(h) - h)},$$

$$I_1 = -2\frac{(\cos^2(h) + h\sin(h) - 1)}{(2\sin(h)\cos(h) - h\cos(h) - h)},$$

$$I^{<-1>} = \frac{(-h\cos(h) + 2\sin(h) - h)}{(2\sin(h)\cos(h) - h\cos(h) - h)}.$$

Let us solve the problem:

$$y' = -150(y - \cos(x)), y(0) = 0, x \in [0, 1].$$

The exact solution is the following:

$$y(x) = \frac{22500}{22501}\cos(x) + \frac{150}{22501}\sin(x) - \frac{22500}{22501}\exp(-150\,x).$$

Let us take $h = 0.001$. The errors of the solution of the Cauchy problem by methods (b) and (c) are represented in Figures 3 and 4.

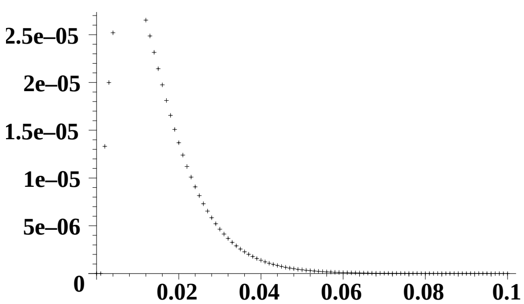

Fig. 3. Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ by the method (b), h=0.001

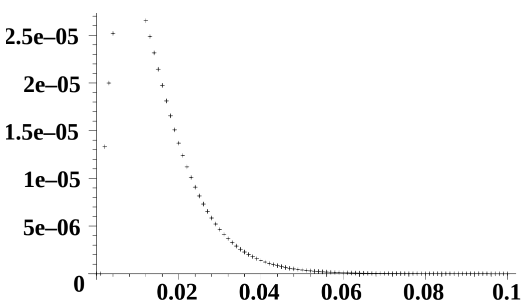

Fig. 4. Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ by the method (c), h=0.001

Table 2 shows the actual errors $R = |y_k - y(kh)|$, $h = 0.001$. Calculations were done in Maple, Digits=25.

Table 2.

k	method b	method c	method a
10	$0.29408 \cdot 10^{-4}$	$0.29409 \cdot 10^{-4}$	$0.29405 \cdot 10^{-4}$
20	$0.13693 \cdot 10^{-4}$	$0.13694 \cdot 10^{-4}$	$0.13692 \cdot 10^{-4}$
100	$0.43470 \cdot 10^{-9}$	$0.43490 \cdot 10^{-9}$	$0.43529 \cdot 10^{-9}$

Let us solve the problem:

$$y' = -2(y - sin(x)) + cos(x), y(0) = 0, x \in [0, 100].$$

The errors of the solution of the Cauchy problem by the method 1 are represented in Figures 5, 6.

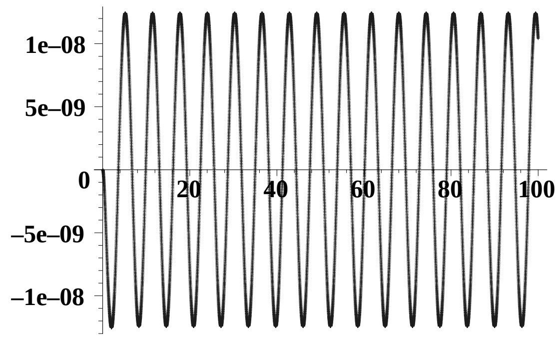

Fig. 5. Graph of the error of the solution of the problem $y' = -2(y - sin(x)) + cos(x), y(0) = 0$, $h = 0.01$, method b.

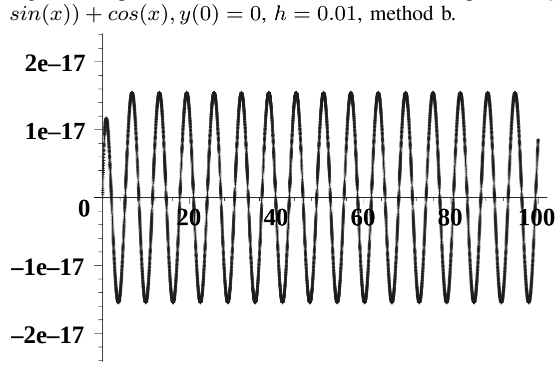

Fig. 6. Graph of the error of the solution of the problem $y' = -2(y - sin(x)) + cos(x), y(0) = 0$, $h = 0.01$, method c.

B. Numerical method 2

Now let us approximate function $u(x)$ by

$$\tilde{u}(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x) + \left(\int_{x_{j-1}}^{x_j} u(x)dx\right)\omega_j^{<-1>}(x) + \left(\int_{x_{j-2}}^{x_j} u(x)dx\right)\ \omega_j^{<-2>}(x)$$

on $[x_j, x_{j+1}]$. Here $\omega_j(x)$, $\omega_{j+1}(x)$, $\omega_j^{<-1>}(x)$, $\omega_j^{<-2>}(x)$ we determine from the equations:

$$\tilde{u}(x) = u(x),\ u(x) = \varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \varphi_4(x).$$

So we have:

$$y_{j+1} = y_j\left(1 + I^{<-1>} + I^{<-2>}\right) - y_{j-1}I^{<-1>} - y_{j-2}I^{<-2>} + f(x_j, y_j)I_0 + f(x_{j+1}, y_{j+1})I_1.$$

a) In the polynomial case $\varphi_1(x) = 1, \varphi_2(x) = x, \varphi_3(x) = x^2, \varphi_4(x) = x^3$, we have:

$$I^{<-2>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-2>}(x)dx = 1/17,$$

$$I^{<-1>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -9/17,$$

$$I_0 = \int_{x_j}^{x_{j+1}} \omega_j(x)dx = 18h/17,$$

$$I_1 = \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = 6h/17,$$

$$y_{j+1} = y_j\left(\frac{9}{17}\right) - y_{j-1}\left(\frac{-9}{17}\right) - y_{j-2}\left(\frac{1}{17}\right) +$$
$$+ f(x_j, y_j)\left(\frac{18h}{17}\right) + f(x_{j+1}, y_{j+1})\left(\frac{6h}{17}\right).$$

The error has the form:

$$|R| \leq \; Kh^5\|y^{\mathrm{V}}\|_{[x_{j-2},x_{j+1}]}, K > 0.$$

Let us solve the problem:

$$y' = -150(y - \cos(x)), y(0) = 0, x \in [0, 1].$$

The errors of the solution of the Cauchy problem by the method 2 are represented in Figures 7 and 8.

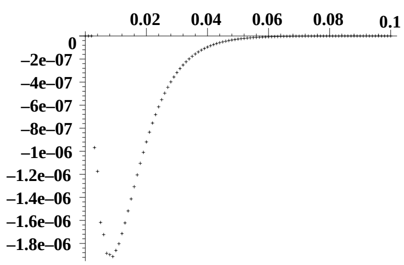

Fig. 7. Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ $(h = 0.001)$

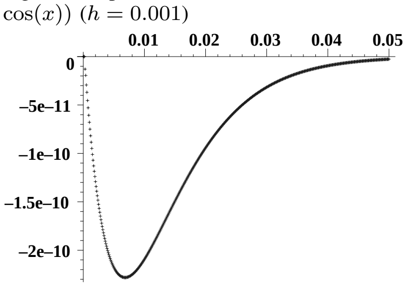

Fig. 8. Graph of the error of the solution of the problem $y' = -150(y - \cos(x))$ $(h = 0.0001)$

b) In the case $\varphi_1(x) = 1$, $\varphi_2(x) = x$, $\varphi_3(x) = exp(x)$, $\varphi_4(x) = exp(-x)$, we have:

$I^{<-2>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-2>}(x)dx = (-hexp(5h) + 2exp(5h) - 2hexp(4h) - 13exp(3h)h - 2exp(3h) + 6exp(3h)h^2 + 13hexp(2h) - 2exp(2h) + 6exp(2h)h^2 + 2hexp(h) + h + 2)/(hexp(5h) - 6hexp(4h) + 2exp(4h) + 2exp(3h)h^2 + 5exp(3h)h - 2exp(3h) - 5hexp(2h) + 2exp(2h)h^2 - 2exp(2h) + 2exp(h) + 6hexp(h) - h)$,

$I^{<-1>} = \int_{x_j}^{x_{j+1}} \omega_j^{<-1>}(x)dx = -(-exp(3h)h + 2exp(3h) - 2exp(2h) - 5hexp(2h) + 2exp(2h)h^2 + 5hexp(h) - 2exp(h) + 2exp(h)h^2 + h + 2)exp(h)/(hexp(5h) - 6hexp(4h) + 2exp(4h) + 2exp(3h)h^2 + 5exp(3h)h - 2exp(3h) - 5hexp(2h) + 2exp(2h)h^2 - 2exp(2h) + 2exp(h) + 6hexp(h) - h)$,

$I_0 = \int_{x_j}^{x_{j+1}} \omega_j(x)dx = -(3exp(5h) - 2hexp(5h) - 9exp(4h) + 4hexp(4h) - 2exp(3h)h + 6exp(3h) + 2hexp(2h) + 6exp(2h) - 9exp(h) - 4hexp(h) + 3 + 2h)h/(hexp(5h) - 6hexp(4h) + 2exp(4h) + 2exp(3h)h^2 + 5exp(3h)h - 2exp(3h) - 5hexp(2h) + 2exp(2h)h^2 - 2exp(2h) + 2exp(h) + 6hexp(h) - h)$,

$I_1 = \int_{x_j}^{x_{j+1}} \omega_{j+1}(x)dx = -(-exp(5h) + 3exp(4h) + 2hexp(4h) - 2exp(3h) - 6exp(3h)h + 6hexp(2h) - 2exp(2h) + 3exp(h) - 2hexp(h) - 1)h/(hexp(5h) - 6hexp(4h) + 2exp(4h) + 2exp(3h)h^2 + 5exp(3h)h - 2exp(3h) - 5hexp(2h) + 2exp(2h)h^2 - 2exp(2h) + 2exp(h) + 6hexp(h) - h)$,

The error has the form:

$$|R| \leq \; Kh^5\|y^{\mathrm{III}} - y^{\mathrm{V}}\|_{[x_{j-2},x_{j+1}]}, K > 0.$$

We solve the problem: $y' = -150(y - \cos(x)), y(0) = 0$, $x \in [0, 1]$, $h \leq 0.001$.

Table 3 shows the actual errors $R = |y_k - y(kh)|$, $h = 0.001$. Calculations were done in Maple, $Digits = 45$.

Table 3.

k	method a $h = 0.001$	method b $h = 0.001$
10	$0.186087 \cdot 10^{-5}$	$0.186079 \cdot 10^{-5}$
20	$0.191919 \cdot 10^{-6}$	$0.919144 \cdot 10^{-6}$
1000	$0.69494 \cdot 10^{-16}$	$0.13899 \cdot 10^{-15}$

Table 4 shows the actual errors $R = |y_k - y(kh)|$, $h = 0.0001$. Calculations were done in Maple, $Digits = 45$.

Table 4.

k	method a $h = 0.0001$	method b $h = 0.0001$
10	$0.678129 \cdot 10^{-10}$	$0.678099 \cdot 10^{-10}$
20	$0.129152 \cdot 10^{-9}$	$0.129146 \cdot 10^{-9}$
10000	$0.69511 \cdot 10^{-20}$	$0.13738 \cdot 10^{-19}$

Now let us solve the next problem:
$y' = -2(y - \sin(x)) + \cos(x)$, $y(0) = 0$.

The exact solution is $y = \sin(x)$. The error of solution of the Cauchy problem by the method a is represented on Figure 9. The error of solution of the Cauchy problem by the method b is represented on Figure 10.

VI. Conclusion

The results explained in the previous sections show that the numerical methods for a Cauchy problem could be used on practical calculations. The approximation by the nonpolynomial splines and numerical methods for Cauchy problem by the nonpolynomial splines may be better then by polynomial splines, but the values of Digits must be large enough.

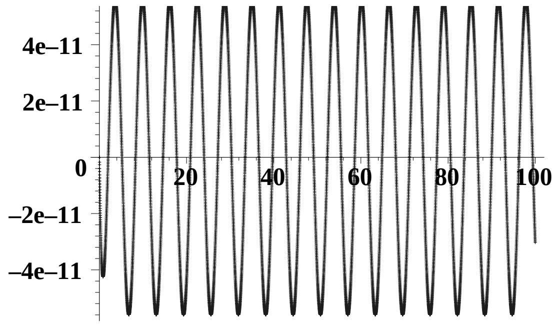

Fig. 9. Graph of the error of the solution of the problem $y' = -2(y - \sin(x)) + \cos(x)$, $y(0) = 0$ ($h = 0.01$) method a

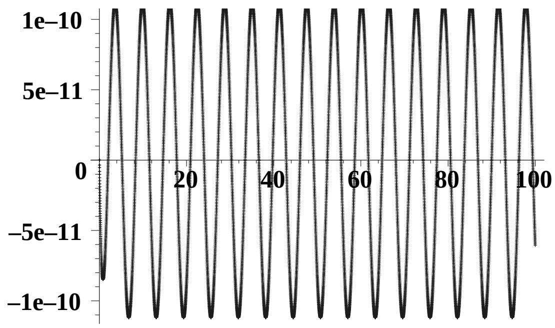

Fig. 10. Graph of the error of the solution of the problem $y' = -2(y - \sin(x)) + \cos(x)$, $y(0) = 0$ ($h = 0.01$), method b

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