Deterministic, stochastic and fractional technology diffusion model with distributed time delay

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Abstract– In this paper, we analyze a technology diffusion model with distributed time delays. In the process of adoption a technology, a firm takes into account the advertising effectiveness, government policy and production costs. The average time for making the final decision is introduced. The mathematical model is described by a system with three nonlinear differential equations with distributed time delays. Two types of kernels are taken into account: Dirac and weak. When the distributions are Dirac, the cases, with one delay and two different delays, are considered. The conditions for the existence of a Hopf bifurcation are given. The stochastic and fractional technology diffusion models are introduced and their orbits are displayed. The last part of the paper includes numerical simulations and conclusions.

Keywords– advertising diffusion model, bifurcation theory, Hopf bifurcation, stability

I. Introduction

The time delay has a high impact on the dynamics of a system, which can not only cause the loss of stability, but also induce various oscillations and periodic solutions [25], [10]. Many models from economics contain time delay, because often there is delay between an economic action and a consequence.

Recently, the diffusion of innovations has been intensively studied. Many models explaining the spread of a new product among a population of potential customers have been provided. Bass (1969) proposed a model which explains the sales of a new durable product which each household purchase only once [22]. Simon and Sebastian (1987) suggested several alternative ways integrating advertising in the basic Bass model [22]. Feichtinger et al. (1995) proved that if a firm has a poor advertising effectiveness, their new product is doomed to failure. Lately, Wang et al. (2006) extended the Bass model by introducing stage structures: the awareness and decision making stages, which leads to the introduction of time delay [24]. In 2010, Dhar et al. a time delay three compartment model consisting of adopter class, thinker class and non-adopter class has been formulated and analyzed [5]. A similar model has been detailed analyzed in [19], [20]. In 2012, Fanelli and Maddalena provided a mathematical model with time delay to describe the process of diffusion of a new technology, that requires great initial investments and public subsidies [7]. A continuation of the study is given in [2].

Following the above studies, in [17] we consider a technology diffusion model, where word of mouth, advertising effectiveness, government policy and production costs are taken into consideration. In the present paper we develop our past study and furthermore, the stochastic and fractional aspects are considered.

When a new technology is introduced into a market, it has to be advertised in order to persuade firms to adopt it. Also, the government policy is important. The adoption process is delayed, because there are some previous steps to follow: knowledge (the individual is exposed to innovation), persuasion (the individual forms an attitude toward the innovation), decision, implementation and confirmation or adoption [7]. Therefore, to take into account the time delay is imperative [17].

We consider the average time for a firm to evaluate whether to adopt a technology or not, thus the distributed time delay is introduced. The mathematical model is described by a system with three nonlinear differential equations with distributed delay.

Regarding the practical situations, for to capture the uncertainty about the environment in which the system is operating, we use the stochastic perturbation framed by a stochastic differential delay system.

When past economic behaviors affect present and futures ones, the fractional calculus should be considered [26]. In this sense, we present the fractional technology diffusion model.

The paper is organized as follows. In Section 2 the deterministic model with distributed time delay is described. The stability analysis for different types of kernels is analyzed in Section 3. In section 4, the stochastic technology diffusion model is introduced. The fractional technology diffusion model is presented in Section 5. Numerical simulations are carried out in Section 6. Finally, concluding remarks are given in Section 7.

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II. THE MATHEMATICAL MODEL

Let $x_1(t)$ be the number of non adopters (the number of firms that do not adopt the technology), $x_2(t)$ be the number of thinkers (potential adopters) and $x_3(t)$ be the number of adopters at moment $t \in \mathbb{R}$.

Contacts between non adopters and adopters are random and through word of mouth. Then the rate of generations of adopters is $\alpha x_1(t)x_3(t)$, where α is the word of mouth effectiveness. In [22] it is assumed that $\alpha = a_2 x_3(t)$, where a_2 reflects the advertising effectiveness.

If a_1 represents the number of newly non-adopters who enter the market, then the dynamics of $x_1(t)$ is given by:

$$\dot{x_1}(t) = a_1 - a_2 x_1(t) x_3(t)^2 + a_3 x_3(t),$$

where $a_3 x_3(t)$ is the rate of adopters who become non-adopters.

In the present paper we consider distributed time delay in the thinker class, because a firm needs time for adoption the technology or not.

The number of firms that know about the technology and decide to adopt it after a period of time, is given by:

$$\begin{array}{l}\left(e^{b_1(b_2-b_3)} + a_4 \int_{-\infty}^t h_3(t-s)x_3(s)ds\right)\cdot \\ \cdot\left(\int_{-\infty}^t h_2(t-s)x_2(s)ds\right),\end{array} \tag{1}$$

where $e^{b_1(b_2-b_3)}$ is an external factor of influence, where b_1 is a positive constant; b_2 is a government incentive and b_3 represents the production costs; a_4 is a positive constant.

Contacts between adopters at time t and past adopters can lead to the migration from the adopter class to the thinker class. Thus, the dynamics of $x_2(t)$ is described by:

$$\begin{array}{l}\dot{x_2}(t) = a_2 x_1(t) x_3(t)^2 - \\ -\left(e^{b_1(b_2-b_3)} + a_4 \int_{-\infty}^t h_3(t-s)x_3(s)ds\right)\cdot \\ \cdot\left(\int_{-\infty}^t h_2(t-s)x_2(s)ds\right) + \\ +a_5 x_3(t) \int_{-\infty}^t h_3(t-s)x_3(s)ds.\end{array} \tag{2}$$

If $a_6 x_3(t)$ is the rate of the adopters that are lost forever and taking into account the above considerations, then we have:

$$\begin{array}{l}\dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4 \int_{-\infty}^t h_3(t-s)x_3(s)ds\right)\cdot \\ \cdot\left(\int_{-\infty}^t h_2(t-s)x_2(s)ds\right) - \\ -a_5 x_3(t) \int_{-\infty}^t h_3(t-s)x_3(s)ds - (a_3+a_6)x_3(t).\end{array} \tag{3}$$

Thus, the deterministic model is given by the following non-linear differential system with distributed delay:

$$\begin{array}{l}\dot{x_1}(t) = a_1 - a_2 x_1(t) x_3(t)^2 + a_3 x_3(t), \\ \dot{x_2}(t) = a_2 x_1(t) x_3(t)^2 - \\ -\left(e^{b_1(b_2-b_3)} + a_4 \int_{-\infty}^t h_3(t-s)x_3(s)ds\right)\cdot \\ \cdot\left(\int_{-\infty}^t h_2(t-s)x_2(s)ds\right) + \\ +a_5 x_3(t) \int_{-\infty}^t h_3(t-s)x_3(s)ds, \\ \dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4 \int_{-\infty}^t h_3(t-s)x_3(s)ds\right)\cdot \\ \cdot\left(\int_{-\infty}^t h_2(t-s)x_2(s)ds\right) - \\ -a_5 x_3(t) \int_{-\infty}^t h_3(t-s)x_3(s)ds - (a_3+a_6)x_3(t),\end{array} \tag{4}$$

where $a_i > 0,\ i = 1,2,3,4,6$ are positive real numbers.

The functions $h_i : [0,\infty) \to [0,\infty),\ i = 2,3,$ are piecewise continuous and

$$\int_0^\infty h_i(s)ds = 1, \int_0^\infty s h_i(s)ds < \infty, i = 2,3.$$

The functions $h_i,\ i = 2,3$ are called kernels. In this paper we consider two types of kernels:

weak kernel (exponential distribution):

$$h_i(s) = d_i e^{-d_i s}, d_i > 0, i = 2,3$$

Dirac kernel (Dirac distribution):

$$h_i(s) = \delta(s-\tau_i), \tau_i \geq 0, i = 2,3$$

where δ is the Dirac function.

III. STABILITY ANALYSIS

System (4) has only one non trivial positive equilibrium point $E(x_{10}, x_{20}, x_{30})$ obtained as:

$$x_{10} = \frac{(a_3+a_6)a_6}{a_1 a_2}, x_{20} = \frac{(a_2 x_{10} + a_5)x_{30}^2}{e^{b_1(b_2-b_3)} + a_4 x_{30}}, x_{30} = \frac{a_1}{a_6}. \tag{5}$$

By carrying out the translation $u_1(t) = x_1(t) - x_{10}$, $u_2(t) = x_2(t) - x_{20}$, $u_3(t) = x_3(t) - x_{30}$, we obtain the linearized system of (4) at the equilibrium point:

$$\begin{array}{l}\dot{u}_1(t) = a_{11}u_1(t) + a_{13}u_3(t), \\ \dot{u}_2(t) = a_{21}u_1(t) + a_{23}u_3(t) + \\ +b_{22}\int_{-\infty}^0 h_2(t-s)u_2(s)ds + \\ +b_{23}\int_{-\infty}^0 h_3(t-s)u_3(s)ds, \\ \dot{u}_3(t) = a_{33}u_3(t) + b_{32}\int_{-\infty}^0 h_2(t-s)u_2(s)ds + \\ +b_{33}\int_{-\infty}^0 h_3(t-s)u_3(s)ds,\end{array} \tag{6}$$

where

$$\begin{array}{l}a_{11} = -a_2 x_{30}^2, a_{13} = a_3 - 2a_2 x_{10} x_{30}, a_{21} = -a_{11}, \\ a_{23} = 2a_2 x_{10} x_{30} + a_5 x_{30}, a_{33} = -a_5 x_{30} - a_3 - a_6, \\ b_{22} = e^{b_1(b_2-b_3)} - a_4 x_{30}, b_{23} = -a_4 x_{20} + a_5 x_{30}, \\ b_{32} = -b_{22}, b_{33} = -b_{23}.\end{array} \tag{7}$$

Using the identities [13]:

$$\int_{-\infty}^t h_i(t-s)u_i(s)ds = \int_{-\infty}^0 h_i(-s)u_i(t+s)ds, i = 2,3$$

and

$$\int_{-\infty}^{0} h_i(-s)e^{\lambda(t+s)}ds = e^{\lambda t}\int_{-\infty}^{0} h_i(s)e^{\lambda s}ds, i = 2,3$$

the characteristic equation of (6) is:

$$\begin{array}{l}\lambda^3+m_2\lambda^2+m_1\lambda+\\ +(n_2\lambda^2+n_1\lambda+n_0)\int_{-\infty}^{0} e^{\lambda s}h_2(-s)ds+\\ +(p_2\lambda^2+p_1\lambda)\int_{-\infty}^{0} e^{\lambda s}h_3(-s)ds = 0,\end{array} \quad (8)$$

where

$$\begin{array}{l} m_2 = a_2x_{30}^2 + a_5x_{30} + a_3 + a_6,\\ m_1 = a_2x_{30}^2(a_5x_{30} + a_3 + a_6),\\ n_2 = e^{b_1(b_2-b_3)} - a_4x_{30}, n_1 = b_{22}(a_{11}+a_{23}),\\ n_0 = a_{11}b_{22}(a_{13}-b_{23}), p_2 = -a_4x_{20}+a_5x_{30},\\ p_1 = a_2x_{30}^2(-a_4x_{20} + a_5x_{30}).\end{array} \quad (9)$$

For the analysis of system (4), there we consider four cases:

Case 1. $h_i(s) = \delta(s-\tau_i)$, $i = 2,3$;
Case 2. $h_2(s) = d_2e^{-d_2s}$, $h_3(s) = \delta(s-\tau_3)$;
Case 3. $h_2(s) = \delta(s-\tau_2)$, $h_3(s) = d_3e^{-d_3s}$;
Case 4. $h_2(s) = d_2e^{-d_2s}$, $h_3(s) = d_3e^{-d_3s}$.

A. *Case 1.* $h_i(s) = \delta(s-\tau_i)$, $i = 2,3$

Case 1.1. $\tau_2 = \tau_3 = 0$**.**

The characteristic equation (8) becomes:

$$\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0 = 0, \quad (10)$$

where

$$\alpha_2 = m_2 + n_2 + p_2, \alpha_1 = m_1 + n_1 + p_1, \alpha_0 = n_0. \quad (11)$$

Applying Routh-Hurwitz criterion, we have:

Proposition 1: ([19], [20]) If $\tau_2 = \tau_3 = 0$ and

$$\alpha_2 > 0, \alpha_1 > 0, \alpha_0 > 0, \alpha_1\alpha_2 - \alpha_0 > 0 \quad (12)$$

then eq. (10) has the roots with negative real part. The equilibrium point E is locally asymptotically stable.

Case 1.2. $\tau_2 \neq 0$**,** $\tau_3 = 0$**.**

The characteristic equation (8) becomes:

$$\lambda^3+\beta_2\lambda^2+\beta_1\lambda+(n_2\lambda^2+n_1\lambda+n_0)e^{-\lambda\tau_2} = 0, \quad (13)$$

with

$$\beta_2 = m_2 + p_2, \beta_1 = m_1 + p_1.$$

For the occurrence of the Hopf bifurcation, a critical time delay τ_{20} must exist such that $\lambda_{1,2}(\tau_{20}) = \pm i\omega_{20}$, $\omega_{20} > 0$ and all other eigenvalues have negative real part at $\tau_2 = \tau_{20}$ and

$$\text{Re}\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)|_{\tau_2=\tau_{20}} \neq 0. \quad (14)$$

Assume that a pair of imaginary roots exists for (13) and $\lambda = i\omega$, $\omega > 0$ is one of its roots.

Substituting in (13) and separating the real and imaginary parts, we obtain:

$$\begin{array}{l}(n_0 - n_2\omega^2)\cos(\omega\tau_2) + \omega n_1\sin(\omega\tau_2) = \beta_2\omega^2\\ (n_0 - n_2\omega^2)\sin(\omega\tau_2) - \omega n_1\cos(\omega\tau_2) = \beta_1\omega - \omega^3.\end{array} \quad (15)$$

Eliminating $\sin(\omega\tau_2)$ and $\cos(\omega\tau_2)$ from (15), the following polynomial equation in ω can be obtained

$$\omega^6 + q_4\omega^4 + q_2\omega^2 + q_0 = 0 \quad (16)$$

where $q_4 = \beta_2^2 - 2\beta_1 - n_2^2$, $q_2 = \beta_1^2 - n_1^2 + 2n_0n_2$, $q_0 = -n_0^2$.

If $\omega^2 = z$, then (16) becomes:

$$f_1(z) := z^3 + q_4z^2 + q_2z + q_0 = 0. \quad (17)$$

Because $q_0 < 0$, eq. (17) has at least one positive root and the characteristic equation (13) has at least a pair of purely imaginary roots.

Now, we establish condition (14).

Let $\lambda(\tau_2) = \alpha(\tau_2)+i\omega(\tau_2)$ be a root of (8) satisfying $\alpha(\tau_{20}) = 0$, $\omega(\tau_{20}) = \omega_{20}$.

Lemma 1. ([9]) *If* $f_1'(\omega_{20}^2) \neq 0$ *then* $\frac{d}{d\tau_2}$*Re*$(\lambda(\tau_2))|_{\tau_2=\tau_{20}} > 0$.

Theorem 1. ([17]) *Suppose that* $\alpha_2 > 0$, $\alpha_1 > 0$, $\alpha_0 > 0$, $\alpha_1\alpha_2 - \alpha_0 > 0$. *The equilibrium point E of (4) is locally asymptotically stable when* $\tau_2 < \tau_{20}$ *and unstable when* $\tau_2 > \tau_{20}$, *where* τ_{20} *is defined by:*

$$\begin{array}{l}\tau_{20} =\\ = \frac{1}{\omega_{20}}(arccos(\frac{\beta_2\omega_{20}^2(n_0-n_2\omega_{20}^2)+}{(n_0-n_2\omega_{20}^2)^2+\omega_{20}^2n_1^2})+\\ +\frac{\omega_{20}n_1(\omega_{20}^3-\beta_1\omega_{20})}{(n_0-n_2\omega_{20}^2)^2+\omega_{20}^2n_1^2}) + 2m\pi), m = 0,1,2,.....\end{array} \quad (18)$$

In addition, if $f_1'(\omega_{20}^2) \neq 0$ then a Hopf bifurcation occurs when $\tau_2 = \tau_{20}$.

Case 1.3. $\tau_2 = \tau_3 = \tau$

We analyze system (4) in the case $\tau_2 = \tau_3 = \tau$. System (4) becomes:

$$\begin{array}{l}\dot{x_1}(t) = a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t),\\ \dot{x_2}(t) = a_2x_1(t)x_3(t)^2-\\ -\left(e^{b_1(b_2-b_3)} + a_4x_3(t-\tau)\right)x_2(t-\tau)+\\ +a_5x_3(t)x_3(t-\tau),\\ \dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4x_3(t-\tau_1)\right)x_2(t-\tau)-\\ -a_5x_3(t-\tau)x_3(t) - (a_3+a_6)x_3(t),\end{array} \quad (19)$$

where $a_i > 0$, $i = 1,2,3,4,6$ are positive real numbers.

System (19) has a unique positive equilibrium point $E(x_{10}, x_{20}, x_{30})$, where x_{10}, x_{20}, x_{30} are given by (5).

By carrying out the translation $u_i(t) = x_i(t) - x_{i0}$, $i = 1,2,3$, we obtain the linear system of (19):

$$\begin{array}{l}\dot{u}_1(t) = a_{11}u_1(t) + a_{13}u_3(t),\\ \dot{u}_2(t) = a_{21}u_1(t) + a_{23}u_3(t) + b_{22}u_2(t-\tau)+\\ +b_{23}u_3(t-\tau),\\ \dot{u}_3(t) = a_{33}u_3(t)+b_{32}u_2(t-\tau)+b_{33}u_3(t-\tau),\end{array} \quad (20)$$

where a_{11},a_{13}, a_{21}, a_{23}, a_{33}, b_{22}, b_{23}, b_{32}, b_{33} are given by (7).

The characteristic equation of (20) is:

$$\begin{aligned}&\lambda^3+m_2\lambda^2+m_1\lambda+((n_2+p_2)\lambda^2+\\&+(n_1+p_1)\lambda+n_0)e^{-\lambda\tau}=0,\end{aligned} \tag{21}$$

where m_2, m_1, n_2, n_1, n_0, p_1, p_2 are given by (9).

Using Proposition 1, we have:

Proposition 2: If $\tau = 0$ and (12) holds, then the equilibrium point E is locally asymptotically stable.

Let $\tau > 0$. For the occurrence of the Hopf bifurcation, a critical time delay τ_0 must exist such that $\lambda_{12}(\tau_0) = \pm i\omega_0$, $\omega_0 > 0$ and for all other eigenvalues have negative real part for eq. (21) and

$$\text{Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)|_{\tau=\tau_0} \neq 0. \tag{22}$$

Assume that a pair of imaginary roots exists for (21) and $\lambda = i\omega$, $\omega > 0$ is one of its roots.

Substituting in (21) and separating the real and imaginary parts we obtain:

$$\begin{aligned}&(n_0-(n_2+p_2)\omega^2)\cos(\omega\tau)+(n_1+p_1)\omega\sin(\omega\tau)=\\&=m_2\omega^2,\\&(n_0-(n_2+p_2)\omega^2)\sin(\omega\tau)-(n_1+p_1)\omega\cos(\omega\tau)=\\&=m_1\omega-\omega^3.\end{aligned} \tag{23}$$

Eliminating $\sin(\omega\tau)$ and $\cos(\omega\tau)$, the following polynomial equation in ω can be obtained:

$$\omega^6 + r_4\omega^4 + r_2\omega^2 + r_0 = 0, \tag{24}$$

where

$$\begin{aligned}&r_4 = m_2^2 - 2m_1 - (n_2+p_2)^2, r_0 = -n_0^2,\\&r_2 = m_1^2 + 2n_0(n_2+p_2) - (n_1+p_1)^2.\end{aligned} \tag{25}$$

If $\omega^2 = z$. then (24) becomes:

$$f_2(z) := z^3 + r_4z^2 + r_2z + r_0 = 0. \tag{26}$$

Because $r_0 < 0$, (26) has at least one positive root and the characteristic equation (21) has at least one pair of purely imaginary roots.

Let $\lambda(\tau) = \alpha(\tau)+i\omega(\tau)$ be a root of (21) and $\tau_0 > 0$ so that $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0 > 0$.

Lemma 2. If $f_2'(\omega_0^2) \neq 0$, then $\text{Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)|_{\tau=\tau_0} > 0$.

By Lemma 2, we have the following theorem:

Theorem 2. *Suppose that*

$$\begin{aligned}&m_1+n_1+p_1>0, m_2+n_2+p_2>0, n_0>0,\\&(m_1+n_1+p_1)(m_2+n_2+p_2)-n_0>0.\end{aligned} \tag{27}$$

The equilibrium point E of system (19) is locally asymptotically stable, when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, where τ_0 is defined by:

$$\tau_0 = \frac{1}{\omega_0}\left(arccos\left(\frac{H_0}{L_0}\right)\right) + 2n\pi, n = 0, 1, 2, ...$$

where $H_0 = m_2\omega_0^2(n_0-(n_2+p_2))\omega_0^2+(n_1+p_1)\omega_0(\omega^3-m_1\omega_0)$, $L_0 = (n_0-(n_2+p_2)\omega_0^2)^2+(n_1+p_1)^2\omega_0^2$.

In addition, if $f_2'(\omega_0^2) \neq 0$, then a Hopf bifurcation occurs when $\tau = \tau_0$.

Case 1.4 $\tau_2 \geq 0$**,** $\tau_3 \geq 0$**.**

The characteristic equation (8) becomes:

$$\begin{aligned}&\lambda^3+m_2\lambda^2+m_1\lambda+(p_2\lambda^2+p_1\lambda)e^{-\lambda\tau_3}+\\&+(n_2\lambda^2+n_1\lambda+n_0)e^{-\lambda\tau_2}=0.\end{aligned} \tag{28}$$

We consider τ_2 in its stable interval $[0, \tau_{20}]$ and regard τ_3 as a parameter. Without loss of generality, we assume that the conditions from Proposition 1 hold. Let $\lambda = i\omega$ ($\omega(\tau_2) > 0$) be a root of (28). Then, we can obtain:

$$\begin{aligned}&\omega^6+\varepsilon_5\omega^5+\varepsilon_4\omega^4+\varepsilon_3\omega^3+\\&+\varepsilon_2\omega^2+\varepsilon_1\omega+\varepsilon_0=0,\end{aligned} \tag{29}$$

where

$$\begin{aligned}&\varepsilon_5(\omega) = -2n_2 sin(\omega\tau_2),\\&\varepsilon_4(\omega) = -p_2^2+n_2^2+m_2^2-2m_1+\\&+(2m_2n_2-2n_1)\cos(\omega\tau_2),\\&\varepsilon_3(\omega) = 2(n_2m_1+n_0-n_1m_2)\sin(\omega\tau_2),\\&\varepsilon_2(\omega) = -p_1^2-2n_0n_2+n_1^2+m_1^2+\\&+(-2m_2n_0+2m_1n_1)\cos(\omega\tau_2),\\&\varepsilon_1(\omega) = -2m_1n_0\sin(\omega\tau_2),\\&\varepsilon_0(\omega) = n_0^2.\end{aligned}$$

Denote

$$\begin{aligned}&K(\omega) = \omega^6+\varepsilon_5\omega^5+\varepsilon_4\omega^4+\\&+\varepsilon_3\omega^3+\varepsilon_2\omega^2+\varepsilon_1\omega+\varepsilon_0.\end{aligned} \tag{30}$$

We assume that there exists $\omega_{31} > 0$ so that $K(\omega_{31}) = 0$. For ω_{31}, there exists a sequence $\{\tau_{3j},\ j = 1, 2...\}$ such that (29) holds.

Let $\tau_{31} = \max\{\tau_{3i},\ i = 1, ...n\}$. When $\tau_3 = \tau_{31}$, (28) has a pair of purely imaginary roots $\pm i\omega_{31}$, for $\tau_2 \in [0, \tau_{20})$.

In the following we assume that

$$\left(\frac{d\text{Re}(\lambda(\tau_3))}{d\tau_3}\right)^{-1}_{\lambda=i\omega_{31}} \neq 0.$$

Thus, by the general Hopf bifurcation theorem ([10]), we have the following result on stability and Hopf bifurcation for system (4):

Theorem 3: ([17]) Assume that the above condition is satisfied and that $\tau_2 \in [0, \tau_{20})$. Then, the equilibrium point E is locally asymptotically stable when $\tau_3 \in [0, \tau_{31})$. Moreover, when $\tau_3 = \tau_{31}$ the system (4) undergoes a Hopf bifurcation at E.

If ω_{31} is a positive root of (29), then τ_{3j} is given by

$$\tau_{3j} = \frac{1}{\omega_{31}}\left[\arccos\left(\frac{H_{31}}{L_{31}}\right) + 2j\pi\right], \quad j = 0, 1, 2..., \tag{31}$$

where

$$H_{31} = (-m_2\omega_{31}^2 + n_1\omega_{31}\sin(\omega_{31}\tau_2)+ \\ +(n_0 - n_2\omega_{31}^2)\cos(\omega 31\tau_2))p_2\omega_{31}^2+ \\ +(\omega_{31}^3 - m_1\omega_{31} + (n_0 - n_2\omega_{31}^2)\sin(\omega_{31}\tau_2)- \\ -n_1\omega_{31}\cos(\omega_{31}\tau_2))p_1\omega_{31}, \\ L_{31} = (p_2\omega_{31}^2)^2 + (p_1\omega_{31})^2.$$

B. *Case 2.* $h_2(s) = d_2e^{-d_2s}$, $h_3(s) = \delta(s-\tau_3)$ $d_2 > 0$, $\tau_3 \geq 0$

System (4) becomes:

$$\begin{aligned} &\dot{x_1}(t) = a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t) \\ &\dot{x_2}(t) = a_2x_1(t)x_3(t)^2 - (e^{b_1(b_2-b_3)}+ \\ &+a_4x_3(t-\tau_3))y_4(t)+ \\ &+a_5x_3(t)x_3(t-\tau_3), \\ &\dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4x_3(t-\tau_3)\right)y_4(t)- \\ &-a_5x_3(t)x_3(t-\tau_3) - (a_3+a_6)x_3(t), \\ &\dot{y_4}(t) = d_2(x_2(t) - y_4(t)). \end{aligned} \tag{32}$$

The analysis of the equilibrium point E for (32) can be done studying the roots of the equation:

$$\lambda^4 + (m_2+d_2)\lambda^3 + (m_1 + m_2d_2 + d_2n_2)\lambda^2+ \\ +(m_1+n_1)d_2\lambda+ \\ +d_2n_0 + e^{-\lambda\tau_3}(p_2\lambda^3 + (p_1+d_2p_2)\lambda^2 + d_2p_1\lambda) = 0. \tag{33}$$

The analysis of (33) can be done in a similar way as in Case 1.

C. *Case 3.* $h_3(s) = d_3e^{-d_3s}$, $h_2(s) = \delta(s-\tau_2)$, $d_3 > 0$, $\tau_2 \geq 0$

System (4) becomes:

$$\begin{aligned} &\dot{x_1}(t) = a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t) \\ &\dot{x_2}(t) = a_2x_1(t)x_3(t)^2 - (e^{b_1(b_2-b_3)}+ \\ &+a_4y_4(t))x_2(t-\tau_2) + a_5x_3(t)y_4(t), \\ &\dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4y_4(t)\right)x_2(t-\tau_2)- \\ &-a_5x_3(t)y_4(t) - (a_3+a_6)x_3(t), \\ &\dot{y_4}(t) = d_3(x_3(t) - y_4(t)). \end{aligned} \tag{34}$$

The analysis of the equilibrium point E for (34) can be done studying the roots of the equation:

$$\lambda^4 + (m_2+d_3)\lambda^3 + (m_1 + m_2d_3 + d_3p_2)\lambda^2+ \\ +(m_1+p_1)d_2\lambda + e^{-\lambda\tau_2}(n_2\lambda^3+ \\ +(n_1+d_3n_2)\lambda^2 + (n_0+d_3n_1)\lambda + d_3n_0) = 0. \tag{35}$$

The analysis of (37) can be done in a similar way as in Case 1.

D. *Case 4.* $h_2(s) = d_2e^{-d_2s}$, $h_3(s) = d_3e^{-d_3s}$, $d_2 > 0$, $d_3 > 0$

System (4) becomes:

$$\begin{aligned} &\dot{x_1}(t) = a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t) \\ &\dot{x_2}(t) = a_2x_1(t)x_3(t)^2 - (e^{b_1(b_2-b_3)}+ \\ &+a_4y_5(t))y_4(t) + a_5x_3(t)y_5(t), \\ &\dot{x_3}(t) = \left(e^{b_1(b_2-b_3)} + a_4y_5(t)\right)y_4(t)- \\ &-a_5x_3(t)y_5(t) - (a_3+a_6)x_3(t), \\ &\dot{y_4}(t) = d_2(x_2(t) - y_4(t)), \\ &\dot{y_5}(t) = d_3(x_3(t) - y_5(t)) \end{aligned} \tag{36}$$

The analysis of the equilibrium point E for (36) can be done studying the roots of the equation:

$$\lambda^5 + (m_2+d_2+d_3)\lambda^4 + (m_1 + m_2(d_2+d_3)+ \\ +d_2d_3)\lambda^3 + (m_1(d_2+d_3) + d_2d_3m_2 + d_2n_2+ \\ +d_3p_2)\lambda^2 + (d_2d_3m_1 + d_2n_1 + d_3p_1)\lambda + d_2n_0 = 0. \tag{37}$$

The analysis of (37) can be done in a similar way as in Case 1.

IV. TECHNOLOGY DIFFUSION STOCHASTIC MODEL

Let the probability space $(\Omega, \mathcal{F}, P)$ be given, and $w(t) \in R$ be a scalar Wiener process defined on Ω having independent stationary Gaussian increments with $w(0) = 0$, $E(w(t) - w(s)) = 0$ and $E(w(t)w(s)) = min(t,s)$. The symbol E denotes the mathematical expectation. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval [11].

We are interested in knowing the effect of the noise perturbation on the equilibrium $E(x_{10}, x_{20}, x_{30})$. The stochastic differential system with delay is:

$$\begin{aligned} &dx_1(t) = (a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t))dt+ \\ &+\sigma_1(x_1(t) - x_{10})dw(t), \\ &dx_2(t) = (a_2x_1(t)x_3(t)^2- \\ &-\left(e^{b_1(b_2-b_3)} + a_4x_3(t)\right)x_2(t)+ \\ &+a_5x_3(t)x_3(t))dt + \sigma_2(x_2(t) - x_{20})dw(t), \\ &dx_3(t) = ((e^{b_1(b_2-b_3)} + a_4x_3(t))x_2(t)- \\ &-a_5x_3(t)x_3(t) - (a_3+a_6)x_3(t))dt+ \\ &+\sigma_3(x_3(t) - x_{30})dw(t), \end{aligned} \tag{38}$$

where $\sigma_1, \sigma_2, \sigma_3 > 0$ are scalars and we denote by $x_i(t) = x_i(t,\omega)$, $i = 1,2,3$, $\omega \in \Omega$ the components of a stochastic process on the probability space [11], [16].

In the numerical simulation section we can visualize the orbits of (38).

V. TECHNOLOGY DIFFUSION FRACTIONAL MODEL

The fractional-order system associated to (19), the technology diffusion model with $\tau_2 = \tau_3 = \tau$ can be described by:

$$\begin{aligned}
&D^{q_1}x_1(t) = a_1 - a_2x_1(t)x_3(t)^2 + a_3x_3(t),\\
&D^{q_2}x_2(t) = a_2x_1(t)x_3(t)^2 - (e^{b_1(b_2-b_3)}+\\
&+a_4x_3(t-\tau))x_2(t-\tau) + a_5x_3(t)x_3(t-\tau),\\
&D^{q_3}x_3(t) = (e^{b_1(b_2-b_3)} + a_4x_3(t-\tau))x_2(t-\tau)-\\
&-a_5x_3(t-\tau)x_3(t) - (a_3+a_6)x_3(t),
\end{aligned} \tag{39}$$

where D_t^q is defined by ([14], [16]):

$$D_t^q = \begin{cases} \frac{d^q}{dt^q}, & \text{Re}(q) > 0 \\ 1 & \text{Re}(q) = 0 \\ \int_0^t (ds)^{-q} & \text{Re}(q) < 0 \end{cases}$$

and q is the derivative order, that can be a complex number with Re(q) the real part of q. There are many definitions for general fractional derivative. The most frequently used ones are: the Grunwald-Letnikov definition, the Riemann-Liouville and the Caputo definitions [18].

We consider $q_1 = q_2 = q_3 = q \in (0,1)$. In this case, the fractional order system is commesurate-order ([23]).

As in [14], in this paper we use the Caputo definition for the fractional derivative.

Using the fractional order Routh-Hurwitz condition for (21) with $\tau = 0$, we obtain that the equilibrium point $E(x_{10}, x_{20}, x_{30})$ is local asymptotically stable for all $q \in (0,1)$.

VI. Numerical simulations

We use Maple and Matlab for the numerical simulations. For the parameters $a_1 = 0.5$, $a_2 = 0.15$, $a_3 = 0.12$, $a_4 = 0.13$, $a_5 = 0.1089$, $a_6 = 0.2$, $b_1 = 0.6$, $b_2 = 0.8$, $b_3 = 0.2$ all the conditions from Proposition 1 are satisfied.

When both kernels of system (4) are Dirac, $h_i(s) = \delta(s-\tau_i)$, $i = 2,3$ and $\tau_2 \geq 0$, $\tau_3 \geq 0$, the time delay τ_2 is considered in its stable interval $[0, \tau_{20}]$. By (18) we find $\tau_{20} = 1.68$. The equilibrium point E is locally asymptotically stable if $\tau_3 \in [0, \tau_{31})$ and unstable for $\tau_3 > \tau_{31}$. For $\tau_3 = \tau_{31}$, system (4) exhibits a Hopf bifurcation and the solution is periodically. By (31) we obtain $\tau_{31} = 0.16$. In Fig. 1, Fig. 2, Fig. 3, the orbits corresponding to the number of non-adopters ($x_1(t)$), the number of thinkers ($x_2(t)$) and the number of adopters ($x_3(t)$) are periodically.

Fig. 1 For system (4), $(t, x_1(t))$ is periodically when $\tau_2 = 1.68$, $\tau_{31} = 0.16$

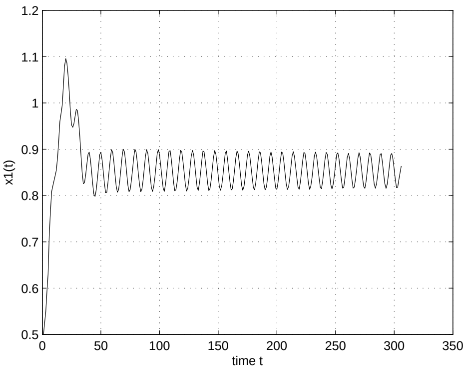


Fig. 2 For system (4), $(t, x_2(t))$ is periodically when $\tau_2 = 1.68$, $\tau_{31} = 0.16$

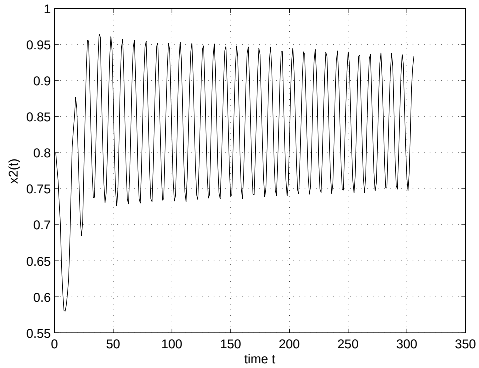
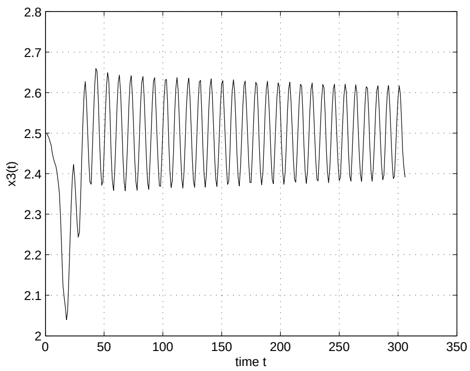


Fig. 3 For system (4), $(t, x_3(t))$ is periodically when $\tau_2 = 1.68$, $\tau_{31} = 0.16$



For system (4), when both kernels are weak, $h_2(s) = d_2e^{-d_2s}$, $h_3(s) = d_3e^{-d_3s}$ with $d_2 = 0.3$ and $d_3 = 0.6$, we obtain the orbits given in Fig. 4, Fig. 5 and Fig.6.

Fig. 4 The orbit $(t, x_1(t))$ of (4) when $d_2 = 0.3$ and $d_3 = 0.6$

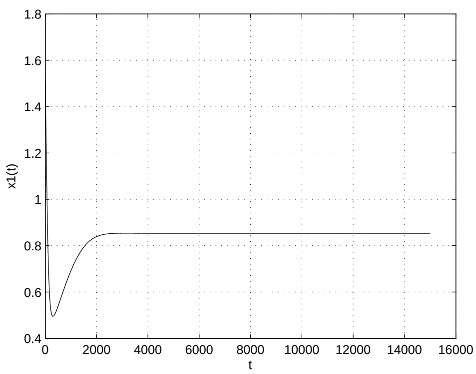


Fig. 5 The orbit $(t, x_2(t))$ of (4) when $d_2 = 0.3$ and $d_3 = 0.6$

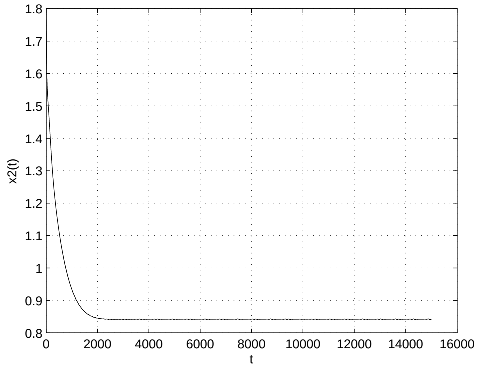
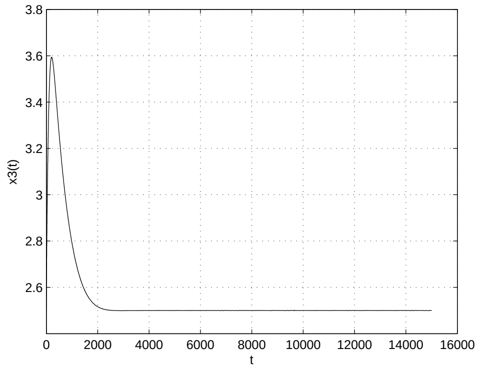


Fig. 6 The orbit $(t, x_3(t))$ of (4) when $d_2 = 0.3$ and $d_3 = 0.6$



The numerical simulation of the stochastic model (38) is done using Milstein algorithm in Matlab for $\tau = 0$. The orbits are given in Fig. 7, Fig. 8 and Fig. 9.

Fig. 7 The orbit $(j, x_1(j, \omega))$ of (38) when $\sigma_i = 1.2, i = 1, 2, 3$

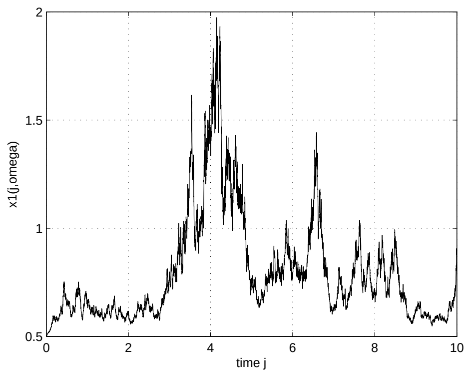


Fig. 8 The orbit $(j, x_2(j, \omega))$ of (38) when $\sigma_i = 1.2, i = 1, 2, 3$

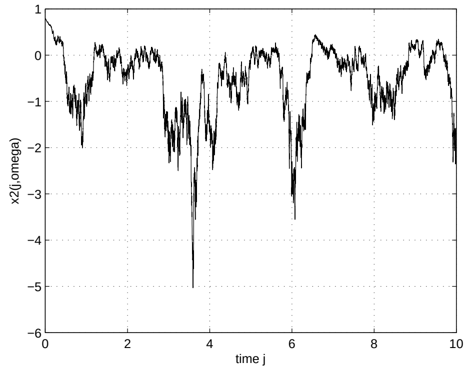
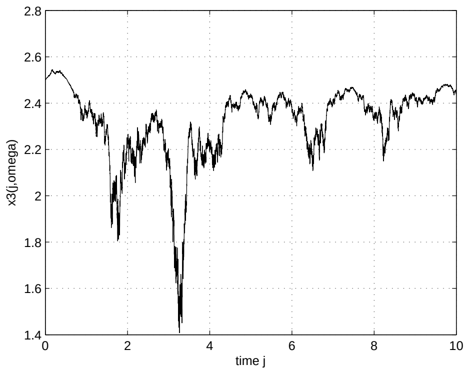


Fig. 9 The orbit $(j, x_3(j, \omega))$ of (38) when $\sigma_i = 1.2, i = 1, 2, 3$



Using the numerical method from [18], for $\tau = 0$ and $q_1 = q_2 = q_3 = 0.98$, we obtain the following orbits:

Fig. 10 The orbit $(j, x_1(j))$ of (39) when $\tau = 0$ and $q_1 = q_2 = q_3 = 0.98$

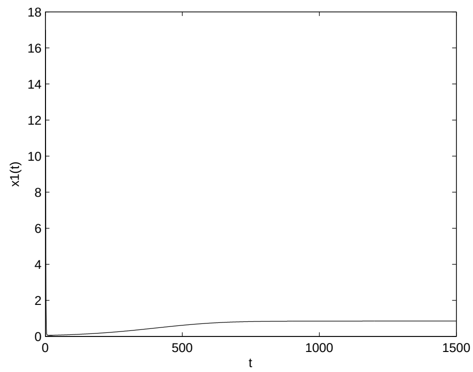


Fig. 11 The orbit $(j, x_2(j))$ of (39) when $\tau = 0$ and $q_1 = q_2 = q_3 = 0.98$

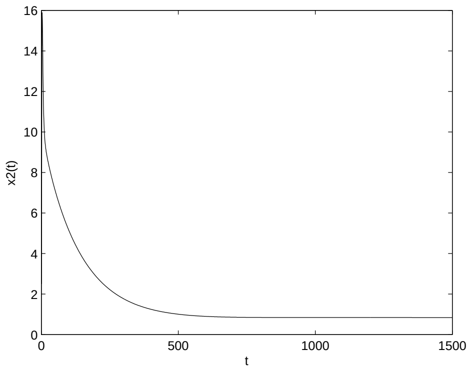
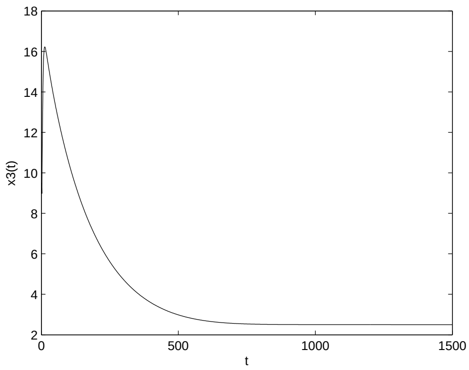


Fig. 12 The orbit $(j, x_3(j))$ of (39) when $\tau = 0$ and $q_1 = q_2 = q_3 = 0.98$



Using the numerical method from [15], for $\tau = 0.4$ and $q_1 = q_2 = q_3 = 0.98$, we obtain the following orbits:

Fig. 13 The orbit $(j, x_1(j))$ of (39) when $\tau = 0.4$ and $q_1 = q_2 = q_3 = 0.98$

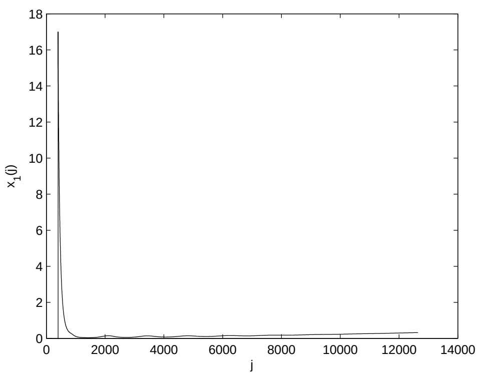


Fig. 14 The orbit $(j, x_2(j))$ of (39) when $\tau = 0.4$ and $q_1 = q_2 = q_3 = 0.98$

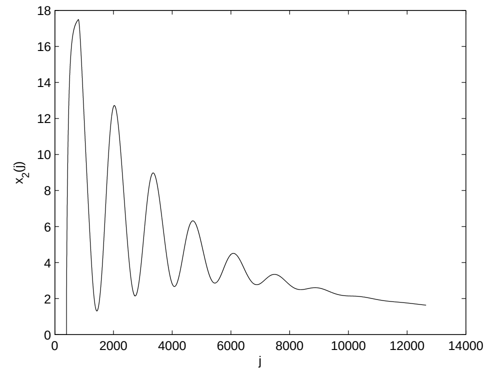
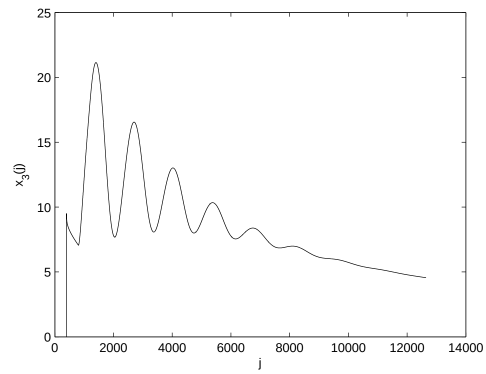


Fig. 15 The orbit $(j, x_3(j))$ of (39) when $\tau = 0.4$ and $q_1 = q_2 = q_3 = 0.98$



The numerical simulations verify the theoretical results.

VII. Conclusion

In the present paper a mathematical model that describes the process of diffusion of a technology has been considered, in the deterministic, stochastic and fractional cases.

There are three variables corresponding to the adopter class, the thinker class and the non-adopter class. It is assumed that there is an average time for a firm to decide whether to adopt the technology or not. The mathematical model is described by a non-linear differential system with distributed time delay. The stability analysis has been carried out about the unique positive equilibrium point.

If there is no delay, the system is locally asymptotically stable under some conditions involving parameters of system (4). Introducing the distributed delay we have considered two types of kernels: weak and Dirac.

When there is distributed time delay, we have prezented the cases: both kernels are Dirac, one kernel is weak and the other is Dirac and both kernels are weak. When both kernels are Dirac, we have investigated the existence of the Hopf-bifurcation for one time delay and for two different time delays, as well. For one time delay, in some conditions of the parameters, we have proved that a family of periodic solutions bifurcates from the equilibrium point when the bifurcation parameter passes through a critical value. For two different time delays, we fix the first one in its stable interval, and with respect to the second one, we find sufficient conditions for the existence of a change in local stability of the stationary state, from stable to unstable.

A similar analysis can be done for other types of kernel: strong, uniform or Gauss.

We have also considered the stochastic and the fractional approaches of the deterministic model, where the corresponding orbits have been displayed.

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