Estimation of the maximum error induced by the uncertainties of the model of a process in the determination of a control law

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Abstract**—** This paper presents an approach which enables to check the quality of the control law for a process whose evolution is described in the state space and the control is made from accessible information about the process and its desired evolution. For that it deals with the definition of the attractors characterizing the precision of control laws for a nonlinear process in presence of uncertainties and/or bounded perturbations This approach is based on aggregation techniques for stability study and on the choice of the state representations of the process.

Keywords**—** Attractor, aggregation techniques, comparison systems, nonlinear systems.

I. INTRODUCTION

This study deals the determination of attractors and attraction domains of nonlinear continuous systems, using the aggregation techniques for stability study, and by the choice of adequate representations of the studied systems. .
The aggregation techniques by the use of vector norms are applied to the definition of comparison systems of a complex non-linear and / or non-stationary model for a given process with possible uncertainties.
The study of the stability of this comparison system makes it easy to study the stability of the initial process and to define an attractor, when the system is locally unstable. The use of comparison system enables to determine an attractor of the studied system with an estimation of the attraction domain.
A classical way to control nonlinear systems is to compute a linear controller using the first-order approximation of the system dynamics for example around the origin $x = 0$, which gives a local linear approximation of the system. A non-approximating method consists to define a nonlinear feedback control law which realizes an input-output decoupling by linearization [1]-[5]. For an ill-defined model or a perturbed process the determination of a suitable control appears more difficult and is usually achieved by a precise and non-perturbed model. In such a case it appears very important to estimate by overvaluation the error induced by this simplification.

For high order systems, generally described in the state space, the stability conditions are obtained for the complete system or sub-systems with their interactions [6-9].
The influence of the choice of the state space description and of the characteristic matrix, on the determination of the stability domain, appears in the implementation of the stability criteria based on the work of Borne and Gentina [6,7], [10-12] with the use of vector norms and defining comparison systems for complex process [10-14]

In this paper, we study the stability of controlled system by determination of the attractor, which localizes the induced error, and is achieved by using aggregation techniques and stability criteria, with the use of vector norms and of comparison systems [6]-[11]. In section 2, we propose a method for studying the stability of nonlinear continuous systems by determination of a comparison system and of local attractors. The determination of the attractors using the method of diagonalization of the locally linearized system is studied in section 3. The stability study of a third order nonlinear complex system is proposed to illustrate the efficiency of this approach. In section 4, we propose athe stability study of a nonlinear process controlled by a decoupling linearization method.

II. Determination of Comparison systems and Attractors

For stability study, several criteria based on different Lyapunov's theorems are developed. If applied to the comparison system, they enable to determine the attractor of the process and its field of attraction [12], [16-20].
Let us consider the following system (S) described by

$$(S): \frac{dx}{dt} = A(x)x + B(x,t), \forall\, t \in \tau_0 = \left[t_0, +\infty\right] \qquad (1)$$

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where A is an $n \times n$ matrix and B an n vector.

We can now study the stability of the controlled system by determination of the attractor, possibly reduced to the origin $x=0$, by using aggregation techniques and definition of a comparison system such that the instantaneous matrix of the non-perturbed system be the opposite of an M-matrix.

For the vector norm $p(x) = \left[|x_1|, |x_2| \ldots, |x_n|\right]^T$, we obtain by overvaluation the linear comparison system:

$$z \in \mathbb{R}^n \,/\, \dot{z}(t) = Mz(t) + N \tag{2}$$

Notation $M(.) = \{m_{i,j}(.)\}$ such that:

$$\begin{cases} m_{i,i}(.) = a_{i,i}(.) & \forall i = 1, 2, \ldots n \\ m_{i,j}(.) = \left|a_{i,j}(.)\right| & \forall i \neq j \end{cases} \tag{3}$$

and the vector $N(.)$ defined by

$$N(.) = |B(.)| \tag{4}$$

we can define the constant matrices M and N by

$$\begin{aligned} M &= \{m_{i,j}\}; \quad m_{i,j} = \max\{m_{i,j}(.)\} \\ N &= \{n_i\}; \quad\;\;\; n_i = \max\{n_i(.)\} \end{aligned} \tag{5}$$

If M is the opposite of an M-matrix, it exists an attractor D asymptotically stable such that

$$D = \left\{ x \in R^n ; p(x) \leq -M^{-1}N \right\} \tag{6}$$

with

$$z(t) \geq p(x(t)) \tag{7}$$

$\forall t \in \tau_0 = [t_0; +\infty]$; we have:

$$\lim_{t\to\infty} z(t) = z(+\infty) = -M^{-1}N \text{ then } \lim_{t\to+\infty} p(x) \leq -M^{-1}N \tag{8}$$

III. Determination of Attractors Using the Method of Diagonalization

This section presents the determination of an attractor by diagonalization of the characteristic matrix around an operating point, for this step we must firstly determine the linearized model of the initial system defined around the operating point, secondly the change of basis for which the linearized model is diagonal at the operating point and is diagonal dominant in its neighborhood.

For this study, we must first find the linearized system (2) of (1) defined around the operating point x_0.

$$\dot{x} = A(x_0, t)x + B(x_0, t) \tag{9}$$

If, matrix $A(x_0, t)$, is constant and diagonalizable it exists an invertible matrix P such that

$$x = \mathrm{P}y \tag{10}$$

which diagonalize the matrix $A(x_0, t)$, $\mathrm{D}(x_0, t) = P^{-1}A(x_0, t)P$ then have

$$\mathrm{D}(x,t) = \mathrm{P}^{-1}A(x,t)\mathrm{P} \tag{11}$$

where $\mathrm{D}(x_0, t)$ is a diagonal matrix. We obtain the new description of the system:

$$\dot{y} = \mathrm{P}^{-1}A(x,t)\mathrm{P}y(t) + \mathrm{P}^{-1}B(x,t) \tag{12}$$

and we can apply the previous attractor determination approach.

A. *Illustration of the method on a third order nonlinear complex system*

To illustrate the proposed approach for attractors determination, let us consider the system (S) defined by (2), with

$$A(x(t), t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{13}$$

$$a_{11} = 4\mathrm{sat}x_1 \cos t - 2\cos x_2 - 8.5$$

$$a_{12} = 0.08\cos x_1 - 0.16\sin x_3 + 0.04\mathrm{sat}x_1 \cos t - 0.08$$

$$a_{13} = 0.08\cos x_1 - 0.16\sin x_3 + 0.24\mathrm{sat}x_1 \cos t - 0.08$$

$$a_{21} = 0.1\cos x_3 - 0.15\frac{x_1x_2}{x_1^2x_2^2}\sin x_1 - 0.1$$

$$\begin{aligned} a_{22} &= 0.2e^{-x_1^2} - 0.18e^{-x_2^2} + 1.2\sin x_3 - 0.8\sin x_1 \cos t \\ &\quad + 1.2\cos x_3 e^{-x_2^2} - 4.62 \end{aligned}$$

$$\begin{aligned} a_{23} &= 1.2e^{-x_1^2} - 1.08e^{-x_2^2} + 1.2\sin x_3 - 4.8\sin x_1 \cos t \\ &\quad + 1.2\cos x_3 e^{-x_2^2} + 2.28 \end{aligned}$$

$$a_{31} = -0.1\cos x_3 - 0.025\frac{x_1x_2}{x_1^2x_2^2}\sin x_1 + 0.1$$

$$\begin{aligned} a_{32} &= -0.2e^{-x_1^2} + 0.03e^{-x_2^2} - 0.2\sin x_3 + 0.8\sin x_1 \cos t \\ &\quad - 0.2\cos x_3 e^{-x_2^2} - 0.23 \end{aligned}$$

$$\begin{aligned} a_{33} &= -1.2e^{-x_1^2} + 0.18e^{-x_2^2} - 0.2\sin x_3 + 4.8\sin x_1 \cos t \\ &\quad - 0.2\cos x_3 e^{-x_2^2} - 6.38 \end{aligned}$$

and

$$B(x(t)) = \begin{bmatrix} 0.5\sin t \\ 0.4\cos t \\ \frac{1}{6}(3\mathrm{sat}x_1 + 1.8) \end{bmatrix} \tag{14}$$

with $\mathrm{sat}\, x_i = x_i$, if $|x_i| < 1$, and else $\mathrm{sat}\, x_i = \mathrm{sign}\; x_i$

Let us consider the operating point $x_0 = 0$, so the matrix $A(0)$ of (13) defined by

$$A(0)=\begin{bmatrix}-10.5 & 0 & 0\\ 0 & -3.4 & 3.6\\ 0 & -0.6 & -7.6\end{bmatrix};B(0)=\begin{bmatrix}0.5\sin t\\ 0.4\cos t\\ 0.3\end{bmatrix} \quad (15)$$

can be diagonalized with the matrix of change of base P.

$$\mathrm{P}=\begin{bmatrix}2 & 0 & 0\\ 0 & -3 & -1\\ 0 & 0.5 & 1\end{bmatrix} \quad (16)$$

So it comes the matrix $\mathrm{D}(x(t))$ defined in (11)

$$\mathrm{D}\big(x(\mathrm{t})\big)=$$

$$\begin{bmatrix}\begin{pmatrix}-2\cos x_2\\ +4\mathrm{sat}\, x_1\cos t\\ -8.5\end{pmatrix} & \begin{pmatrix}0.2\sin x_3\\ -0.1(\cos x_1-1)\end{pmatrix} & 0.1\mathrm{sat}\, x_1\cos t\\ 0.1\dfrac{x_1x_2}{x_1^2x_2^2}\sin x_1 & \begin{pmatrix}-5\\ +e^{-x_2^2}\cos x_3\\ +\sin x_3\end{pmatrix} & 0.3(e^{-x_2^2}-1)\\ 0.2(1-\cos x_3) & 0 & \begin{pmatrix}-6+4\mathrm{sat}\, x_1\cos t\\ -e^{-x_1^2}\end{pmatrix}\end{bmatrix} \quad (17)$$

and

$$\mathrm{P}^{-1}B(x(t))=\begin{bmatrix}\dfrac{1}{4}\sin t\\ -\dfrac{1}{5}\mathrm{sat}x_1-\dfrac{4}{25}\cos t-\dfrac{3}{25}\\ \dfrac{3}{2}\mathrm{sat}x_1+\dfrac{2}{25}\cos t+\dfrac{9}{25}\end{bmatrix} \quad (18)$$

and we have the description

$$\dot{y}=\mathrm{P}^{-1}A(x(t))\mathrm{P}y(t)+\mathrm{P}^{-1}B(x(t)) \quad (19)$$

The use of the vector norm $p(y)$ defined by $p(y)=\left[|y_1|,|y_2|,|y_3|\right]^T$ enables by overvaluation, to define the comparison system defined on (2)

$$M_d=M\big(\mathrm{D}(x(\mathrm{t}))\big)=\begin{bmatrix}d_{11} & |d_{12}| & |d_{13}|\\ |d_{21}| & d_{22} & |d_{23}|\\ |d_{31}| & |d_{32}| & d_{33}\end{bmatrix} \quad (20)$$

and

$$N_d=N(\mathrm{P}^{-1}B(x(t)))=\begin{bmatrix}\left|\dfrac{1}{4}\sin t\right|\\ \left|-\dfrac{1}{5}\mathrm{sat}x_1-\dfrac{4}{25}\cos t-\dfrac{3}{25}\right|\\ \left|\dfrac{3}{5}\mathrm{sat}x_1+\dfrac{2}{25}\cos t+\dfrac{9}{25}\right|\end{bmatrix} \quad (21)$$

In this case, the comparison system from (2) is described by

$$\dot{z}=\begin{bmatrix}-2.5 & 0.4 & 0.1\\ 0.05 & -3 & 0.3\\ 0.4 & 0 & -2\end{bmatrix}z+\begin{bmatrix}0.25\\ 0.48\\ 1.04\end{bmatrix} \quad (22)$$

The following conditions:

$$\begin{cases}-(-2.5)\succ 0\\ 2.5\times 3-0.05\times 0.4\succ 0\\ (-1)^3\det M_d\succ 0\end{cases}$$

imply that M_d is the opposite of a M-matrix

It comes from (7) and (8)

$$z_\infty=\begin{bmatrix}0.1569\\ 0.2178\\ 0.5514\end{bmatrix}\geq\begin{bmatrix}|x_1|\\ |x_2|\\ |x_3|\end{bmatrix} \quad (23)$$

as

$$y=\mathrm{P}^{-1}x(t)=\begin{bmatrix}0.5 & 0 & 0\\ 0 & -0.4 & -0.4\\ 0 & 0.2 & 1.2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \quad (26)$$

$$\lim_{t\to+\infty}p(y)=\lim_{t\to+\infty}p(\begin{bmatrix}0.5x_1\\ -0.4x_2-0.4x_3\\ 0.2x_2+1.2x_3\end{bmatrix})\leq\begin{bmatrix}0.1569\\ 0.2178\\ 0.5514\end{bmatrix} \quad (27)$$

$$\Leftrightarrow\begin{cases}|x_1|\prec 0.3124\\ |-x_2-x_3|\prec 0.5445\\ |x_2+6x_3|\prec 2.7570\end{cases}$$

$$D_1=\begin{Bmatrix}x,|x_1|\prec 0.3124\ ,\\ |-x_2-x_3|\prec 0.5445,\\ |x_2+6x_3|\prec 2.757\end{Bmatrix} \quad (28)$$

In this domain D_1, we repeat the overvaluation approach to determine a new attractor taking into account the condition (28) The domain D_1 is defined such that

$$|y|\leq\begin{bmatrix}0.3124\\ 0.5445\\ 2.757\end{bmatrix}$$

and $\mathrm{sat}x_1=x_1\leq 0.3124$
then in D_1 we can write

$$D'(x(t)) =$$

$$\begin{bmatrix} \begin{pmatrix} -2\cos x_2 \\ +1.2496\cos t \\ -8.5 \end{pmatrix} & \begin{pmatrix} 0.2\sin x_3 \\ -0.0048 \end{pmatrix} & 0.0312\cos t \\ \begin{pmatrix} 0.1\dfrac{x_1x_2}{x_1^2x_2^2}\sin x_1 \\ -\dfrac{0.3124}{5} \end{pmatrix} & \begin{pmatrix} -5+e^{-x_2^2}\cos x_3 \\ +\sin x_3 \end{pmatrix} & 0.3(e^{-x_2^2}-1) \\ \begin{pmatrix} 0.2(1-\cos x_3) \\ +\dfrac{3}{5} \end{pmatrix} & 0 & \begin{pmatrix} 6.907 \\ +1.2496\cos t \end{pmatrix} \end{bmatrix} \quad (29)$$

and

$$P^{-1}B'(x(t)) = \begin{bmatrix} \dfrac{1}{4}\sin t \\ -\dfrac{4}{25}\cos t - \dfrac{3}{25} \\ \dfrac{2}{25}\cos t + \dfrac{9}{25} \end{bmatrix} \quad (30)$$

In this case, the comparison system from (2) is described by

$$\dot{z} = \begin{bmatrix} -5.2504 & 0.2048 & 0.0312 \\ 0.1125 & -3 & 0.3 \\ 1 & 0 & -5.6574 \end{bmatrix} z + \begin{bmatrix} 0.25 \\ 0.28 \\ 0.44 \end{bmatrix} \quad (31)$$

M_d is the opposite of an M-matrix, then

$z(t_0) \geq p(x_0)$; then $\lim_{t\to+\infty} p(z) \leq z_\infty = -M_d^{-1}N_d$

$$z_\infty = \begin{bmatrix} 0.0523 \\ 0.1064 \\ 0.0870 \end{bmatrix} \geq \begin{bmatrix} |x_1| \\ |x_2| \\ |x_3| \end{bmatrix} \quad (32)$$

or

$$\lim_{t\to+\infty} p(y) = \lim_{t\to+\infty} (P^{-1}x(t))$$

$$= \lim_{t\to+\infty} p\left(\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.4 & -0.4 \\ 0 & 0.2 & 1.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \leq -M_d^{-1}N_d$$

$$\lim_{t\to+\infty} p(y) = \lim_{t\to+\infty} p\left(\begin{bmatrix} 0.5x_1 \\ -0.4x_2 - 0.4x_3 \\ 0.2x_2 + 1.2x_3 \end{bmatrix}\right) \leq \begin{bmatrix} 0.0523 \\ 0.1064 \\ 0.0870 \end{bmatrix} \quad (33)$$

$$D_2 : \begin{cases} |x_1| \leq 0.1046 \\ |-x_2 - x_3| \leq 0.266 \\ |x_2 + 6x_3| \leq 0.435 \end{cases} \quad (34)$$

D_2 is the attractor, with

$$D_2 = \begin{Bmatrix} x, |x_1| \leq 0.1046\ , |-x_2 - x_3| \leq 0.266, \\ |x_2 + 6x_3| \leq 0.435 \end{Bmatrix} \quad (36)$$

Attractors and state space variables evolutions of the system in the domain D_2 is given in Fig. 1.

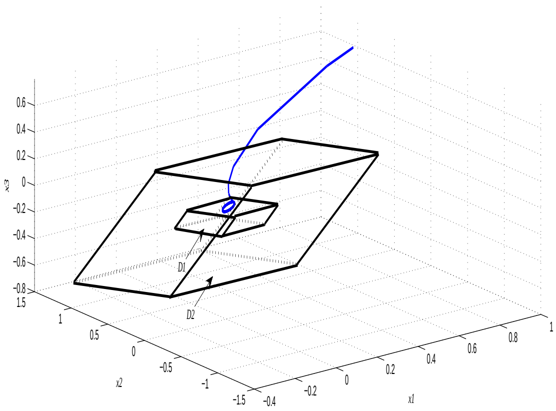

Fig. 1. Attraction zones and state space variables evolutions of the systems in the domain D_1 and D_2

IV. Decoupling Linearization-Basic Idea

This section deals with an approach to the study of the error estimation in the decoupling of ill-defined and/or perturbed nonlinear processes.

Let us consider a smooth nonlinear control affine system whose evolution is described by the equations:

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned} \quad (37)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ the output vector and $u \in \mathbb{R}^m$ the control vector.

The method consists of deriving each component y_i of the output vector to display the control vector.

With the notation $y_i^* = y_i^{(d_i+1)} \quad \forall i = 1, 2, \cdots, m,$ it comes [21]:

$$y^* = f^*(x) + G^*(x)u \quad (38)$$

If G^* is invertible in the domain of evolution of the state vector we can define the control law:

$$u = G^{*^{-1}}(x)(v - f^*(x)) \quad (39)$$

which leads to the relations:

$$y_i^{(d_i+1)} = v_i, \forall i = 1, 2, \cdots, m \quad (40)$$

This representation is valid only if the instable variables of the initial system are observable.

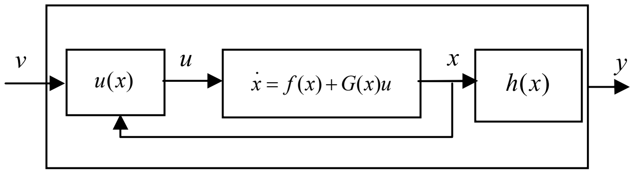

Fig. 2 Representation of the decoupled system

A. Attractor determination to a Third order Nonlinear System in presence of disturbances and/or uncertainties

Let us consider the system (1) submitted to the bounded errors δf and δG such that $|\delta G| \leq \Delta G$ and $|\delta f| \leq \Delta f$.

it comes

$$\begin{aligned} \dot{x} &= f(x) + \delta f + (G(x) + \delta G)u \\ &= f(x) + G(x)u + \delta f + \delta G u \end{aligned} \tag{41}$$

Let us consider the nonlinear system

$$(S): \begin{cases} \dot{x}_1 = -2x_1 \cos x_2 + x_3 + (1 + e^{-x_1^2})u_1 + u_2 \\ \dot{x}_2 = -3x_2 + \sin x_3 + 3u_1 \\ \dot{x}_3 = x_1 - x_2^2 x_3 - 0.4x_3 - 0.1\text{sat}x_3 \end{cases} \tag{42}$$

$$\text{sat v} = \text{v} \ \text{ if } \ |\text{v}| \leq 1, \text{sat v} = \text{sign v} \ \text{ if } \ |\text{v}| \succ 1 \tag{43}$$

with the outputs

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{44}$$

It comes the following notations:

$$f(x) = \begin{bmatrix} -2x_1 \cos x_2 + x_3 \\ -3x_2 + \sin x_3 \\ x_1 - x_2^2 x_3 - 0.4x_3 - 0.1\text{sat}x_3 \end{bmatrix}; G(x) = \begin{bmatrix} 1 + e^{-x_1^2} & 1 \\ 3 & 0 \\ 0 & 0 \end{bmatrix} \tag{45}$$

and

$$h(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{46}$$

To verify if the system admits an input-output linearization, we must determine the relative degree. We obtain by derivation of the outputs:

$$\begin{cases} \dot{y}_1 = \dot{x}_1 = -2x_1 \cos x_2 + x_3 + (1 + e^{-x_1^2})u_1 + u_2 \\ \dot{y}_2 = \dot{x}_2 = -3x_2 + \sin x_3 + 3u_1 \end{cases} \tag{47}$$

as $h_{i_x}^T G \neq 0$, we have

$$h^*(x) = f(x) = \begin{bmatrix} -2x_1 \cos x_2 + x_3 \\ -3x_2 + \sin x_3 \end{bmatrix}, G^* = \begin{bmatrix} 1 + e^{-x_1^2} & 1 \\ 3 & 0 \end{bmatrix} \tag{48}$$

G^* being invertible, we can impose the input u defined by (39)

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{3x_2 - \sin x_3 + v_2}{3} \\ 2x_1 \cos x_2 - x_3 + v_1 - \dfrac{1}{3}(1 + e^{-x_1^2})(3x_2 - \sin x_3 + v_2) \end{bmatrix} \tag{49}$$

$$\begin{cases} \dot{y}_1 = \dot{x}_1 = -2x_1 \cos x_2 + x_3 + (1 + e^{-x_1^2})(\dfrac{3x_2 - \sin x_3 + v_2}{3}) \\ \quad + (2x_1 \cos x_2 - x_3 + v_1 - \dfrac{1}{3}(1 + e^{-x_1^2})(3x_2 - \sin x_3 + v_2)) \\ \dot{y}_2 = \dot{x}_2 = -3x_2 + \sin x_3 + 3(\dfrac{3x_2 - \sin x_3 + v_2}{3}) \end{cases} \tag{50}$$

it comes

$$\begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \end{cases} \tag{51}$$

Let us choose

$$\begin{aligned} v_1 &= \frac{1}{\tau_1}(y_1^c - y_1) \\ v_2 &= \frac{1}{\tau_2}(y_2^c - y_2) \end{aligned} \tag{52}$$

For constant inputs $y_j^c \quad \forall j = 1,2$

it comes:

$$\begin{aligned} \dot{v}_1 &= -\frac{\dot{y}_1}{\tau_1} = -\frac{\dot{x}_1}{\tau_1} = -\frac{1}{\tau_1} v_1 \\ \dot{v}_2 &= -\frac{\dot{y}_2}{\tau_2} = -\frac{\dot{x}_2}{\tau_2} = -\frac{1}{\tau_2} v_2 \end{aligned} \tag{53}$$

If we now take into account, the presence of uncertainties on f and G, it comes

$$\begin{aligned} \dot{x}_1 &= v_1 + \delta f_1 + G_{11}u_1 + \delta G_{12}u_2 \\ \dot{x}_2 &= v_2 + \delta f_2 + G_{21}u_2 + \delta G_{22}u_2 \\ \dot{x}_3 &= \delta f_3 + x_1 - x_2^2 x_3 - 0.4x_3 - 0.1\text{sat}x_3 \end{aligned} \tag{54}$$

$$\text{with:} |\delta G| \leq \begin{bmatrix} 0.01 & 0.01 \\ 0.03 & 0.01 \\ 0 & 0 \end{bmatrix} \text{ and } |\delta f| \leq \begin{bmatrix} 0.01 \\ 0.02 \\ 0.02 \end{bmatrix} \tag{55}$$

$$\begin{cases} \dot{x}_1 = -\dfrac{x_1}{\tau_1} + \delta f_1 + \delta G_{11}u_1 + \delta G_{12}u_2 \\ \dot{x}_2 = -\dfrac{x_2}{\tau_2} + \delta f_2 + \delta G_{21}u_1 + \delta G_{22}u_2 \\ \dot{x}_3 = \delta f_3 + x_1 - x_2^2 x_3 - 0.4x_3 - 0.1\text{sat}x_3 \end{cases} \tag{56}$$

for the vector norm $p(x)=\left[|x_1|,|x_2|,|x_3|\right]^T$, it comes the overvaluation:

$$\begin{aligned}
\frac{d|x_1|}{dt} &\le -\frac{|x_1|}{\tau_1} + Max|\delta f_1| + Max|\delta G_{11}u_1| + Max|\delta G_{12}u_2| \\
\frac{d|x_2|}{dt} &\le -\frac{|x_2|}{\tau_2} + Max|\delta f_2| + Max|\delta G_{21}u_2| + Max|\delta G_{22}u_2| \\
\frac{d|x_3|}{dt} &\le Max|\delta f_3| + |x_1| - x_2^2|x_3| - 0.4|x_3| - 0.1|\mathrm{sat}x_3|
\end{aligned} \tag{60}$$

$$\begin{aligned}
\frac{d|x_1|}{dt} &\le -\frac{|x_1|}{\tau_1} + Max|\delta f_1| + Max\left|0.01\left(\frac{3x_2-\sin x_3+v_2}{3}\right)\right| \\
&+Max\left|0.01\left(2x_1\cos x_2 - x_3 + v_1 - \frac{1}{3}(1+e^{-x_1^2})(3x_2-\sin x_3+v_2)\right)\right| \\
\frac{d|x_2|}{dt} &\le -\frac{|x_2|}{\tau_2} + Max|\delta f_2| + Max\left|0.03\left(\frac{3x_2-\sin x_3+v_2}{3}\right)\right| \\
&+Max\left|0.01\left(2x_1\cos x_2 - x_3 + v_1 - \frac{1}{3}(1+e^{-x_1^2})(3x_2-\sin x_3+v_2)\right)\right|
\end{aligned}$$

$$\frac{d|x_3|}{dt} \le Max|\delta f_3| + |x_1| - x_2^2|x_3| - 0.4|x_3| - 0.1|\mathrm{sat}x_3| \tag{61}$$

For

$$\tau_1 = 1, \tau_2 = 0.5 \tag{62}$$

it comes

$$\begin{aligned}
\frac{d|x_1|}{dt} &\le -|x_1| + Max|\delta f_1| + Max\left|0.01\left(\frac{x_2-\sin x_3}{3}\right)\right| \\
&+ Max\left|\begin{matrix}0.01\left(x_1(2\cos x_2 - 1) - x_3\right) \\ -\frac{0.01}{3}\left((1+e^{-x_1^2})(x_2-\sin x_3)\right)\end{matrix}\right| \\
\frac{d|x_2|}{dt} &\le -2|x_2| + Max|\delta f_2| + Max\left|0.03\left(\frac{x_2-\sin x_3}{3}\right)\right| \\
&+ Max\left|\begin{matrix}0.01\left(x_1(2\cos x_2 - 1) - x_3\right) \\ -\frac{0.01}{3}\left((1+e^{-x_1^2})(x_2-\sin x_3)\right)\end{matrix}\right| \\
\frac{d|x_3|}{dt} &\le Max|\delta f_3| + |x_1| - x_2^2|x_3| - 0.4|x_3| - 0.1\left|\frac{\mathrm{sat}x_3}{x_3}\right||x_3|
\end{aligned} \tag{63}$$

Taking into account inequalities (55) we obtain:

$$\begin{aligned}
\frac{d|x_1|}{dt} &\le |x_1|\left(-1+0.01|2\cos x_2 - 1|\right) \\
&+|x_2|\left(\frac{0.01}{3}+\frac{1}{3}(1+e^{-x_1^2})\right)+0.01+\left|\frac{-0.01\sin x_3}{3}\right| \\
&-0.01|x_3| + \frac{0.01}{3}\left|(1+e^{-x_1^2})(\sin x_3)\right| + 0.01 \\
\frac{d|x_2|}{dt} &\le |x_2|\left(-2+0.01+\frac{0.01}{3}(1+e^{-x_1^2})\right)+0.02 \\
&+\left|\frac{0.03\sin x_3}{3}\right| + 0.01|x_3| + \frac{0.01}{3}\left|(1+e^{-x_1^2})(\sin x_3)\right| \\
&+|x_1|(0.02\cos x_2 - 0.01) + 0.02 \\
\frac{d|x_3|}{dt} &\le 0.02 + |x_1| - x_2^2|x_3| - 0.4|x_3| - 0.1\left|\frac{\mathrm{sat}x_3}{x_3}\right||x_3|
\end{aligned} \tag{64}$$

it comes

$$\frac{dp(x)}{dt} \le M(x)p(x) + N(x) \tag{65}$$

with

$$M(x) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\begin{aligned}
m_{11} &= -1 + 0.01(2\cos x_2 - 1) \\
m_{12} &= \left|\frac{0.01}{3} + \frac{1}{3}(1+e^{-x_1^2})\right| \\
m_{13} &= |-0.01| \\
m_{21} &= |0.02\cos x_2 - 0.01| \\
m_{22} &= -2 + 0.01 - \frac{0.01}{3}(1+e^{-x_1^2}) \\
m_{23} &= |-0.01| \\
m_{31} &= 1 \\
m_{32} &= 0 \\
m_{33} &= -0.4 - x_2^2 - 0.1\left|\frac{\mathrm{sat}x_3}{x_3}\right|
\end{aligned} \tag{66}$$

for the linear comparison system it comes

$$M = \begin{bmatrix} -0.99 & 0.67 & 0.01 \\ 0.03 & -1.993 & 0.01 \\ 1 & 0 & -0.4 \end{bmatrix} \tag{67}$$

The conditions for M to be the opposite of an M-matrix are:

$$\begin{cases} -0.99 < 0 \\ (-0.99 \times -1.993) - (0.03 \times 0.67) > 0 \\ (-1)^3 \det(M) \prec 0 \Rightarrow (-1)^3 \times (-0.7348) > 0 \end{cases} \quad (68)$$

The overvaluation of $N(x)$

$$N(x) = \begin{bmatrix} 0.01 + \left|\frac{-0.01 \sin x_3}{3}\right| + \frac{0.01}{3}\left|(1+e^{-x_1^2})(\sin x_3)\right| \\ 0.02 - \left|\frac{0.03 \sin x_3}{3}\right| + \frac{0.01}{3}\left|(1+e^{-x_1^2})(\sin x_3)\right| \\ 0.02 \end{bmatrix} \quad (69)$$

gives:

$$N = \begin{bmatrix} 0.02 \\ 0.0367 \\ 0.02 \end{bmatrix} \quad (70)$$

It comes out:

$$\lim_{t \to +\infty} p(z) \leq \begin{bmatrix} 0.0349 \\ 0.0196 \\ 0.1372 \end{bmatrix} = D_3 \quad (71)$$

The following figure shows the evolution of the error between the process with and without uncertainties

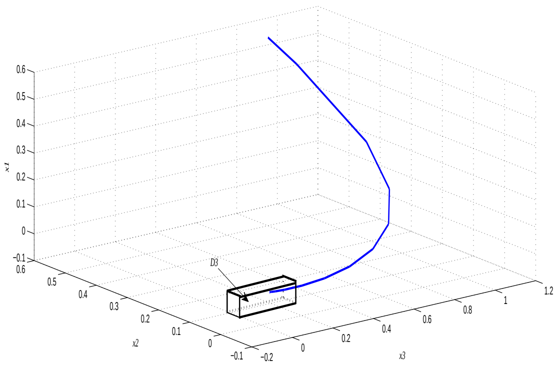

Fig. 3. Attractor D_3 and evolution of the state vector

CONCLUSION

The use of aggregation techniques and of comparison systems enables to estimate by overvaluation the maximum error induced by the use of a non-perturbed model for the determination of the control law of a nonlinear process in presence of uncertainties and/or of bounded perturbations.

APPENDIX

Appendix A. Vector Norms Definition

Definition1 : Let $E = R^n$ and $E_1, E_2 \ldots E_k$ be subspaces of the space E, $E = E_1 \cup E_2 \ldots \cup E_k$.

Let x be an n vector defined on E and $x_i = P_i x$ the projection of x on E_i, where P_i is a projection operator from E into E_i, p_i a scalar norm (i=1,2,…, k) defined on the subspace E_i and p denotes a vector norm of dimension k and with its component $p_i(x) = p_i(x_i)$, $p(x): R^n \to R_+^k$

Let y be another vector in space E, with $y_i = P_i y$, we have the following properties

$$\begin{cases} p_i(x_i) \geq 0, \forall x_i \in E_i \forall i = 1,2,\ldots,k \\ p_i(x_i) = 0 \leftrightarrow x_i = 0, \forall i = 1,2,\ldots,k \\ p_i(x_i + y_i) \leq p_i(x_i) + p_i(y_i), \forall x_i, y_i \in E_i \forall i = 1,2,\ldots,k \\ p_i(\lambda x_i) = |\lambda| p_i(x_i), \forall x_i \forall i = 1,2,\ldots,k, \forall \lambda \cup R \end{cases}$$

If k-1 of the subspaces E_i are insufficient to define the whole space E, the vector norm is surjective. If in addition the subspaces E_i are in disjoint pairs, $E_i \cap E_j = \varnothing$, $\forall i \neq j = 1,2,\ldots,k$, the vector norm p is said to be regular.

Appendix B. Overvaluing and comparison systems

Let the differential equation $\dot{x} = A(x,t)x$. The overvaluing system is defined by the use of the vector norm $p(x)$ of the state vector x and the use of the right-band derivation $D^+ p_i(x_i)$ proposed by [22, 23] $D^+ p_i(x_i)$ is taken along the motion of x in the subspace E_i and $D^+ p(x)$ along the motion of x in E.

Definition 2: The matrix $M(x,t)$ defines an overvaluing system of S with respect to the vector norm p if and only if the following inequality is verified for each corresponding component: $D^+ p(x) \leq M(x,t)\, p(x)$

If for the same system we can define a constant overvaluing matrix M, we have $M \geq M(x,t)$ and with $\dot{z} = M z$ we have $z(t) \geq p(x(t))$ for $t \geq t_0$ as soon as this property is satisfied at the origin t_0

When an overvaluing matrix $M(x,t)$ of a matrix $A(x,t)$ is defined with respect to a regular vector norm p we have the following properties:

- The off- diagonal elements of matrix $M(x,t)$ are non negative.
- If we denote by $\mathrm{Re}(\lambda_M)$ the real part of the eigenvalue of the maximum real part of $M(x,t)$ the following inequality is verified

$\mathrm{Re}(\lambda_A) \leq \mathrm{Re}(\lambda_M) = \lambda_M \quad \forall t, x \in \tau \times \mathbb{R}^n$,

whatever the eigenvalue λ_A of matrix $A(x,t)$

- When all the real parts of the eigenvalues of $M(x,t)$ are negative this matrix is the opposite of an M-matrix and it admits an inverse whose elements are all non positive.

- When due to perturbations and/or uncertainties it is not possible to define an homogeneous overvaluing system we can define a non homogeneous overvaluing system of the form $D^+ p(x) \le M(x,t)\, p(x) + N(x,t)$, where all the elements of vector norm nonnegative and when M and N are constant, we can define the comparison system $\dot{z} = Mz + N$

Remark 1. With $M(.) = \{m_{ij}(.)\}$ the verification of the Kotelyanski lemma by the matrix $M(.)$ prove that $M(.)$ is the opposite of an M-matrix

$$m_{1,1} \prec 0, \begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \succ 0, \ldots, (-1)^k \begin{vmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,k} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ m_{k,1} & m_{k,2} & \cdots & m_{k,k} \end{vmatrix} \succ 0$$

Remark 2. A less conservative approach consists to use a vector norm of size k=n, for example $p(x) = \left[|x_1|, |x_2|, \ldots, |x_n|\right]^T$

Remark 3. If $M(.)$ is an overvaluing matrix of a matrix $A(.)$, $M(.) + M^*$ where the elements of M^* are all non negative is also an overvaluing matrix of $A(.)$. This property can be used to simplify the determination of an overvaluing matrix of $A(.)$ when some elements of $A(.)$ are ill defined or subject to uncertainties.

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