A New Block Method of Order Nine for Solving Fourth Order Ordinary Differential Equations Directly

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Abstract— This paper considers the derivation of a new block method of order nine for the solution of fourth order ordinary differential equations directly. The multistep collocation approach is adopted in developing the new method where the use of power series approximate solution as an interpolation polynomial and its fourth derivative as a collocation equation are considered. The properties of the new developed method which include zero-stability, order, error constant, consistency, convergence and region of absolute stability are verified. Furthermore, the accuracy of the method is tested by solving some fourth order initial value problems which confirmed the superiority of the method over the existing methods.

Keywords—Interpolation, Multistep collocation, Block method, fourth order initial value problems.

I. Introduction

The general fourth order initial value problems of ordinary differential equations of the form

$$y^{iv} = f(x,y,y',y'',y''') \,,\, y^{s}(a) = y_{s}, s = 0(1)3 \quad x \in [a,b] \qquad (1)$$

is examined in this paper.

The reduction of equation (1) to its equivalent system of first order ordinary differential equations has been found having some setbacks which include wastage in computer time and burden of computing which at times lead to lower accuracy of the method. Direct method of solving (1) had been developed by some authors like Omar [1], Awoyemi [2], Kayode [3] and Adesanya, et.al.[4] in order to overcome the setbacks in reduction method. Several attempts have been made by many scholars in the derivation of block method for the solution of (1) directly.

The use of numerical integration approach in developing block methods with step-length *k=8* for solving equation (1) directly was considered by Omar [1]. In the differential problems solved to examine the accuracy of the methods, it is noted that the errors are too large. Furthermore, in the work done by Olabode [5] and Mohammed [6], interpolation and collocation technique was used in the derivation of the block methods having a step-length *k=6* for direct solution of equation (1). It is observed that the accuracy of the methods is very low.

The new proposed order nine block method with a step-length *k=8* for solving (1) directly is developed to address these observed setbacks mentioned above. The new block method was derived through multistep collocation approach and the numerical results generated from the method are compared with the existing method of the same step-length.

II. Derivation of the Method

We assume power series of the form

$$y(x) = \sum_{j=0}^{k+4} a_j x^j \qquad (2)$$

as an approximate solution to the general fourth order of the form (1). The first, second, third and fourth derivatives of (2) are

$$y'(x) = \sum_{j=1}^{k+4} j a_j x^{j-1} \qquad (3)$$

$$y''(x) = \sum_{j=2}^{k+4} j(j-1) a_j x^{j-2} \qquad (4)$$

$$y'''(x) = \sum_{j=3}^{k+4} j(j-1)(j-2) a_j x^{j-2} \qquad (5)$$

$$y^{iv}(x) = \sum_{j=4}^{k+4} j(j-1)(j-2)(j-3) a_j x^{j-4} \qquad (6)$$

Equation (2) is interpolated at the points $x = x_{n+i}, i = 3(1)6$ while equation (6) is collocated at the points $x = x_{n+i}, i = 0(1)8$.

As a result, the following is obtained

$$AX = B \tag{7}$$

where

$$X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}]^T$$

$$B = [y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8}]^T$$

and A is shown in Appendix 1

The Gaussian elimination method is applied to (7) in order to find the values of the unknown coefficients $a_j's$ which are substituted into (2) to give a continuous implicit scheme of the form

$$y(z) = \sum_{j=3}^{k-2} \alpha_j(z) y_{n+j} + h^4 \sum_{j=0}^{k} \beta_j(z) f_{n+j} \tag{8}$$

where $x = zh + x_n + 7h,$

$$\begin{pmatrix} \alpha_3(z) \\ \alpha_4(z) \\ \alpha_5(z) \\ \alpha_6(z) \end{pmatrix} = \begin{pmatrix} -1 & -\frac{11}{6} & -1 & -\frac{1}{6} \\ 4 & 7 & \frac{7}{2} & \frac{1}{2} \\ -6 & -\frac{19}{2} & -4 & -\frac{1}{2} \\ 4 & \frac{13}{3} & \frac{3}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} \beta_0(z) \\ \beta_1(z) \\ \beta_2(z) \\ \beta_3(z) \\ \beta_4(z) \\ \beta_5(z) \\ \beta_6(z) \\ \beta_7(z) \\ \beta_8(z) \end{pmatrix} = Q \begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \\ z^7 \\ z^8 \\ z^9 \\ z^{10} \\ z^{11} \\ z^{12} \end{pmatrix} \tag{10}$$

where the value of Q can be seen in Appendix 1.

Equation (8) is evaluated at the non-interpolating points $x = x_{n+i}, i = 0,1,2,7 \text{ and } 8$ to produce the discrete schemes while it derivatives are evaluated at all the grid points $x = x_{n+i}, i = 0(1)8$ to give the derivatives of the discrete schemes. These discrete schemes and its derivatives are combined in a matrix form and then both sides of y and f function are multiplied by the inverse of the coefficients $y_{n+i}, i = 1(1)8$. This produces a block of the form

$$\begin{aligned} A^0 y_N &= a y_{N-1} + h a' y'_{N-1} + h^2 b y''_{N-1} + h^3 b' y'''_{N-1} \\ &\quad + h^4 (e F_N + f F_{N-1}) \end{aligned} \tag{11}$$

where

$y_N = [y_{n+1}, y_{n+2} .., y_{n+8}]^T$, $y_{N-1} = [y_{n-7}, y_{n-6} ..., y_n]^T$, $y'_{N-1} = [y'_{n-7}, y'_{n-6} .., y'_n]^T$, $y''_{N-1} = [y''_{n-7}, y''_{n-6} .., y''_n]$, $y'''_{N-1} = [y'''_{n-7}, y'''_{n-6} .., y'''_n]^T$, $F_{N-1} = [f_{n-7}, f_{n-6} ..., f_n]^T$, $F_N = [f_{n+1}, f_{n+2} .., f_{n+8}]^T$,

$$A^0 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$a = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$a' = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&2 \\ 0&0&0&0&0&0&0&3 \\ 0&0&0&0&0&0&0&4 \\ 0&0&0&0&0&0&0&5 \\ 0&0&0&0&0&0&0&6 \\ 0&0&0&0&0&0&0&7 \\ 0&0&0&0&0&0&0&8 \end{pmatrix},$$

$$b=\begin{pmatrix}0&0&0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&0&0&2\\0&0&0&0&0&0&0&\frac{9}{2}\\0&0&0&0&0&0&0&8\\0&0&0&0&0&0&0&\frac{25}{2}\\0&0&0&0&0&0&0&18\\0&0&0&0&0&0&0&\frac{49}{2}\\0&0&0&0&0&0&0&32\end{pmatrix}$$

$$b'=\begin{pmatrix}0&0&0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&0&0&\frac{4}{3}\\0&0&0&0&0&0&0&\frac{9}{2}\\0&0&0&0&0&0&0&\frac{32}{3}\\0&0&0&0&0&0&0&\frac{125}{6}\\0&0&0&0&0&0&0&36\\0&0&0&0&0&0&0&\frac{343}{6}\\0&0&0&0&0&0&0&\frac{256}{3}\end{pmatrix}$$

$$f=\begin{pmatrix}0&0&0&0&0&0&0&\frac{97}{3809}\\0&0&0&0&0&0&0&\frac{1145}{4137}\\0&0&0&0&0&0&0&\frac{481}{462}\\0&0&0&0&0&0&0&\frac{2957}{1135}\\0&0&0&0&0&0&0&\frac{4389}{835}\\0&0&0&0&0&0&0&\frac{2014}{217}\\0&0&0&0&0&0&0&\frac{449}{30}\\0&0&0&0&0&0&0&\frac{34646}{1533}\end{pmatrix}$$

where e is displayed in Appendix 1.

The first, second and third derivatives of (11) give

$$\begin{pmatrix}y'_{n+1}\\y'_{n+2}\\y'_{n+3}\\y'_{n+4}\\y'_{n+5}\\y'_{n+6}\\y'_{n+7}\\y'_{n+8}\end{pmatrix}=\begin{pmatrix}1&1&\frac{1}{2}\\1&2&2\\1&3&\frac{9}{2}\\1&4&8\\1&5&\frac{25}{2}\\1&6&18\\1&7&\frac{49}{2}\\1&8&32\end{pmatrix}\begin{pmatrix}y'_n\\hy''_n\\h^2y'''_n\end{pmatrix}+h^3G\begin{pmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+7}\\f_{n+8}\end{pmatrix}$$

$$\begin{pmatrix}y''_{n+1}\\y''_{n+2}\\y''_{n+3}\\y''_{n+4}\\y''_{n+5}\\y''_{n+6}\\y''_{n+7}\\y''_{n+8}\end{pmatrix}=\begin{pmatrix}1&1\\1&2\\1&3\\1&4\\1&5\\1&6\\1&7\\1&8\end{pmatrix}\begin{pmatrix}y''_n\\hy'''_n\end{pmatrix}+h^2H\begin{pmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+7}\\f_{n+8}\end{pmatrix}$$

$$\begin{pmatrix}y'''_{n+1}\\y'''_{n+2}\\y'''_{n+3}\\y'''_{n+4}\\y'''_{n+5}\\y'''_{n+6}\\y'''_{n+7}\\y'''_{n+8}\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}\left(y'''_n\right)+h^2I\begin{pmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+7}\\f_{n+8}\end{pmatrix}$$

Where G, H and I are also shown in Appendix 1.

III. ANALYSIS OF THE BLOCK METHOD

A. *The Order of the Block Method*

The method proposed by Lambert [7] is used in finding the order of the block method. That is, taking the Taylor series of (11) at point x gives J (refer to Appendix 2). Comparing the coefficients of h^m and y_n^m. This gives our method to have a uniform order $(9,9,9,9,9,9,9,9)^T$ with the following error constants

$$\left(\frac{29}{75916},\frac{67}{11985},\frac{113}{5009},\frac{137}{2360},\frac{394}{3315},\frac{3305}{15607},\frac{1079}{3141},\frac{1001}{1921}\right)^T$$

B. The Zero Stability of the Block Method

Definition 1**:** A linear multistep method is said to be *zero-stable* if the root of the first characteristics polynomial $\rho(r)$ satisfies $|r_s| \le 1$ and the root $|r| = 1$ having multiplicity not exceeding the order of differential equation. Therefore, in finding the zero stability of the block (11), we only put into consideration the coefficients of *y*-function according to definition (1). That is

$$\rho(r) = \det[rA^0 - a] = 0$$

This implies $r = 0,0,0,0,0,0,0,1$. Hence, the method is zero stable. The method is found to be consistent because it is having an order greater than one. Furthermore, the method is convergent since it is zero-stable and consistent ([7]).

C. Region of Absolute Stability of the block

The boundary locus method proposed by Lambert [1973] and Henrici [1962] is adopted in finding the region of absolute stability of the block method (8). This is given as $\bar{h}(r) = \dfrac{\ell(r)}{\sigma(r)}$

where $\ell(r)$ is the first characteristics polynomial and $\sigma(r)$ is the second characteristics polynomial.

Therefore, by the use of the above approach, the region of absolute stability of (11) gives $(0,\ 40145.67)$.This is shown in the diagram below

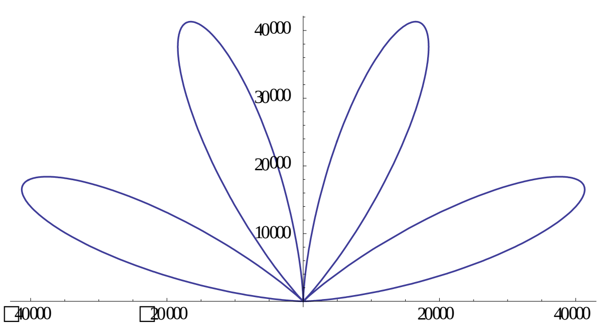

Figure 1: Region of absolute stability for 8-step block method for fourth order ordinary differential equations.

IV. TEST PROBLEMS

In order to examine the accuracy of the method, the differential problems below are solved.

Problem1: $y^{iv} = (x^4 + 14x^3 + 49x^2 + 32x - 12)e^x,$

$$y(0) = y'(0) = 0,\ y''(0) = 2,$$

$$y'''(0) = -6, \quad 0 \le x \le 1$$

Exact Solution: $y(x) = x^2(1-x)^2 e^x$

Problem 2: $y^{iv} = y,\ y(0) = y'(0) = y''(0) = 1$

$$y'''(0) = 1, \quad 0 \le x \le 1$$

Exact Solution: $y(x) = e^x$

The problems stated above were solved in Omar [1] using a step-length *k=8* whereby selection of maximum errors were considered. The new method is also applied to the same problems and the results generated are compared in terms of error. These are shown in Tables 1 and 2 (refer to Appendix 2) below.

The following notations are used in the tables

S2PEB	Sequential implementation of the 2-Point Explicit Block Method.
P2PEB	Parallel implementation of the 2-Point Explicit Block Method
S3PEB	Sequential implementation of the 3-Point Explicit Block Method
P3PEB	Parallel implementation of the 3-Point Explicit Block Method

V. CONCLUSION

An accurate block method for solving fourth order initial value problems of ordinary differential equations has been developed in this paper. It is observed in the two tables displayed above that the new method which is developed via interpolation and collocation approach performs better than Omar [1] whereby the use of numerical integration in developing the method was considered. Therefore, based on the results shown in the TABLES 1 and 2(refer to Appendix 2), the numerical method derived via interpolation and collocation has a better accuracy than the numerical method developed through numerical integration.

REFERENCES

[1] Z. Omar, Z. "Parallel block methods for solving higher order ordinary differential equations directly". Thesis Submitted in Fulfilment of the Requirement for the Degree of Doctor of Philosophy in the Faculty of Science and Environmental studies *Universiti Putra Malaysia* (unpublished), 19999.

[2] D. O. Awoyemi "A new sixth-order algorithm for general second order ordinary differential equations". *International Journal of Computer Mathematics*, *77*(1), 117- 124, 2001.

[3] S. J. Kayode "An efficient zero-stable numerical method for fourth-order differential equations". *International Journal of Mathematics and Mathematical Sciences*.1 – 10, 2008.

[4] A. O. Adesanya, M. O. Udo and A. M. Alkali "A New Block-Predictor Corrector Algorithm for the Solution of

y'''= f (x, y, y',y'')" *American Journal of Computational Mathematics*, *2*(04), 341 – 344, 2012.

[5] B. T. Olabode "A six-step scheme for the solution of fourth order ordinary differential equation " *The Pacific Journal of Science and Technology*, *10*(1), 143-148, 2009.

[6] U. Mohammed "A six step block method for solution of fourth order ordinary differential equations". *The Pacific Journal of Science and Technology*. 11(1): 259 – 265, 2010.

[7] J. D. Lambert "Computational Methods in Ordinary Differential Equations "*Introductory Mathematics for Scientists and Engineers*.Wiley, 1973.

[8] P. Henrici. "Discrete variable methods in Ordinary Differential Equations", 1962.

Appendix 1

$$A=\begin{pmatrix}
1 & x_{n+3} & x_{n+3}^{2} & x_{n+3}^{3} & x_{n+3}^{4} & x_{n+3}^{5} & x_{n+3}^{6} & x_{n+3}^{7} & x_{n+3}^{8} & x_{n+3}^{9} & x_{n+3}^{10} & x_{n+3}^{11} & x_{n+3}^{12}\\
1 & x_{n+4} & x_{n+4}^{2} & x_{n+4}^{3} & x_{n+4}^{4} & x_{n+4}^{5} & x_{n+4}^{6} & x_{n+4}^{7} & x_{n+4}^{8} & x_{n+4}^{9} & x_{n+4}^{10} & x_{n+4}^{11} & x_{n+4}^{12}\\
1 & x_{n+5} & x_{n+5}^{2} & x_{n+5}^{3} & x_{n+5}^{4} & x_{n+5}^{5} & x_{n+5}^{6} & x_{n+5}^{7} & x_{n+5}^{8} & x_{n+5}^{9} & x_{n+5}^{10} & x_{n+5}^{11} & x_{n+5}^{12}\\
1 & x_{n+6} & x_{n+6}^{2} & x_{n+6}^{3} & x_{n+6}^{4} & x_{n+6}^{5} & x_{n+6}^{6} & x_{n+6}^{7} & x_{n+6}^{8} & x_{n+6}^{9} & x_{n+6}^{10} & x_{n+6}^{11} & x_{n+6}^{12}\\
0 & 0 & 0 & 0 & 24 & 120x_{n} & 360x_{n}^{2} & 840x_{n}^{3} & 1680x_{n}^{4} & 3024x_{n}^{5} & 5040x_{n}^{6} & 7920x_{n}^{7} & 11880x_{n}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^{2} & 840x_{n+1}^{3} & 1680x_{n+1}^{4} & 3024x_{n+1}^{5} & 5040x_{n+1}^{6} & 7920x_{n+1}^{7} & 11880x_{n+1}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^{2} & 840x_{n+2}^{3} & 1680x_{n+2}^{4} & 3024x_{n+2}^{5} & 5040x_{n+2}^{6} & 7920x_{n+2}^{7} & 11880x_{n+2}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^{2} & 840x_{n+3}^{3} & 1680x_{n+3}^{4} & 3024x_{n+3}^{5} & 5040x_{n+3}^{6} & 7920x_{n+3}^{7} & 11880x_{n+3}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^{2} & 840x_{n+4}^{3} & 1680x_{n+4}^{4} & 3024x_{n+4}^{5} & 5040x_{n+4}^{6} & 7920x_{n+4}^{7} & 11880x_{n+4}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x_{n+5}^{2} & 840x_{n+5}^{3} & 1680x_{n+5}^{4} & 3024x_{n+5}^{5} & 5040x_{n+5}^{6} & 7920x_{n+5}^{7} & 11880x_{n+5}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+6} & 360x_{n+6}^{2} & 840x_{n+6}^{3} & 1680x_{n+6}^{4} & 3024x_{n+6}^{5} & 5040x_{n+6}^{6} & 7920x_{n+6}^{7} & 11880x_{n+6}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+7} & 360x_{n+7}^{2} & 840x_{n+7}^{3} & 1680x_{n+7}^{4} & 3024x_{n+7}^{5} & 5040x_{n+7}^{6} & 7920x_{n+7}^{7} & 11880x_{n+7}^{8}\\
0 & 0 & 0 & 0 & 24 & 120x_{n+8} & 360x_{n+8}^{2} & 840x_{n+8}^{3} & 1680x_{n+8}^{4} & 3024x_{n+8}^{5} & 5040x_{n+8}^{6} & 7920x_{n+8}^{7} & 11880x_{n+8}^{8}
\end{pmatrix}$$

$$Q=\begin{pmatrix}
\frac{-54120}{w} & \frac{76510}{w} & \frac{132823}{w} & \frac{-233750}{w} & 0 & \frac{-142560}{w} & \frac{-68904}{w} & \frac{3960}{w} & \frac{12573}{w} & \frac{4400}{w} & \frac{726}{w} & \frac{60}{w} & \frac{2}{w}\\
\frac{54120}{u} & \frac{76150}{u} & \frac{-160741}{u} & \frac{-275011}{u} & 0 & -\frac{166320}{u} & -\frac{79068}{u} & -\frac{5346}{u} & -\frac{14553}{u} & -\frac{4950}{u} & -\frac{792}{u} & -\frac{63}{u} & -\frac{2}{u}\\
-\frac{299640}{v} & -\frac{401962}{v} & \frac{1458421}{v} & \frac{2342934}{v} & 0 & -\frac{1397088}{v} & -\frac{648648}{v} & \frac{53064}{v} & -\frac{120681}{v} & \frac{39490}{v} & \frac{6072}{v} & \frac{462}{v} & \frac{14}{v}\\
-\frac{25080}{u} & -\frac{123410}{u} & -\frac{2106673}{u} & -\frac{3000371}{u} & 0 & -\frac{1746360}{u} & -\frac{781704}{u} & -\frac{80586}{u} & -\frac{147411}{u} & -\frac{45650}{u} & -\frac{6666}{u} & -\frac{483}{u} & -\frac{14}{u}\\
-\frac{16595304}{w} & \frac{30227870}{w} & \frac{19023001}{w} & \frac{6746674}{w} & 0 & -\frac{2328480}{w} & -\frac{977592}{w} & -\frac{135432}{w} & \frac{187407}{w} & \frac{53460}{w} & \frac{7326}{w} & \frac{504}{w} & \frac{14}{w}\\
\frac{78422520}{u} & \frac{151550770}{u} & \frac{85710089}{u} & \frac{10435095}{u} & 0 & \frac{3492720}{u} & \frac{1272348}{u} & -\frac{266310}{u} & -\frac{248391}{u} & -\frac{63250}{u} & -\frac{8052}{u} & -\frac{525}{u} & -\frac{14}{u}\\
\frac{42056520}{v} & \frac{119368550}{v} & \frac{124268317}{v} & \frac{52255742}{v} & 0 & -\frac{6985440}{v} & -\frac{1380456}{v} & \frac{578952}{v} & \frac{340461}{v} & \frac{75350}{v} & \frac{8844}{v} & \frac{546}{v} & \frac{14}{v}\\
-\frac{349800}{u} & \frac{696106}{u} & \frac{4648877}{u} & \frac{71307831}{u} & \frac{4989600}{u} & \frac{1589544}{u} & \frac{4224}{u} & -\frac{179982}{u} & -\frac{68013}{u} & -\frac{12870}{u} & -\frac{1386}{u} & -\frac{81}{u} & -\frac{2}{u}\\
\frac{262680}{w} & \frac{174530}{w} & -\frac{914201}{w} & -\frac{1251162}{w} & 0 & \frac{997920}{w} & \frac{862488}{w} & \frac{371448}{w} & \frac{95733}{w} & \frac{15400}{w} & \frac{1518}{w} & \frac{84}{w} & \frac{2}{w}
\end{pmatrix}$$

$$e=\begin{pmatrix}
\frac{269}{7060} & \frac{-875}{15851} & \frac{416}{5937} & \frac{-123}{1916} & \frac{4358}{107497} & \frac{-32}{1911} & \frac{101}{24781} & \frac{-13}{29296}\\
\frac{944}{1299} & \frac{-417}{497} & \frac{1831}{1742} & \frac{-1915}{2003} & \frac{1767}{2939} & \frac{-712}{2875} & \frac{82}{1363} & \frac{-44}{6727}\\
\frac{1407}{403} & \frac{-1133}{358} & \frac{29698}{7047} & \frac{-8661}{2251} & \frac{2021}{834} & \frac{-926}{927} & \frac{596}{2455} & \frac{-73}{2765}\\
\frac{14458}{1457} & \frac{-6831}{953} & \frac{1799}{164} & \frac{-800}{81} & \frac{1531}{246} & \frac{-5795}{2258} & \frac{632}{1013} & \frac{-53}{781}\\
\frac{5381}{249} & \frac{-1268}{99} & \frac{1777}{76} & \frac{-5502}{275} & \frac{3167}{249} & \frac{-488}{93} & \frac{1327}{1040} & \frac{-103}{742}\\
\frac{10999}{274} & \frac{-7599}{379} & \frac{27703}{634} & \frac{-3974}{115} & \frac{3649}{160} & \frac{-2420}{259} & \frac{793}{349} & \frac{-332}{1343}\\
\frac{8053}{120} & \frac{-1443}{50} & \frac{10901}{147} & \frac{-18086}{337} & \frac{5059}{133} & \frac{-3311}{221} & \frac{20455}{5554} & \frac{-335}{836}\\
\frac{22901}{220} & \frac{-9524}{243} & \frac{7721}{166} & \frac{-33226}{429} & \frac{18572}{309} & \frac{-8104}{375} & \frac{5905}{1028} & \frac{-1122}{1847}
\end{pmatrix}$$

$$G=\begin{pmatrix}
\frac{3619903}{39916800} & \frac{6779886}{39916800} & -\frac{9359135}{39916800} & \frac{11774146}{39916800} & -\frac{10745445}{39916800} & \frac{6771082}{39916800} & -\frac{2792861}{39916800} & \frac{679110}{39916800} & \frac{-73886}{39916800}\\
\frac{286967}{623700} & \frac{911204}{623700} & -\frac{926646}{623700} & \frac{1173140}{623700} & -\frac{1067950}{623700} & \frac{671628}{623700} & -\frac{276634}{623700} & \frac{67196}{623700} & -\frac{7305}{623700}\\
\frac{183384}{492800} & \frac{711918}{492800} & -\frac{521217}{492800} & \frac{766290}{492800} & -\frac{699885}{492800} & \frac{441306}{492800} & -\frac{182043}{492800} & \frac{44262}{492800} & -\frac{4815}{492800}\\
\frac{321172}{155925} & \frac{1371264}{155925} & -\frac{752480}{155925} & \frac{1435264}{155925} & -\frac{1243200}{155925} & \frac{784768}{155925} & -\frac{323744}{155925} & \frac{78720}{155925} & -\frac{8564}{155925}\\
\frac{5253125}{1596672} & \frac{23702750}{1596672} & -\frac{10296375}{1596672} & \frac{25537250}{1596672} & -\frac{19680625}{1596672} & \frac{12920250}{1596672} & -\frac{5237125}{1596672} & \frac{1295750}{1596672} & -\frac{141000}{1596672}\\
\frac{74034}{15400} & \frac{346248}{15400} & -\frac{123660}{15400} & \frac{385128}{15400} & -\frac{258660}{15400} & \frac{190296}{15400} & -\frac{75348}{15400} & \frac{18360}{15400} & -\frac{1998}{15400}\\
\frac{37701874}{5702400} & \frac{180838518}{5702400} & -\frac{54639557}{5702400} & \frac{206894170}{5702400} & -\frac{122270925}{5702400} & \frac{104842066}{5702400} & -\frac{35782103}{5702400} & \frac{9423582}{5702400} & -\frac{1020425}{5702400}\\
\frac{169624}{155925} & \frac{828928}{155925} & -\frac{216192}{155925} & \frac{970240}{155925} & -\frac{510560}{155925} & \frac{508416}{155925} & -\frac{134528}{155925} & \frac{51712}{155925} & -\frac{4440}{155925}
\end{pmatrix}$$

$$H=\begin{pmatrix}
\frac{1624505}{7257600} & \frac{4124231}{7257600} & -\frac{5225623}{7257600} & \frac{6488191}{7257600} & -\frac{64888311}{7257600} & \frac{3698922}{7257600} & -\frac{1522673}{7257600} & \frac{369744}{7257600} & -\frac{40187}{7257600}\\
\frac{58193}{113400} & \frac{235072}{113400} & -\frac{183708}{113400} & \frac{247328}{113400} & -\frac{227030}{113400} & \frac{143232}{113400} & -\frac{59092}{113400} & \frac{14368}{113400} & -\frac{1563}{113400}\\
\frac{71661}{89600} & \frac{328608}{89600} & -\frac{150624}{89600} & \frac{315000}{89600} & -\frac{281430}{89600} & \frac{177264}{89600} & -\frac{73128}{89600} & \frac{17784}{89600} & -\frac{1935}{89600}\\
\frac{30812}{28350} & \frac{148992}{28350} & -\frac{46400}{28350} & \frac{160256}{28350} & -\frac{118440}{28350} & \frac{76288}{28350} & -\frac{31552}{28350} & \frac{7680}{28350} & -\frac{836}{28350}\\
\frac{398825}{290304} & \frac{1987000}{290304} & -\frac{465000}{290304} & \frac{2294000}{290304} & -\frac{1283750}{290304} & \frac{1020600}{290304} & -\frac{412000}{290304} & \frac{100000}{290304} & -\frac{10875}{290304}\\
\frac{2325}{1400} & \frac{11808}{1400} & -\frac{2196}{1400} & \frac{14208}{1400} & -\frac{6390}{1400} & \frac{7200}{1400} & -\frac{2268}{1400} & \frac{576}{1400} & -\frac{63}{1400}\\
\frac{288533}{1036800} & \frac{1484112}{1036800} & -\frac{225008}{1036800} & \frac{1830248}{1036800} & -\frac{689430}{1036800} & \frac{1009792}{1036800} & -\frac{145432}{1036800} & \frac{84168}{1036800} & -\frac{8183}{1036800}\\
\frac{63296}{28350} & \frac{329728}{28350} & -\frac{44544}{28350} & \frac{419840}{28350} & -\frac{145280}{28350} & \frac{251904}{28350} & -\frac{14848}{28350} & \frac{47104}{28350} & 0
\end{pmatrix}$$

$$I=\begin{pmatrix}
\frac{315273}{1069200} & \frac{1316197}{1069200} & -\frac{1356711}{1069200} & \frac{1648632}{1069200} & -\frac{1482974}{1069200} & \frac{927046}{1069200} & -\frac{380447}{1069200} & \frac{92186}{1069200} & -\frac{10004}{1069200}\\
\frac{32377}{113400} & \frac{182584}{113400} & -\frac{42494}{113400} & \frac{120088}{113400} & -\frac{116120}{113400} & \frac{74728}{113400} & -\frac{31154}{113400} & \frac{7624}{113400} & -\frac{833}{113400}\\
\frac{12881}{44800} & \frac{70902}{44800} & \frac{3438}{44800} & \frac{79934}{44800} & -\frac{56160}{44800} & \frac{34434}{44800} & -\frac{14062}{44800} & \frac{3402}{44800} & -\frac{369}{44800}\\
\frac{8126}{28350} & \frac{45152}{28350} & \frac{488}{28350} & \frac{65504}{28350} & -\frac{18160}{28350} & \frac{18464}{28350} & -\frac{7912}{28350} & \frac{1952}{28350} & -\frac{214}{28350}\\
\frac{41705}{145152} & \frac{230150}{145152} & \frac{7550}{145152} & \frac{318350}{145152} & -\frac{4000}{145152} & \frac{170930}{145152} & -\frac{49150}{145152} & \frac{11450}{145152} & -\frac{1225}{145152}\\
\frac{401}{1400} & \frac{2232}{1400} & \frac{18}{1400} & \frac{3224}{1400} & -\frac{360}{1400} & \frac{2664}{1400} & \frac{158}{1400} & \frac{72}{1400} & -\frac{9}{1400}\\
\frac{149527}{518400} & \frac{816634}{518400} & \frac{48706}{518400} & \frac{1085937}{518400} & \frac{54880}{518400} & \frac{736078}{518400} & \frac{522046}{518400} & \frac{223174}{518400} & -\frac{8183}{518400}\\
\frac{7912}{28350} & \frac{47104}{28350} & -\frac{7424}{28350} & \frac{83968}{28350} & -\frac{36320}{28350} & \frac{83968}{28350} & -\frac{7424}{28350} & \frac{47104}{28350} & \frac{7912}{28350}
\end{pmatrix}$$

Appendix 2

$$J=\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{h^m}{m!}y_n^{(m)}-\frac{24396497}{958003200}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(958003200*m!)}y_n^{(4+m)}\begin{pmatrix}36501816(1)^m-52883276(2)^m+\\ 67126376(3)^m-61500210(4)^m+\\ 38838088(5)^m-16041916(6)^m\\ +3904536(7)^m-425111(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{1035731}{3742200}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(3742200*m!)}y_n^{(4+m)}\begin{pmatrix}2719504(1)^m-3139836(2)^m+\\ 3933392(3)^m-3577790(4)^m+\\ 2249904(5)^m-926764(6)^m\\ +225136(7)^m-24477(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{4104531}{3942400}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(3942400*m!)}y_n^{(4+m)}\begin{pmatrix}13764168(1)^m-12476916(2)^m\\ +16614360(3)^m-15168870(4)^m\\ +9553464(5)^m-3938148(6)^m\\ +957096(7)^m-104085(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{1218688}{467775}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{6h^{4+m}}{(467775*m!)}y_n^{(4+m)}\begin{pmatrix}1160448(1)^m-838240(2)^m+\\ 1282816(3)^m-1155000(4)^m\\ +727808(5)^m-300128(6)^m+\\ 72960(7)^m-7936(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{201421625}{38320128}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(38320128*m!)}y_n^{(4+m)}\begin{pmatrix}828115000(1)^m-490807500(2)^m\\ +895985000(3)^m-766681250(4)^m\\ +487389000(5)^m-201077500(6)^m\\ +48895000(7)^m-5319375(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{285858}{30800}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{6h^{4+m}}{(30800*m!)}y_n^{(4+m)}\begin{pmatrix}206064(1)^m-102924(2)^m\\ +224304(3)^m-177390(4)^m\\ +117072(5)^m-47964(6)^m\\ +11664(7)^m-1269(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{2048300303}{136857600}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(136857600*m!)}y_n^{(4+m)}\begin{pmatrix}9184285992(1)^m-3949712228(2)^m\\ +10148873336(3)^m-7344827070(4)^m\\ +5205732952(5)^m-2050386772(6)^m\\ +504037128(7)^m-54841241(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(8h)^m}{m!}y_n^{(m)}-\sum_{m=0}^{3}\frac{(8h)^m}{m!}y_n^{(m)}-\frac{10571776}{467775}h^4y_n^{iv}-\sum_{m=0}^{\infty}\frac{h^{4+m}}{(467775*m!)}y_n^{(4+m)}\begin{pmatrix}48693248(1)^m-18333696(2)^m\\ +54722560(3)^m-36229120(4)^m\\ +28114944(5)^m-10108928(6)^m\\ +2686976(7)^m-284160(8)^m\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

TABLE 1: THE COMPARISON OF THE NEW METHOD WITH OMAR [1] FOR SOLVING PROBLEM 1

h-values	New Method	Omar [1]	Number of Steps	Error in new Method, k=8	Error in Omar [1], k=8
10^{-2}	8-Step Method	S2PEB	54	1.728040E-11	1.00778E-02
		P2PEB	54	1.728040E-11	1.00778E-02
		S3PEB	39	7.958079E-13	1.00778E-02
		P3PEB	39	7.958079E-13	1.00778E-02
10^{-3}	8-Step	S2PEB	504	8.185452E-12	1.00778E-03
		P2PEB	504	8.185452E-12	1.00778E-02
		S3PEB	339	3.410605E-13	1.00778E-03

	Method	P3PEB	339	3.410605E-13	1.00778E-02
10^{-4}	8-Step Method	S2PEB	5004	1.100034E-09	1.00008E-04
		P2PEB	5004	1.100034E-09	1.00008E-04
		S3PEB	3339	3.997513E-11	1.00008E-04
		P3PEB	3339	3.997513E-11	1.00008E-04
10^{-5}	8-Step Method	S2PEB	50004	1.035278E-09	1.00001E-05
		P2PEB	50004	1.035278E-09	1.00001E-05
		S3PEB	33339	3.454659E-11	1.00001E-05
		P3PEB	33339	3.454659E-11	1.00001E-05

TABLE 2: THE COMPARISON OF THE NEW METHOD WITH OMAR [1] FOR SOLVING PROBLEM 2

h-values	New Method	Omar [1]	Number of Steps	Error in new Method, *k*=8	Error in Omar [1], *k*=8
10^{-2}	8-Step Method	S2PEB	54	1.193712E-11	8.37112E-04
		P2PEB	54	1.193712E-11	8.37112E-04
		S3PEB	39	2.199130E-12	8.37105E-04
		P3PEB	39	2.199130E-12	8.37105E-04
10^{-3}	8-Step Method	S2PEB	504	2.131628E-14	8.34604E-05
		P2PEB	504	2.131628E-14	8.34604E-05
		S3PEB	339	1.776357E-14	8.34604E-05
		P3PEB	339	1.776357E-14	8.34604E-05
10^{-4}	8-Step Method	S2PEB	5004	8.427037E-12	8.34353E-06
		P2PEB	5004	8.427037E-12	8.34353E-06
		S3PEB	3339	1.243450E-13	8.34353E-06
		P3PEB	3339	1.243450E-13	8.34353E-06
10^{-5}	8-Step Method	S2PEB	50004	1.938183E-11	8.34326E-07
		P2PEB	50004	1.938183E-11	8.34326E-07
		S3PEB	33339	1.257483E-12	8.34330E-07
		P3PEB	33339	1.257483E-12	8.34330E-07