A Nonlinear Curve Equation for an Object Moving with Constant Acceleration Components

Mehmet Pakdemirli and İhsan Timuçin Dolapci

***Abstract*—**An ordinary differential equation describing a curve for which the tangential and normal acceleration components of the object remains constant is derived. The equation and initial conditions are expressed in dimensionless form. In its dimensionless form, the curves are effected only by a parameter which represents the ratio of the tangential acceleration to the normal acceleration. For constant velocity case, the equation can be solved analytically yielding a circular arc solution as expected. For nonzero tangential acceleration, closed form solutions are not available. Using a series solution, the curve is approximated by polynomials of arbitrary order. The general recursion relation for the polynomial coefficients are given. Two different perturbation solutions are also presented. In the first perturbation approach, the curve parameter is selected as the perturbation parameter. In the second approach, the depending variable is assumed to be small by introducing an alternative perturbation parameter. It is found that the second perturbation solution yields identical results with the series solutions. The approximate solutions and the numerical solutions are contrasted and within the range of validity, the curves can be successfully approximated by the analytical solutions. Potential application areas can be the design of highway curves, highway exits, railroads, route selection for ships and aircrafts. A practical application to highway exits is considered as an example.

***Keywords*—** Curve Design, Highways, Kinematics, Numerical Solution, Perturbation Solution, Series Solution, Vehicle Routes

I. INTRODUCTION

During transportation, aerial, marine and land vehicles cannot travel always in straight routes. Tracking a curved path is inevitable at least for some portion of the travel. To seek for an ideal curve path becomes then a technological problem. Especially at high speeds, smooth transitions in curvatures are needed when entering curved routes. Abrupt changes in the curvatures affect safety and comfort of the travel negatively.

Usually entering to the curves, the velocity should be reduced and the straight path velocity can no longer be maintained. For a constant tangential deceleration, the goal in this study is to seek a specific curve for which the normal acceleration component throughout the curve remains constant.

In curved parts of roads, a special function named clothoid is used [1-3]. The clothoid has the property that its curvature varies linearly with its arc length. Since they are transcendental functions, they have been approximated by polynomials, power series, continued fractions and rational functions [2]. Clothoids are especially useful in transportation engineering, since they can be navigated at constant speed by linear steering and a constant rate of angular acceleration [3]. The parametric representations of clothoids are also used in optics [4]. The aesthetic aspect of logarithmic spiral, clothoid and involute curves were studied [5]. Curve generation algorithms were discussed in [6]. The curves derived in this study are not clothoids, since the basic assumption is not a constant velocity with a constant angular acceleration, rather the assumption is a constant tangential and normal (centripetal) acceleration components with respect to the curve.

The equation determining the curve is derived using basic principles of kinematics. Equation and initial conditions are expressed in dimensionless form. The curves depend on a single parameter which is the ratio of the tangential acceleration to the normal acceleration. For vanishing of the parameter, the curve is a circular arc for which constant normal acceleration with constant velocity implies constant radius of curvature. For non-zero parameters, closed form solutions do not exist. The next best choice is to find approximate analytical solutions. A polynomial series solution is constructed to approximate the curve function. Furthermore, the curve parameter is selected as the perturbation parameter and a first order uniform perturbation solution is also presented. Finally, numerical solutions are calculated using a variable step size Runge-Kutta algorithm. It is found that the numerical solutions can be replaced with the approximate solutions in a wide range of the interval.

II. DERIVATION OF THE CURVE EQUATION

Assume that the object enters a curve with initial radius of curvature ρ_0 and velocity v_0. The object has a constant deceleration a_0 throughout the curve. s is the length coordinate along the curve with s=0 representing the entrance and cartesian coordinates are selected as shown in Figure 1.

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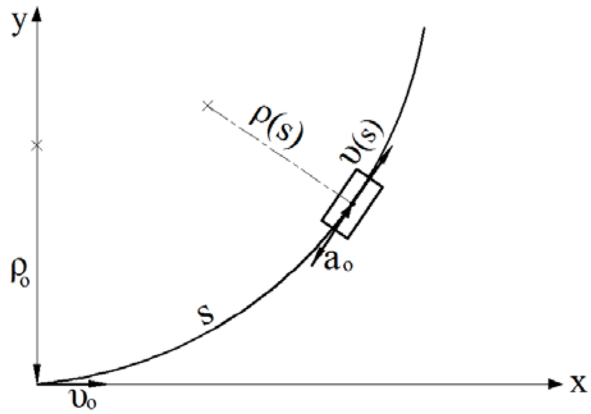

Figure 1- Sketch of the curve

If the normal (centripetal) acceleration [7] remains constant within the curve

$$\frac{v(s)^2}{\rho(s)} = \frac{v_0^2}{\rho_0} \tag{1}$$

where $v(s)$ and $\rho(s)$ represent velocity and radius of curvature at distance s from the entrance. For a constant tangential deceleration component a_0, the reduced speed at location s is

$$v(s)^2 = v_0^2 - 2a_0 s \ . \tag{2}$$

From calculus, the length of a curve and the radius of curvature are given as [8]

$$s = \int_0^x \sqrt{1+y'^2}\,dx \tag{3}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}} \tag{4}$$

where prime denotes differentiation with respect to x. Upon substitution of (2)-(4) into (1),

$$\left(v_0^2 - 2a_0 \int_0^x \sqrt{1+y'^2}\,dx \right) \frac{y''}{(1+y'^2)^{3/2}} = \frac{v_0^2}{\rho_0} \tag{5}$$

solving for the parenthesis, differentiating once to eliminate the integral, and rearranging yields

$$\left(1+y'^2\right)y''' - \left(3y' + \frac{2a_0\rho_0}{v_0^2}\right)y''^2 = 0 \tag{6}$$

which is the differential equation determining a constant normal and tangential acceleration curve. For the specific coordinates chosen, the initial conditions are

$$y(0)=0, \quad y'(0)=0, \quad y''(0)=\frac{1}{\rho_0} \tag{7}$$

The first condition is evident from the origin of coordinate location, the second condition requires a tangent slope at the entrance for smooth transition and the last condition is due to the initial curvature of the function. For universality of results, the system is represented in dimensionless form by defining

$$x^* = \frac{x}{\rho_0}, \quad y^* = \frac{y}{\rho_0} \tag{8}$$

and substituting into (6) and (7)

$$\left(1+y'^2\right)y''' - \left(3y' + 2\varepsilon\right)y''^2 = 0 \tag{9}$$

$$y(0)=0, \quad y'(0)=0, \quad y''(0)=1 \tag{10}$$

where

$$\varepsilon = \frac{a_0\rho_0}{v_0^2} = \frac{a_0}{v_0^2/\rho_0} \ . \tag{11}$$

For simplicity, the symbol star is not shown on the variables keeping in mind that the variables are all dimensionless. The above differential system defines a constant tangential and normal acceleration curve. The family of curves depend on only one parameter ε which is the ratio of the tangential acceleration to the normal acceleration. Rather than choosing separately the accelerations, radius of curvatures and velocities, it is sufficient to choose ε, the combination of all parameters in the analysis which reduces substantially the calculations and presentations in the form of figures. Note that the equation is highly nonlinear possessing quadratic and cubic nonlinearities.

III. Analytical Solutions

Analytical solutions of the model are presented in this section. The degenerate case of ε=0 can be solved in closed form functions. However, $\varepsilon \neq 0$ case cannot be solved in closed form functions and approximations are inevitable. A series solution as well as two different perturbation type solutions are presented in this section.

For the degenerate case, the equation is

$$\left(1+y'^2\right)y''' - 3y'y''^2 = 0 \tag{12}$$

$$y(0)=0, \quad y'(0)=0, \quad y''(0)=1 \ . \tag{13}$$

A straightforward calculation by employing reduction of order and successive integrations yield

$$y = 1 - \sqrt{1-x^2} \tag{14}$$

which represents a circular arc since ε=0 corresponds to no tangential acceleration and the normal component of the acceleration remains constant only in a circular path if the speed is constant.

Series Solution

Assume a power series solution for the problem with nonzero ε,

$$y(x)=\sum_{i=0}^{\infty}a_i x^i \tag{15}$$

Inserting into (9), performing the necessary algebra, the recursive relationship between the coefficients is

$$\begin{aligned}&(i-2)(i-3)(i-4)a_{i-2}\\&+\sum_{j=0}^{i}\sum_{k=0}^{j}k(k-1)(k-2)(j-k)(i-j)a_k a_{j-k}a_{i-j}\\&-3\sum_{j=0}^{i}\sum_{k=0}^{j}k(k-1)(j-k)(j-k-1)(i-j)a_k a_{j-k}a_{i-j}\\&-2\varepsilon\sum_{j=0}^{i-1}j(j-1)(i-j-1)(i-j-2)a_j a_{i-j-1}=0\end{aligned}\qquad i=5,6,7,.. \tag{16}$$

The leading coefficients are

$$a_3=\frac{2a_2^2(3a_1+2\varepsilon)}{3(1+a_1^2)} \tag{17}$$

$$a_4=\frac{a_2^3+2a_1a_2a_3+2\varepsilon a_2a_3}{1+a_1^2} \tag{18}$$

$$a_5=\frac{13a_2^2a_3+4a_1a_2a_4+6a_1a_3^2+\varepsilon(8a_2a_4+6a_3^2)}{5(1+a_1^2)} \tag{19}$$

$$a_6=\frac{30a_2^2a_4+30a_1a_3a_4+45a_2a_3^2+\varepsilon(36a_3a_4+20a_2a_5)}{15(1+a_1^2)} \tag{20}$$

$$\begin{aligned}a_7=&\big(45a_3^3+40a_1a_4^2+200a_2a_3a_4+50a_1a_3a_5+50a_2^2a_5\\&-20a_1a_2a_6+\varepsilon(48a_4^2+80a_3a_5+40a_2a_6)\big)/35(1+a_1^2)\end{aligned} \tag{21}$$

With the aid of (16), calculations can be carried to any arbitrary order using a symbolic computation program. The higher order coefficients are not given here for brevity. Initial conditions (13) require

$$a_0=0,\quad a_1=0,\quad a_2=\frac{1}{2} \tag{22}$$

Substituting (22) into (17)-(21) yields

$$\begin{aligned}&a_3=\frac{1}{3}\varepsilon,\ a_4=\frac{1}{8}+\frac{1}{3}\varepsilon^2,\ a_5=\frac{19}{60}\varepsilon+\frac{2}{5}\varepsilon^3,\\&a_6=\frac{1}{16}+\frac{29}{45}\varepsilon^2+\frac{8}{15}\varepsilon^4,\ a_7=\frac{81}{280}\varepsilon+\frac{388}{315}\varepsilon^3+\frac{16}{21}\varepsilon^5\end{aligned} \tag{23}$$

Hence the polynomial approximation is

$$\begin{aligned}y(x)=&\frac{1}{2}x^2+\frac{1}{3}\varepsilon x^3+\left(\frac{1}{8}+\frac{1}{3}\varepsilon^2\right)x^4+\left(\frac{19}{60}\varepsilon+\frac{2}{5}\varepsilon^3\right)x^5\\&+\left(\frac{1}{16}+\frac{29}{45}\varepsilon^2+\frac{8}{15}\varepsilon^4\right)x^6\\&+\left(\frac{81}{280}\varepsilon+\frac{388}{315}\varepsilon^3+\frac{16}{21}\varepsilon^5\right)x^7+O(x^8)\end{aligned}\ . \tag{24}$$

For vanishing curve parameter, the Taylor expansion of the circular solution (14) is obtained. The original circular solution is an even power polynomial and deformations from this solution with the curve parameter introduces the odd powers also.

Perturbation Solution 1

If the curve parameter is our perturbation parameter, an approximate solution

$$y(x)=y_0(x)+\varepsilon y_1(x)+O(\varepsilon^2) \tag{25}$$

can be constructed. Substituting the expansion into (9) and (10), separating at different orders yields

$$\begin{aligned}O(1):\quad &(1+y_0'^2)y_0'''-3y_0'y_0''^2=0\\&y_0(0)=0,\quad y_0'(0)=0,\quad y_0''(0)=1\end{aligned} \tag{26}$$

$$\begin{aligned}O(\varepsilon):\quad &(1+y_0'^2)y_1'''-3y_1'y_0''^2-6y_0'y_0''y_1''+2y_0'y_1'y_0'''-2y_0''^2=0\\&y_1(0)=0,\quad y_1'(0)=0,\quad y_1''(0)=0\end{aligned} \tag{27}$$

The first order solution is the circular arc solution presented before

$$y_0=1-\sqrt{1-x^2} \tag{28}$$

Substituting this solution to the next order

$$\begin{aligned}&y_1'''-\frac{6x}{1-x^2}y_1''+3\frac{2x^2-1}{(1-x^2)^2}y_1'=\frac{2}{(1-x^2)^2}\\&y_1(0)=0,\quad y_1'(0)=0,\quad y_1''(0)=0\end{aligned} \tag{29}$$

and solving yields

$$y_1=-\frac{2x-\pi}{\sqrt{1-x^2}}-\frac{2\cosh^{-1}(x)}{\sqrt{x^2-1}} \tag{30}$$

The approximate solution is

$$y(x)=1-\sqrt{1-x^2}-\varepsilon\left(\frac{2x-\pi}{\sqrt{1-x^2}}+\frac{2\cosh^{-1}(x)}{\sqrt{x^2-1}}\right)+O(\varepsilon^2) \tag{31}$$

Since the function is not defined near x=1, the singularity at this point is unimportant.

Perturbation Solution 2

An alternative perturbation solution can be constructed by assuming the dependent variable to be small. If α is the perturbation parameter, the smallness of the dependent variable is represented by the transformation

$$y(x) = \alpha u(x) \tag{32}$$

and the equation in terms of this transformation becomes

$$\left(1+\alpha^2 u'^2\right)u''' - \left(3\alpha^2 u' + 2\varepsilon\alpha\right)u''^2 = 0 \tag{33}$$

$$u(0)=0, \quad u'(0)=0, \quad u''(0)=1/\alpha \tag{34}$$

The expansion in terms of the perturbation parameter is

$$u(x) = u_0(x) + \alpha u_1(x) + \alpha^2 u_2(x) + O(\alpha^3) \tag{35}$$

Substituting and separation at different orders yields

$$O(1): \quad u_0''' = 0 \qquad u_0(0)=0, \quad u_0'(0)=0, \quad u_0''(0)=1/\alpha \tag{36}$$

$$O(\alpha): \quad u_1''' - 2\varepsilon u_0''^2 = 0 \qquad u_1(0)=0, \quad u_1'(0)=0, \quad u_1''(0)=0 \tag{37}$$

$$O(\alpha^2): \quad u_2''' + u_0'^2 u_0''' - 3u_0' u_0''^2 - 4\varepsilon u_0'' u_1'' = 0 \qquad u_2(0)=0, \quad u_2'(0)=0, \quad u_2''(0)=0 \tag{38}$$

The equations can be solved consecutively

$$u_0 = \frac{1}{2\alpha}x^2, \quad u_1 = \frac{1}{3\alpha^2}\varepsilon x^3, \quad u_2 = \frac{1}{\alpha^3}\left(\frac{1}{8}+\frac{1}{3}\varepsilon^2\right)x^4 \tag{39}$$

Substituting into (35) and back transforming to the original variable *y(x)*

$$y(x) = \frac{1}{2}x^2 + \frac{1}{3}\varepsilon x^3 + \left(\frac{1}{8}+\frac{1}{3}\varepsilon^2\right)x^4 + O(x^5) \tag{40}$$

which is the same solution with the series solution up to the approximation considered. The first perturbation solution which assumes the curve parameter to be small is of functional type and the solution presented here is of polynomial type. Since this second perturbation solution is similar to the series solution, it will not be considered further in numerical comparisons.

IV. COMPARISONS WITH THE NUMERICAL SOLUTIONS

The series solution and the first perturbation solution is compared with the numerical solution. Equation (9) and (10) is cast into a suitable form first by defining $y_1 = y,\, y_2 = y',\, y_3 = y''$ In terms of the new variables

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= \frac{\left(3y_2 + 2\varepsilon\right)y_3^2}{1+y_2^2} \\ y_1(0) &= 0, \quad y_2(0)=0, \quad y_3(0)=1 \end{aligned} \tag{41}$$

the system is reduced to a system with three equations of first order. The above system is solved by employing a variable step size Runge-Kutta algorithm. Figure 2 shows that as the number of terms in the series solution increases, convergence to the numerical solution is achieved. Note that the figure is drawn for a fairly large curve parameter of ε=1. Since the curve parameter is the ratio of tangential acceleration to the normal one, ε=1 corresponds to equal acceleration components.

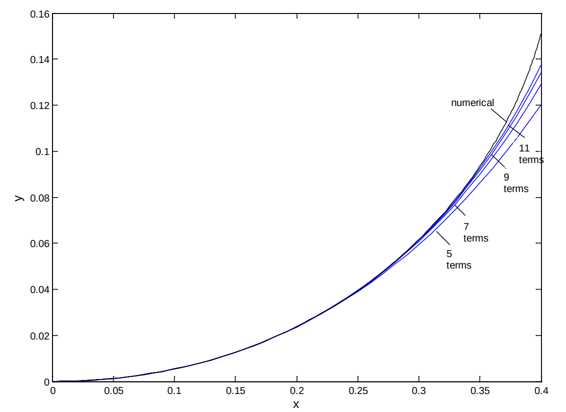

Figure 2- Convergence of series solutions to numerical solution (ε=1)

For ε=0.2, the 7 and 11-term series solution and the perturbation solution is contrasted with the numerical solution in Figure 3. The one correction term perturbation solution (i.e. equation 31) performs slightly better than the 7-term series solution.

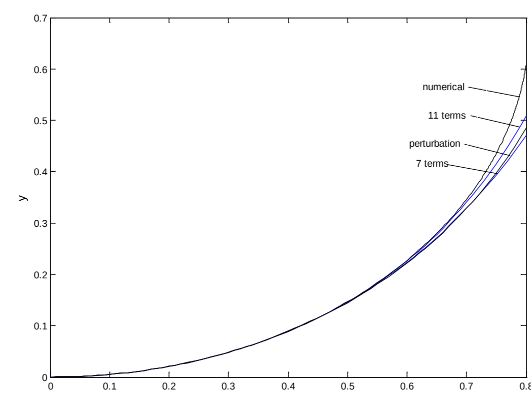

Figure 3- Comparisons of the series, perturbation and numerical solutions (ε=0.2)

Finally an intermediate value of ε=0.6 is considered in Figure 4. Five-term series solution performs slightly better than the perturbation solution. In conclusion, perturbation solution can replace the numerical solution for small curve parameter values. For larger parameter values, the series solution with sufficient number of terms better approximates the numerical solutions.

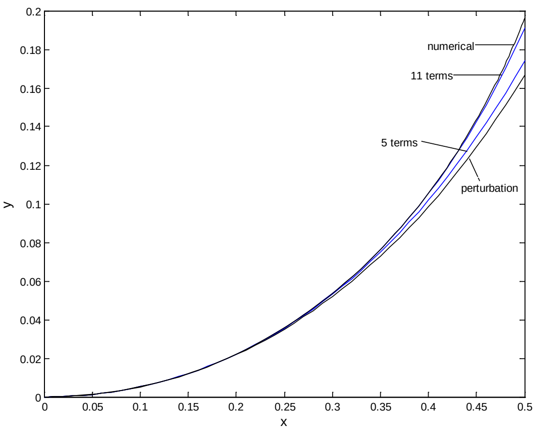

Figure 4- Comparisons of the series, perturbation and numerical solutions (ε=0.6)

In practical calculations, back substitution to dimensional quantities should be done as a final step.

The error analysis is also done for the three curve parameters considered. In Figure 5, the residual error corresponding to ε=1 is presented. As the number of terms increase, the residual error decrease in most of the domain. However, in a narrow region at the right, a reverse behavior is observed and as the number of terms increase, the residual error increases. For larger *x* values, the higher order polynomial terms added is the reason of this residual error. As can be seen from Figure 2, the absolute error is still smaller for higher term polynomials in this region also.

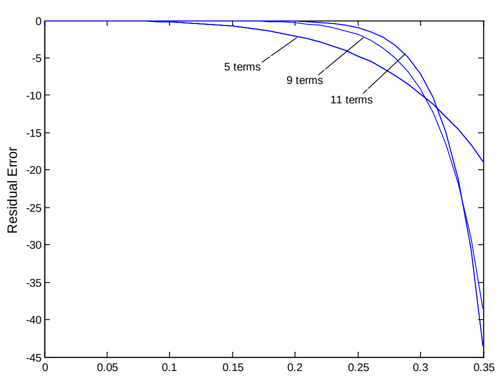

Figure 5- The residual error of polynomial solution (ε=1)

For a smaller value of the curve parameter (i.e. ε=0.2), the residual errors of 7 and 11-term solutions are contrasted. A similar behavior is observed. Adding terms reduces the residual errors in most of the domain except in a narrow region of higher *x* values.

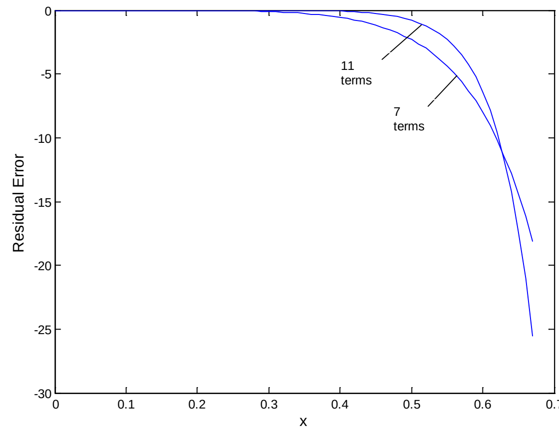

Figure 6- The residual error of polynomial solution (ε=0.2)

Finally, residual error analysis for ε=0.6 is presented in Figure 7. The qualitative behavior is the same with the previous figures.

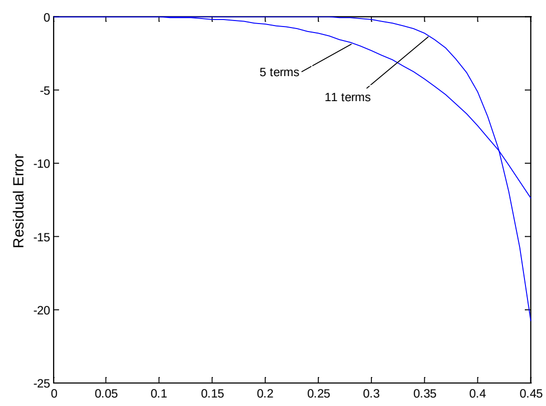

Figure 7- The residual error of polynomial solution (ε=0.6)

In conclusion, to decrease the absolute and residual errors, the series solution should be truncated and not recommended for usage for high *x* values.

V. APPLICATION TO HIGHWAY EXITS

One of the potential application areas might be the design of curves of highway exits and entrances. In Figure 8, a sketch of a typical highway exit and entrance to another highway perpendicular to the previous one is given.

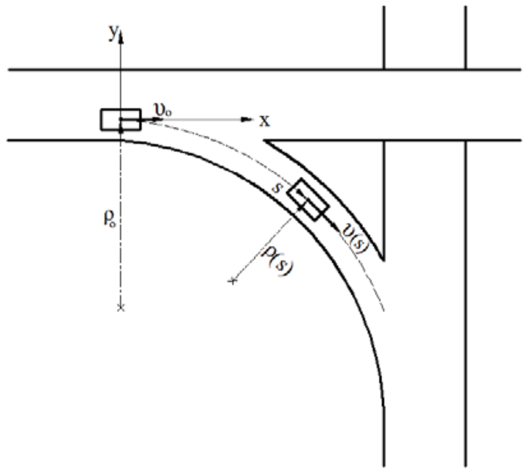

Figure 8-Sketch of a highway exit and entrance

The vehicle cannot maintain its initial velocity of v_0 after entering the exit. So, to the middle of the route, there will be a constant deceleration phase and after the middle point there will be an equivalent acceleration phase.

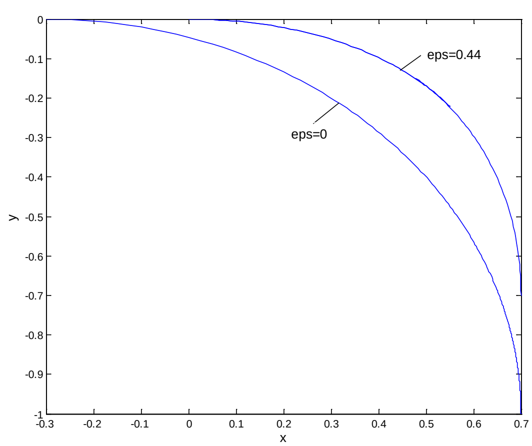

Figure 9-The curve design of the exit and entrance

In Figure 9, the curve is designed with dimensionless quantities for the curve parameter selected as ε=0.44. The exit diverges from the first highway at point (0,0) and connects the other highway at point (0.7,-0.7). To compare with the circular path having tangency to the both highways, the curve corresponding to ε=0 is also drawn. The circular path starts from (-0.3,0) and ends at (0.7,-1). It can be seen that, the new curve proposed occupies a smaller area compared to the circular path. The circular path is not realistic because the vehicles do not travel at speed limits of the highway and a reduction of speed is inevitable.

For an efficient usage of the curves, the recommended reduced speeds should be indicated on the traffic signs at sufficient number of locations. Nevertheless, for most of the time, the drivers intuitively adjust their speeds for the curves and the proposed curves will help better during steering than the alternatives.

Although, for land vehicles, the routes are predetermined and the vehicles have no choice but to follow the given paths, it is not the case for marine and aerial vehicles. Within the given loose constraints, they can choose a path from many alternatives. Especially, when there is a need to follow a curved path, the proposed curves might be a good alternative among the others.

VI.CONCLUDING REMARKS

A new family of curves which can be used in highways, routes of marine and aerial vehicles is derived. Throughout the curve, the tangential and normal accelerations remain constant, providing a comfortable transport within the vehicle. Although the curves are derived under the assumption of tangential deceleration, if they are tracked from the reverse, the curves will represent motion with tangential acceleration. An application might be the highway exits and entrances to other highways, where the vehicle should reduce its speed in entering the exit and then has to accelerate to adjust its speed to the next highway.

Usually, only a portion of the curve is used in the applications. For such analysis, rather than using the numerically calculated curves, the polynomial approximations or the perturbation solutions can be used within the given validity ranges.

Similar to clothoids which find vast application areas from transportation to optics and manufacturing engineering, the proposed curves outlined may find other application areas as well in the future.

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