A compact mathematical model of the World System economic and demographic growth, 1 CE – 1973 CE

Andrey V. Korotayev and Artemy Y. Malkov

Abstract**—**Extremely simple mathematical models are shown to be able to account for 99.2–99.91 per cent of all the variation in economic and demographic macrodynamics of the world for almost two millennia of its history. In this article we show that it is in no way coincidental that the world GDP dynamics in 1–1973 is approximated so well with a quadratic hyperbola, whereas the world population one does with a simple hyperbola. This appears to suggest a novel approach to the formation of the general theory of social macroevolution.

Keywords**—**Complex systems, demographic macrodynamics, differential equations, economic macrodynamics, finite-time singularity, power-law behavior, superexponential growth, the World System.

I. Introduction

In 1960 von Foerster, Mora, and Amiot conducted a statistical analysis of the available world population data and found out that the general shape of the world population (N) growth is best approximated by the curve described by the following hyperbolic equation:

$$N = \frac{C}{t_0 - t}, \tag{1}$$

where C and t_0 are constants, whereas t_0 corresponds to an absolute limit of such a trend at which N would become infinite, and thus logically implies the certainty of the empirical conclusion that further increases in the growth trend will cease well before that date, which von Foerster wryly called the "doomsday" implication of power-law growth (he refers tongue-in-cheek to the estimated t_0 as "Doomsday, Friday, 13 November, A.D. 2026" [1]) (Of course, von Foerster and his colleagues did not imply that the world population on that day could actually become infinite. The real implication was that the world population growth pattern that was followed for many centuries prior to 1960 was about to come to an end and be transformed into a radically different pattern. Note that this prediction began to be fulfilled only in a few years after the "Doomsday" paper was published, as since the 1970s the world population growth began to diverge more and more from the blow-up regime, and now it is not hyperbolic any more (see, *e.g.*, [2], where we present a compact mathematical model that describes both the hyperbolic development of the World System in the period prior to the early 1970s, and its withdrawal from the blow-up regime in the subsequent period, see also [3]).

Note that if von Foerster, Mora, and Amiot had had at their disposal, in addition to the world population data, also the data on the world GDP dynamics for 1–1973 (published, however, only in 2001 by Maddison [4]) they could have made another striking "prediction" – that on Saturday, 23 July, 2005 an "economic doomsday" would take place, that is on that day the world GDP would become infinite. They would have also found that in 1–1973 CE the world GDP growth had followed quadratic-hyperbolic rather than simple hyperbolic pattern.

Indeed, Maddison's estimates of the world GDP dynamics for 1–1973 CE are almost perfectly approximated by the following equation:

$$G = \frac{C}{(t_0 - t)^2}, \tag{2}$$

where G is the world GDP, C = 17355487.3 and t_0 = 2005.56 (see Fig. 1).

In this article we show that it is in no way coincidental that the world GDP dynamics in 1–1973 is approximated so well with a quadratic hyperbola, whereas the world population growth is as well approximated with a simple hyperbola. We also suggest a compact explanatory mathematical model describing the world population and GDP growth in 1–1973 CE and discuss its implications.

We believe that the most significant progress towards the development of a compact mathematical model describing the world GDP growth has been achieved by Michael Kremer [5] whose model will be discussed next (for the other mathematical models describing the world hyperbolic growth see [6–18]; of special interest are the mathematical models developed by Anders Johansen and Didier Sornette [19], [20]; note also that the explanatory logic used by Michael Kremer appears to be first suggested by Rein Taagepera [21–23]).

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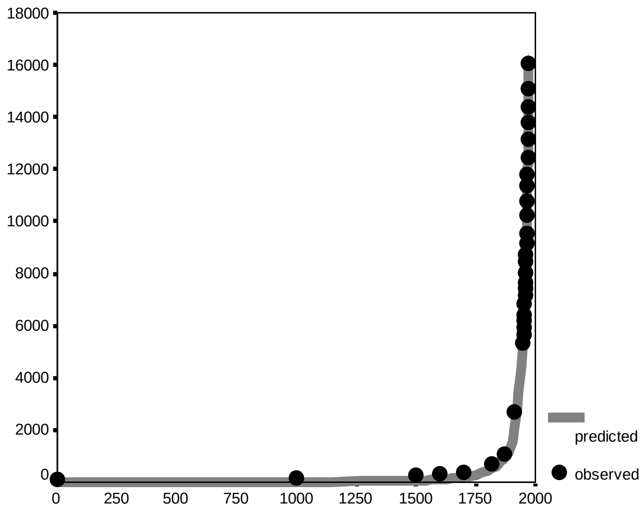

Fig. 1 World GDP dynamics, 1–1973 CE (in billions of 1990 international dollars, PPP): the fit between predictions of quadratic-hyperbolic model and the observed data. $r = 0.9993$, $R^2 = 0.9986$, $p << 0.0001$. The black markers correspond to Maddison's [4] estimates (Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of [24–27]). The grey solid line has been generated with equation (2). The best-fit values of parameters C (17749573.1) and t_0 (2006) have been calculated with the least squares method (actually, as was mentioned above, the best fit is achieved with $C = 17355487.3$ and $t_0 = 2005.56$ [which gives just the "doomsday Saturday, 23 July, 2005"], but we have decided to keep hereafter to the integer year numbers)

II. Michael Kremer's Model of the World Demographic and Technological Growth

One of the basic assumptions of Kremer's model was suggested already in the 18th century by Thomas Malthus [28]. It may be worded in the following way: "Population is limited by the available technology, so that the growth rate of population is proportional to the growth rate of technology" [5, pp. 681–682].

The basic model suggested by Kremer assumes that the production output depends on just two factors: technological level and population size. Kremer uses the following symbols to denote respective variables: Y – output, p – population, A – the level of technology, etc.; while describing Kremer's models we will employ symbols (closer to Kapitza's ones) used in our model, naturally, without distorting the sense of Kremer's equations.

Kremer assumes that overall output produced by the world economy equals

$$G = TN^{\alpha}V^{1-\alpha},$$

where G is output, T is the level of technology, V is land, $0 < \alpha < 1$ is a parameter. Actually Kremer uses a variant of the Cobb-Douglas production function. Kremer further qualifies that variable V is normalized to one. The resultant equation for output looks as follows:

$$G = rTN^{\alpha}, \qquad (3)$$

where r and α are constants.

Further Kremer uses the Malthusian assumption, formulating it in the following way: "In this simplified model I assume that population adjusts instantaneously to N^*" [5, p. 685]. Value N^* in this model corresponds to population size, at which it produces equilibrium level of per capita income g^*, whereas "population increases above some steady state equilibrium level of per capita income, g^*, and decreases below it" [5, p. 685].

Thus, equilibrium level of population N^* equals

$$N^* = \left(\frac{g^*}{T}\right)^{\frac{1}{\alpha-1}}. \qquad (4)$$

Hence, the equation for population size is not actually dynamic. In Kremer's model dynamics is put into the equation for technological growth. Kremer uses the following assumption of the Endogenous Technological Growth theory [29–33]: "High population spurs technological change because it increases the number of potential inventors…. All else equal, each person's chance of inventing something is independent of population. Thus, in a larger population there will be proportionally more people lucky or smart enough to come up with new ideas" [5, pp. 685]; thus, "the growth rate of technology is proportional to total population" [5, pp. 682].

As this supposition, up to our knowledge, was first proposed by Simon Kuznets [29], we shall denote the respective type of dynamics as "Kuznetsian", whereas the systems where the "Kuznetsian" population-technological dynamics is combined with "Malthusian" demographic one will be denoted as "Malthusian-Kuznetsian".

This assumption is expressed mathematically by Kremer in the following way:

$$\frac{dT}{dt} : T = bN, \qquad (5)$$

where b is average innovating productivity per person.

Note that this implies that the dynamics of absolute technological growth rate can be described by the following equation:

$$\frac{dT}{dt} = bNT. \qquad (6)$$

Kremer further combines the research and population determination equations in the following way:

"Since population is limited by technology, the growth rate of population is proportional to the growth rate of technology. Since the growth rate of technology is proportional to the level of population, the growth rate of population must also be proportional to the level of population. To see this formally, take the logarithm of the population determination equation, [(4)], and differentiate with respect to time:

$$\frac{dN}{dt} : N = \frac{1}{1-\alpha}\left(\frac{dT}{dt} : T\right).$$

Substitute in the expression for the growth rate of technology from [(5)], to obtain

$$\frac{dN}{dt} : N = \frac{b}{1-\alpha} N \text{"} \quad (7)$$

[5, p. 686].

Note that multiplying both parts of equation (7) by N we get

$$\frac{dN}{dt} = aN^2, \quad (8)$$

where a equals

$$a = \frac{b}{1-\alpha}.$$

Of course, the same equation can be also written as

$$\frac{dN}{dt} = \frac{N^2}{C}, \quad (9)$$

where C equals

$$C = \frac{1-\alpha}{b},$$

whereas algebraic hyperbolic equation (1) is nothing else but the solution of differential equation (9).

Thus, Kremer's model produces precisely the same dynamics as the ones of von Foerster and Kapitza (and, consequently, it has just the same phenomenal fit with the observed data). However, it also provides a very convincing explanation WHY throughout most of the human history the absolute world population growth rate tended to be proportional to N^2. Within both models the growth of population from, say, 10 million to 100 million will result in the growth of dN/dt 100 times. However, von Foerster and Kapitza failed to explain convincingly why dN/dt tended to be proportional to N^2. Kremer's model explains this in what seems to us a rather convincing way (though Kremer himself does not appear to have spelled this out in a sufficiently clear way). The point is that the growth of the world population from 10 to 100 million implies that the human technology also grew approximately 10 times (as it turns out to be able to support a ten times larger population). On the other hand, the growth of population 10 times also implies 10-fold growth of the number of potential inventors, and, hence, 10-fold increase in the relative technological growth rate. Hence, the absolute technological growth will grow 10 x 10 = 100 times (in accordance to equation (6)). And as N tends to the technologically determined carrying capacity ceiling, we have all grounds to expect that dN/dt will also grow just 100 times.

III. World Dynamics as the World System Dynamics

Note that Kremer's model suggests ways to answer one of the main objections raised against the models of the world population hyperbolic growth. Indeed, by the moment the mathematical models of the world population hyperbolic growth have not been accepted by the social science academic community (The title of an article by a social scientist discussing Kapitza's model, "Demographic Adventures of a Physicist" [35], is rather telling in this respect). We believe there are substantial reasons for such a position, and the authors of the respective models are to blame for their rejection to no less extent than social scientists.

Indeed, all the respective models are based on an assumption that the world population can be treated as an integrated system for many centuries, if not millennia before 1492 (Actually, von Foerster, Mora, and Amiot [1] detected the hyperbolic growth pattern for 1–1958 CE; however, Kremer [5] suggests that it can be traced up to 1 million BCE, whereas Kapitza [7] insists that this pattern is even much more ancient). Already, von Foerster, Mora, and Amiot spelled out this assumption in a rather explicit way:

"However, what may be true for elements which, because of lack of adequate communication among each other, have to resort to a competitive, (almost) zero-sum multiperson game may be false for elements that possess a system of communication which enables them to form coalitions until all elements are so strongly linked that the population as a whole can be considered from a game-theoretical point of view as a single person playing a two-person game with nature as its opponent" [1, p. 1292].

However, did, e.g. in 1–1500 CE, the inhabitants of, say, Central Asia, Tasmania, Hawaii, Terra del Fuego, Kalahari etc. (that is, just the world population) really have such an "adequate communication", which made "all elements… so strongly linked that the population as a whole can be considered from a game-theoretical point of view as a single person playing a two-person game with nature as its opponent"? For any historically minded social scientist the answer to this question is perfectly clear. And, of course, this answer is squarely negative. Against this background it is hardly surprising that those social scientists who managed to get across the world population hyperbolic growth models had sufficient grounds to treat them just as "demographic adventures of physicists" (note, that indeed seven out of ten currently known authors of such models are just physicists), as none of the respective authors ([1], [5–10], [19], [20]) has provided any answer to the question above.

However, it is not so difficult to provide such an answer.

The hyperbolic trend observed for the world population growth after 10000 BCE appears to be mostly a product of the growth of the World System, which seems to have originated in the West Asia around that time in direct connection with the Neolithic Revolution (We are inclined to speak together with Frank [36], but not with Wallerstein [37] about the single World System, which originated long before the "long 16th century"). The presence of the hyperbolic trend indicates that the major part of the entity in question had some systemic unity, and the evidence for this unity is readily available. Indeed, we have evidence for the systematic spread of major innovations (domesticated cereals, cattle, sheep, goats, horses, plow, wheel, copper, bronze, and later iron technology, and so on) throughout the whole North African – Eurasian Oikumene for a few millennia BCE (see, e.g., [38] for a synthesis of such evidence). As a result the evolution of societies of this part of the world already at this time cannot be regarded as truly independent. By the end of the 1st millennium BCE we observe a belt of cultures stretching from the Atlantic to the Pacific with an astonishingly similar level of cultural complexity based on agriculture involving production of wheat and other specific cereals, cattle, sheep, goats, based on plow, iron metallurgy, wheeled transport, professional armies with rather similar weapons, cavalries, developed bureaucracies and Axial Age ideologies and so on – this list can be extended for pages). A few millennia before we would find a belt of societies with a similarly strikingly close level and character of cultural complexity stretching from the Balkans up to the Indus Valley outskirts – note that in both cases the respective entities included the major part of the contemporary world population (see, e.g., [39], [40]). We would interpret this as a tangible result of the World System functioning. The alternative explanations would involve a sort of miraculous scenario – that the cultures with a strikingly similar levels and character of complexity somehow developed independently from each other in a very large but continuous zone, whereas nothing like that appeared in the other parts of the world, which were not parts of the World System. We find such an alternative explanation highly implausible.

Thus, we would tend to treat the world population hyperbolic growth pattern as reflecting the growth of quite a real entity, the World System.

A few other points seem to be relevant here. Of course, there would be no grounds to speak about the World System stretching from the Atlantic to the Pacific even at the beginning of the 1st Millennium CE if we applied the "bulk-good" criterion suggested by Wallerstein [37], as there was no movement of bulk goods at all between, say, China and Europe at this time (as we have no grounds not to agree with Wallerstein in his classification of the 1st century Chinese silk reaching Europe as a luxury, rather than bulk good). However, the 1st century CE (and even the 1st millennium BCE) World System would be definitely qualified as such if we apply a "softer" information network criterion suggested by Chase-Dunn and Hall [41]. Note that at our level of analysis the presence of an information network covering the whole of World System is a perfectly sufficient condition, which makes it possible to consider this system as a single evolving entity. Yes, in the 1st millennium BCE any bulk goods could hardly penetrate from the Pacific coast of Eurasia to its Atlantic coast. However, the World System has reached by that time such a level of integration that the iron metallurgy could spread through the whole of the World System within a few centuries.

Yes, in the millennia preceding the European colonization of Tasmania its population dynamics – oscillating around 4000 level (*e.g.*, [38]) were not influenced by the World System population dynamics and did not influence it at all. However, such facts just suggest that since the 10th millennium BCE the dynamics of the world population reflects very closely just the dynamics of the World System population (see [42], [43] for more detail).

IV. A Compact Mathematical Model of the Economic and Demographic Development of the World System

Though Kremer's model provides a virtual explanation how the World System techno-economic development in connection with the demographic dynamics could lead to the hyperbolic population growth, Kremer did not specify his model to such an extent that it could also describe the economic development of the World System and that such a description could be tested empirically.

In fact, it appears possible to propose a very simple mathematical model describing both the economic and demographic development of the World System up to 1973 using the same assumptions as the ones employed by Kremer.

Kremer's analysis suggests the following relationship between per capita GDP and population growth rate (see Fig. 2):

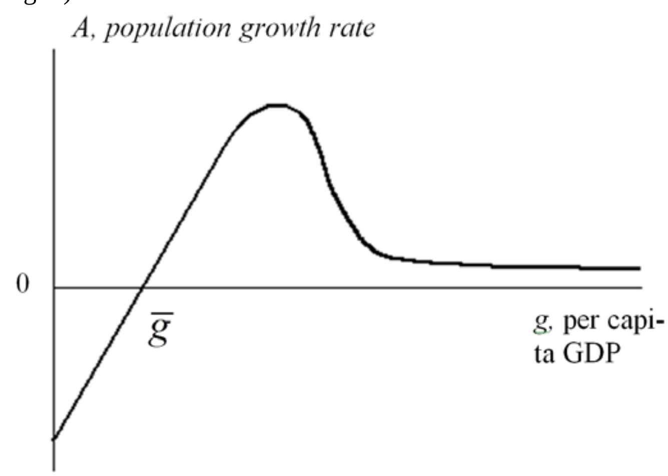

Fig. 2 Relationship between per capita GDP and population growth rate according to Kremer [5]

This suggests that for low per capita GDP range the influence of this variable dynamics on the population growth can be described with the following equation:

$$\frac{dN}{dt} = aSN\,, \tag{10}$$

where S is surplus, which is produced per person over the amount (m), which is minimally necessary to reproduce the population with a zero growth rate in a Malthusian system (thus, $S = g - m$, where g denotes per capita GDP).

As was already noted by Kremer [5, p. 694], in conjunction with equation (3) equation of (10) type "in the absence of technological change [that is if T = const] reduces to a purely Malthusian system, and produces behavior similar to the logistic curve biologists use to describe animal populations facing fixed resources" (actually, we would add to Kremer's biologists those social scientists who model pre-industrial economic-demographic cycles – see, *e.g.*, [44–49]).

Note that with a constant relative technological growth rate ($\frac{\dot{T}}{T} = r_T = const$) within this model (combining equations

(3) and (10)) we will have both constant relative population growth rate ($\frac{\dot{N}}{N} = r_N = const$), and thus the population will grow exponentially) and constant *S*. Note also that the higher value of r_T we take, the higher value of constant *S* we get.

Let us show this formally.

Take the following system:

$$G = rTN^{\alpha} \quad (3)$$

$$\frac{dN}{dt} = aSN \quad (10)$$

$$\frac{dT}{dt} = cT , \quad (11)$$

where $S = \frac{G}{N} - m$.

Equation (11) evidently gives

$$T = T_0 e^{ct} .$$

Thus,

$$G = rT_0 e^{ct} N^{\alpha}$$

and consequently

$$\frac{dN}{dt} = a\left(\frac{rT_0 e^{ct} N^{\alpha}}{N} - m\right)N = arT_0 e^{ct} N^{\alpha} - amN .$$

This equation is known as Bernoulli equation:

$\frac{dy}{dx} = f(x)y + g(x)y^{\alpha}$, which has the following solution:

$$y^{1-\alpha} = Ce^{F(x)} + (1-\alpha)e^{F(x)}\int e^{-F(x)}g(x)dx ,$$

where $F(x) = (1-\alpha)\int f(x)dx$, and *C* is constant.

In the case considered above, we have

$$N^{1-\alpha} = Ce^{F(t)} + (1-\alpha)e^{F(t)}\int e^{-F(t)}arT_0 e^{ct}dt ,$$

where $F(t) = (1-\alpha)\int(-am)dt = (\alpha-1)amt$.

So

$$N^{1-\alpha} = Ce^{(\alpha-1)amt} + (1-\alpha)arT_0 e^{(\alpha-1)amt}\int e^{-(\alpha-1)amt}e^{ct}dt$$

$$N^{1-\alpha} = e^{-(1-\alpha)amt}\left(C + (1-\alpha)arT_0\int e^{(c+(1-\alpha)am)t}dt\right)$$

$$N^{1-\alpha} = e^{-(1-\alpha)amt}\left(C + \frac{(1-\alpha)arT_0}{c+(1-\alpha)am}e^{(c+(1-\alpha)am)t}\right)$$

This result causes the following equation for *S*:

$$S = \frac{rT_0 e^{ct} N^{\alpha}}{N} - m = rT_0 e^{ct} N^{\alpha-1} - m =$$

$$= \frac{rT_0 e^{(c+(1-\alpha)am)t}}{C + \frac{(1-\alpha)arT_0}{c+(1-\alpha)am}e^{(c+(1-\alpha)am)t}} - m$$

$$S = \frac{1}{\frac{C}{rT_0}e^{-(c+(1-\alpha)am)t} + \frac{(1-\alpha)a}{c+(1-\alpha)am}} - m$$

Since $c > 0$ and $(1-\alpha) > 0$, it is clear that $c + (1-\alpha)am > 0$.

Consequently $e^{-(c+(1-\alpha)am)t} \to 0$ as $t \to \infty$.

This means that $S \xrightarrow[t\to\infty]{} \frac{c+(1-\alpha)am}{(1-\alpha)a} - m$, or finally

$S \to \frac{c}{(1-\alpha)a}$ as $t \to \infty$.

This, of course, suggests that within the growing “Malthusian” systems, *S* could be regarded as a rather sensitive indicator of the speed of technological growth. Indeed, within Malthusian systems in the absence of technological growth the demographic growth will lead to *S* tending to 0, whereas a long-term systematic production of *S* will be only possible with systematic technological growth.

Now replace $\frac{\dot{T}}{T} = r_T = const$ with Kremer's technological growth equation (6) and analyze the resultant model:

$$G = rTN^{\alpha} , \quad (3)$$

$$\frac{dN}{dt} = aSN , \quad (10)$$

$$\frac{dT}{dt} = bNT . \quad (6)$$

Within this model, quite predictably, *S* can be approximated as kr_T. On the other hand, within this model, by definition, r_T is directly proportional to *N*. Thus, the model generates an altogether not so self-evident (one could say even a bit unlikely) prediction – that throughout the “Malthusian-Kuznetsian” part of the human history the world per capita surplus production must have tended to be directly proportional to the world population size. This hypothesis, of course, deserves to be empirically tested. In fact, our tests have supported it.

Our test for the whole part of the human history, for which we have empirical estimates for both the world population and the world GDP (that is for 1–2002 CE) has produced the following results: R^2 = 0.98, p << 0.0001, whereas for the period with the most pronounced “Malthusian-Kuznetsian” dynamics (1820–1958) the positive correlation between the two variables is almost perfect (see Fig. 3):

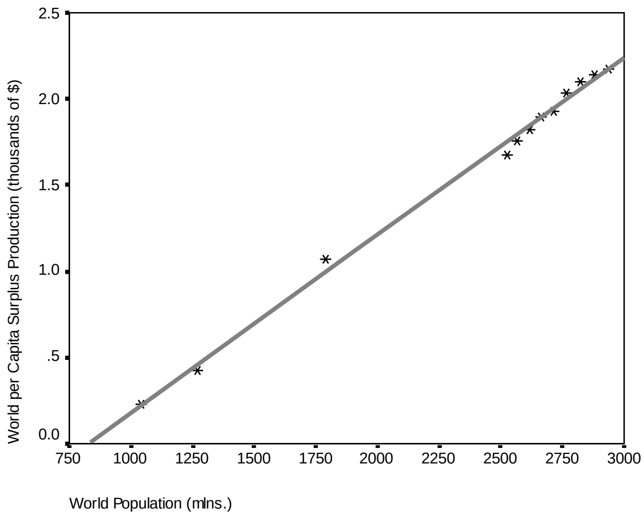

Fig. 3 Correlation between world population and per capita surplus production (1820–1958). $R^2 > 0.996$, $p << 0.0001$. data source – [4].

Note that as within a Malthusian-Kuznetsian system S can be approximated as kN, equation (10) may be appoximated as $dN/dt \sim k_1N^2$, or, of course, as $dN/dt \sim N^2 / C$; thus, Kapitza's equation turns out to be a by-product of the model under consideration.

Thus, we arrive at the following:

$$S \sim k_1 r_T ,$$

$$r_T = k_2 N .$$

Hence, $dS/dt \sim kr_T /dt = k_3 dN/dt$. This implies that for the "Malthusian-Kuznetsian" part of human history dS/dt can be approximates as $k_4 dN/dt$.

On the other hand, as dN/dt in the original model equals aSN, this, of course, suggests that for the respective part of the human history both the economic and demographic World System dynamics may be approximated by the following unlikely simple mathematical model:

$$\frac{dN}{dt} = aSN , \qquad (10)$$

$$\frac{dS}{dt} = bNS , \qquad (12)$$

where N is the world population, and S is surplus, which is produced per person with the given level of technology over the amount, which is minimally necessary to reproduce the population with a zero growth rate.

The world GDP is computed using the following equation:

$$G = mN + SN, \qquad (13)$$

where m denotes the amount of per capita GDP, which is minimally necessary to reproduce the population with a zero growth rate, and S denotes "surplus" produced per capita over m at the given level of the world-system techno-economic development (Note that this model only describes the Malthusian-Kuznetsian World System in a dynamically balanced state (when the observed world population is in a balanced correspondence with the observed technological level). To describe the situations with N disproportionally low or high for the given level of technology (and, hence, disproportionally high or low S) one would need, of course, the unapproximated version of the model ((3) – (10) – (6)). Note, that in such cases N will either grow, or decline up to the dynamic equilibrium level, after which the developmental trajectory will follow the line described by the (10) – (12) model).

Note that this model does not contain any variables, for which we do not have empirical data (at least for 1–1973) and, thus, a full empirical test for this model turns out to be perfectly possible (This refers particularly to the long-range data on the level of world technological development (T), which do not appear to be available till now).

Incidentally, this model implies that the absolute rate of the world population growth (dN/dt) should have been roughly proportional to the absolute rate of the increase in the world per capita surplus production (dS/dt), and, thus (assuming the value of necessary product to be constant) to the absolute rate of the world per capita GDP growth, with which dS/dt will be measured thereafter. Note that among other things this could help us to determine the proportion between coefficients a and b. Thus, if the model suggested by us has some correspondence to reality, one has grounds to expect that in the "Malthusian-Kuznetsian" period of the human history the absolute world population growth rate (dN/dt) was directly proportional to the absolute growth rate of the world per capita surplus production (dS/dt). For the correlation between these two variables see Fig. 4 below. Regression analysis of this dataset has given results displayed in Table I. Note that the constant in this case is very small within the data scale, statistically insignificant, and lies within the standard error from 0, which makes it possible to equate it with 0. In this case regression analysis gives results displayed in Table II.

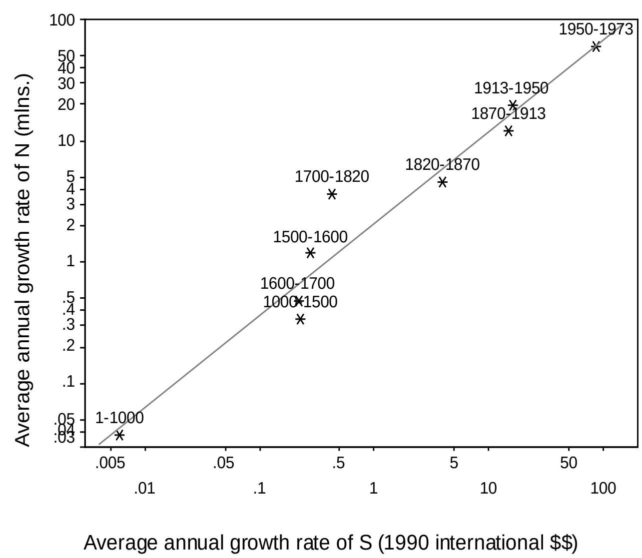

Fig. 4 Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (S, 1990 PPP international dollars) and Average Annual Absolute World Population (N) Growth Rate (1 – 1973 CE), scatterplot in double logarithmic scale with a regression line. Data source – [4].

Table I
Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (*S*, 1990 PPP international dollars) and Average Annual Absolute World Population (N) Growth Rate (1 – 1950 CE), (regression analysis)

	Unstandardized Coefficients		**Standardized Coefficients**	**t**	**Sig.**
Model	**B**	**Std. Error**	**Beta**		
(Constant)	0.820	0.935		0.876	0.414
World Absolute Growth Rate of Per Capita Surplus Production	0.981	0.118	0.959	8.315	<0.001
Dependent variable: Average Annual Absolute World Population (*N*) Growth Rate (mlns. a year)					

NOTE: $R = 0.96$, $R^2 = 0.92$.

Table II
Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (*S*) and Average Annual Absolute World Population (*N*) Growth Rate (1 – 1950 CE), regression analysis, not including constant in equation

	Unstandardized Coefficients		**Standardized Coefficients**	**t**	**Sig.**
Model	**B**	**Std. Error**	**Beta**		
World Absolute Growth Rate of Per Capita Surplus Production	1.04	0.095	0.972	10.94	<0.001
Dependent variable: Average Annual Absolute World Population (*N*) Growth Rate (mlns. a year)					

NOTE: $R = 0.97$, $R^2 = 0.945$.

Thus, just as implied by our second model, in the "Malthusian" period of the human history we do observe a strong linear relationship between the annual absolute world population growth rates (*dN/dt*) and the annual absolute growth rates of per capita surplus production (*dS/dt*). This relationship can be described mathematically with the following equation:

$$\frac{dN}{dt} = 1.04\frac{dS}{dt},$$

where *N* is the world population (in millions), and *S* is surplus (in 1990 PPP international dollars), which is produced per person with the given level of technology over the amount, which is minimally necessary to reproduce the population with a zero growth rate.

Note that according to model (10)-(12),

$$\frac{dN}{dt} = \frac{a}{b}\frac{dS}{dt}.$$

Thus, we get a possibility to express coefficient *b* through coefficient *a*:

$$\frac{a}{b} = 1.04,$$

consequently:

$$b = \frac{a}{1.04} = 0.96a.$$

As a result, for the period under consideration it appears possible to simplify the second compact macromodel, leaving in it just one free coefficient:

$$\frac{dN}{dt} = aSN, \quad (10)$$

$$\frac{dS}{dt} = 0.96aNS, \quad (14)$$

With our two-equation model we start the simulation in the year 1 CE and do annual iterations with difference equations derived from the differential ones:

$$N_{i+1} = N_i + aS_iN_i,$$
$$S_{i+1} = S_i + 0.96aN_iS_i.$$

The world GDP is calculated using equation (13).

We choose the following values of the constants and initial conditions in accordance with historical estimates of Maddison [4]: $N_0 = 230.82$ (in millions); $a = 0.000011383$; $S_0 = 4.225$ (in International 1990 PPP dollars). (The value of S_0 was calculated with equation $S = G/N - m$ on the basis of Maddison's [4] estimates for the year 1 CE. He estimates the world population in this year as 230.82 million, the world GDP as \$102.536 billion (in 1990 PPP international dollars), and hence, the world per capita GDP production as \$444.225 Maddison estimates the subsistence level per capita annual GDP production as \$400 [4, pp. 260, 264]. However, already by 1 CE most population of the world lived in rather complex societies, where the population reproduction even at zero level still required considerable production over subsistence level to maintain various infrastructures (transportation, legal, security, administrative and other subsystems etc.), without which even the simple reproduction of complex societies is impossible almost by definition. Note that the fall of per capita production in complex agrarian societies to subsistence level tended to lead to state breakdowns and demographic collapses (see, e.g., [44–49], [52]). The per capita production to support the above mentioned infrastructures could hardly be lower than 10% of the subsistence level – that is close to Maddison's [4, pp. 259–260) of estimates, which makes it possible to estimate the value of *m* as \$440, and hence, the value of S_0 as \$4.225).

The outcome of the simulation, presented in Fig. 5 indicates that the compact macromodel in question is actually capable of replicating quite reasonably the world GDP estimates of Maddison [4]. The correlation between the predicted and observed values for this simulation looks as follows: $r > 0.999$; $R^2 = 0.9986$; $p << 0.0001$. For the world population these characteristics are also very high: $r = 0.996$; $R^2 = 0.992$; $p << 0.0001$.

According both to our model and the observed data up to the early 1970s we deal with the hyperbolic growth of not only the world population (N), but also per capita surplus production (S) (see Fig. 6). Note that even if S had not been growing, remaining constant, the world GDP would have been growing hyperbolically anyway through the hyperbolic growth of the world population only. However, the hyperbolic growth of S observed during this period of the human history led to the fact that the world population growth correlated with the world GDP growth not lineally, but quadratically (see Fig. 7). Indeed, the regression analysis we have performed has shown here an almost perfect ($R^2 = 0.998$) fit just with the quadratic model (see Fig. 8).

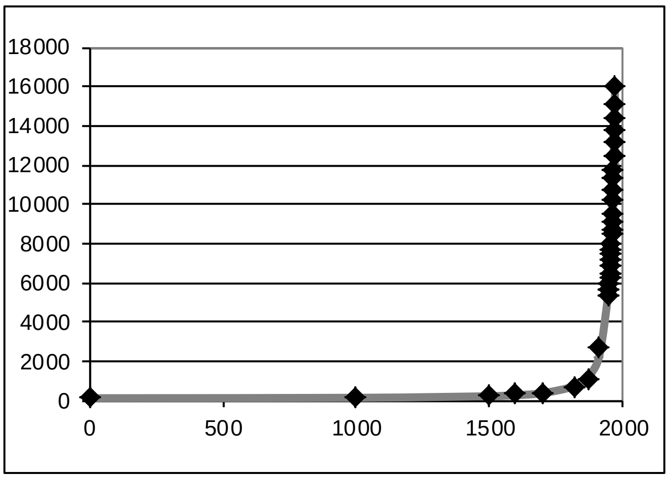

Fig. 5 Predicted and Observed Dynamics of the World GDP Growth, in billions of 1990 PPP international dollars (1 – 1973 CE). The solid grey curve has been generated by the model; black markers correspond to the estimates of world GDP by Maddison [4].

As a result the overall dynamics of the world GDP up to 1973 was not even hyperbolic, but rather quadratic-hyperbolic, leaving far behind the rather impressive hyperbolic dynamics of the world population growth (see Fig. 1 above).

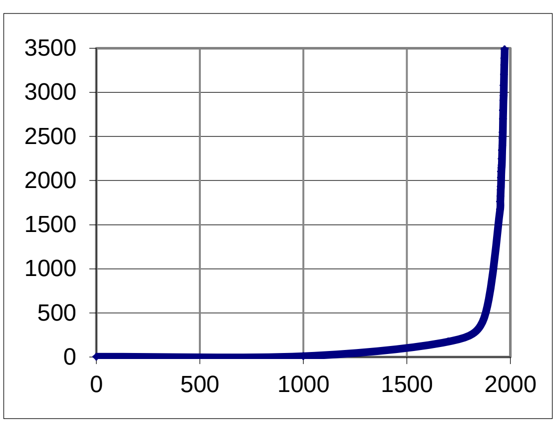

Fig. 6 Hyperbolic Growth of the World Per Capita Surplus Production, in 1990 PPP international dollars (1 – 1973 CE). Data source – [4]

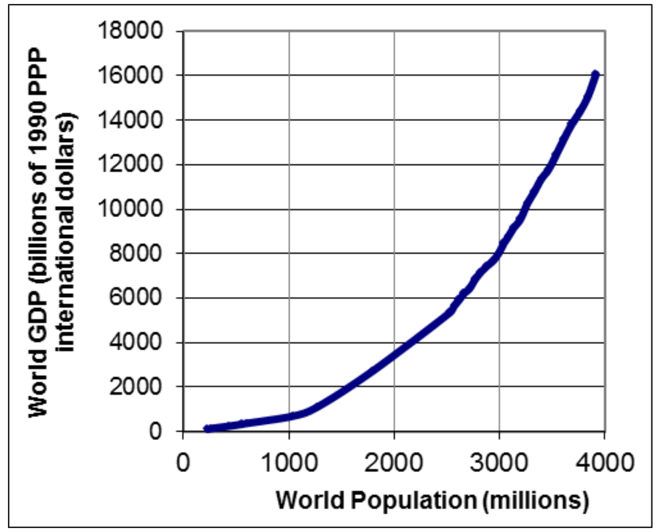

Fig. 7 Correlation between Dynamics of the World Population and GDP Growth (1 – 1973 CE). Data source – [4]Maddison 2001

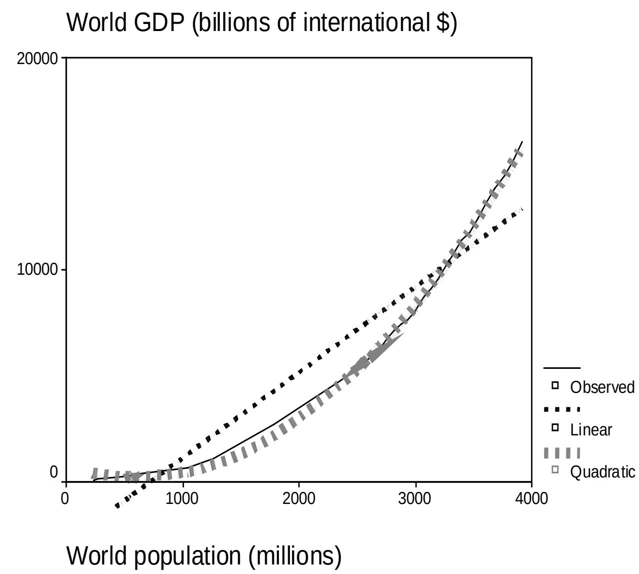

Fig. 8. Correlation between Dynamics of the World Population and GDP Growth (1 – 1973 CE): curve estimations

LINEAR REGRESSION: R2 = .876, p < .001

QUADRATIC REGRESSION: R2 = .998, p < .001

Note also that we have already mentioned that as has been already shown by von Foerster, von Hoerner and Kapitza the world population growth before the 1970s is very well approximated by the following equation:

$$N = \frac{C}{t_0 - t}. \tag{1}$$

As according to the model under consideration S can be approximated as kN, its long term dynamics can be approximated with the following equation:

$$S = \frac{kC}{t_0 - t}. \tag{15}$$

Hence, the long-term dynamics of the most dynamical part of the world GDP, SN, the world surplus product, can be approximated as follows:

$$SN = \frac{kC^2}{(t_0 - t)^2}\,. \tag{16}$$

As we could see at the beginning of this article (see Fig. 1), this approximation does work rather well indeed.

V. Conclusion

Human society is a complex nonequilibrium system that changes and develops constantly. Complexity, multivariability, and contradictoriness of social evolution lead researchers to a logical conclusion that any simplification, reduction, or neglect of the manifolds of factors leads inevitably to multiplication of error and to significantly erroneous understanding of processes under study. The view that any simple general laws are not observed at all with respect to social evolution has become totally predominant within the academic community, especially among those who specialize in the Humanities and who confront directly in their research all the manifolds and unpredictability of social processes. A way to approach human society as an extremely complex system is to recognize differences of abstraction and time scale between different levels. If the main task of scientific analysis is to detect the main acting forces so as to discover fundamental laws at a sufficiently coarse scale, abstracting from details and deviations from general rules at that level, then understanding at that level may help to identify measurable deviations from these laws in finer detail and faster time scales, not as a reductionism but contributing to measurement of deviations that are significant in their own right at finer and faster scales. Modern achievements in the field of mathematical modeling suggest that social evolution can be described with rigorous and sufficiently simple macrolaws.

As is well known in complexity studies – chaotic dynamics at the microlevel can generate a highly deterministic macrolevel behavior [51]. To describe behavior of a few gas molecules in a closed vessel we need very complex mathematical models, which will still be unable to predict long-run dynamics of such a system due to inevitable irreducible chaotic component. However, the behavior of zillions of gas molecules can be described with extremely simple sets of equations, which are capable of predicting almost perfectly the macrodynamics of all the basic parameters (and just because of chaotic behavior at microlevel). Of course, one cannot fail to wonder whether a similar set of regularities is not observed in the human world too, whether very simple regularities accounting for extremely high proportions of all the macrovariation cannot be found just for the largest possible social system – the World System.

Indeed, as we could see, the extremely simple mathematical models specified above can account for 99.2–99.91 per cent of all the variation in economic and demographic macrodynamics of the world for almost two millennia of its history.

In fact, this appears to suggest a novel approach to the formation of the general theory of social macroevolution. The approach prevalent in classical social evolutionism was based on an apparently self-evident assumption that evolutionary regularities of simple systems are significantly simpler than the ones characteristic for complex systems. A rather logical outcome from this almost self-evident assumption is that one should study first evolutionary regularities of simple systems and only after understanding them to move to more complex ones. (Of course, a major exception here is constituted by the world-system approach [*e.g.*, [36], [37], [41]), but the research of world-system students has by now yielded somehow limited results, to a significant extent because they have not used sufficiently standard scientific methods implying that verbal constructions should be converted into mathematical models, whose predictions are to be tested with available data). One wonders if the opposite direction might not be more productive – from the study of simple laws of the development of the most complex social system to the study of the complex regularities of evolution of simple social systems (see also).

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