Visualization Algorithms for the Steady State Sets of a Class of Singularly Disturbed Nonlinear Control Dynamical Systems

Byungik Kahng, Mathew Gomez and Eduardo Padilla

***Abstract*—** It is known that a multiple valued iterative dynamical system (MVIDS) can be used to model a nonlinear disturbed control dynamical systems (DCDS). When the system is subject to large sudden disturbance, which we call the singular disturbance, its maximal invariant set often exhibits a fractal structure. This paper focuses upon the visualization algorithm of the resulting invariant fractal. It discusses the nature of the multiple valuedness of the iterative dynamics in an algorithmic viewpoint. The relevant source codes of our visualization programs will be analyzed.

***Keywords*—** Multiple Valued Iterative Dynamics Model, Nonlinear Disturbed Control Dynamical System, Maximal Invariance, Invariant Fractal

I. INTRODUCTION

THE characterization of the maximal invariant sets and its controllability is a classical topic of control and automation theory, as evidenced by a considerable amount of literature that goes back to the early 1970s. See, for instance, [2] and [3] for a through survey and historical notes. This classic topic, which had once been considered obsolete, is attracting a renewed attention, these days. Some of the modern literature that are directly related to the scope of our research includes, but not restricted to, [1], [5], [22], [23], [24], [27], [26], [29], [28], [30], [31], excluding the author's contributions.

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This paper is based partly upon the undergraduate research by the second author [4] and that of the third author [25], both of which were supervised by the first author. The authors were supported in part by National Research Experience for Undergraduates Program (NREUP), administered by Mathematical Association of America (MAA). Support for this MAA program is jointly provided by the National Science Foundation (Grant DMS-1359016) and the National Security Agency (Grant H98230-15-1-0020).

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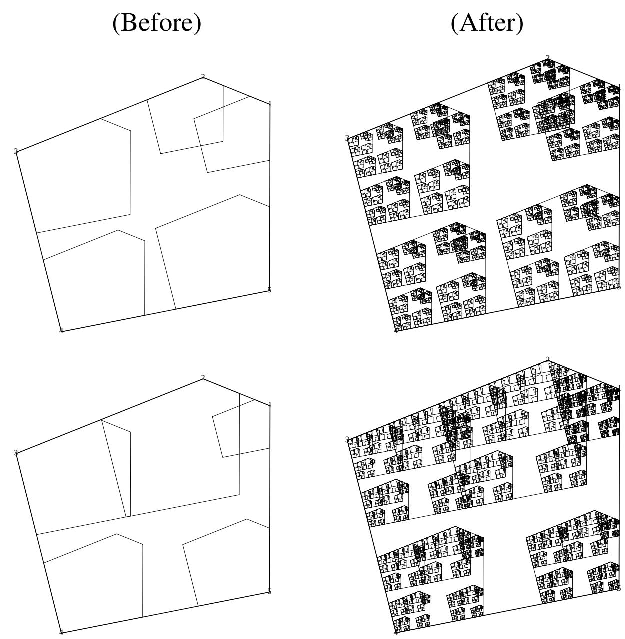

Fig. I.1. Examples of the Maximal Invariant Sets of the MVID Models of Singularly Disturbed Nonlinear Control Dynamical Systems from [16]

A part of the reason is that the recent advancement of computational speed and capacity "made it possible to implement the algorithms for systems of particular interest," as pointed out in [22]. There is another reason, however, that is more directly related to the present paper. The classical adjustments, which will be reviewed briefly in Section II, were devised mostly for the small disturbances that do not affect the qualitative behavior of the system. The bifurcation of the control dynamical systems from sudden large disturbances, on the other hand, requires a fundamentally different approach.

Such sudden disturbance and the resulting bifurcation mentioned in the previous paragraph are common in the application to non-linear physics [15], [14], [18] and also in digital signal processing [7], [8], [10], [9]. It is not out of ordinary, therefore, to consider this line of development in control and automation theory as well.

One way to model the singularly disturbed control dynamical systems is through the iterative dynamics of multiple valued maps. See [11], [12], [13], [16], [17], [6], [21] for the modeling mechanism that we adopt for this paper. See, also, [1], [26] for similar but different approaches. It turns out

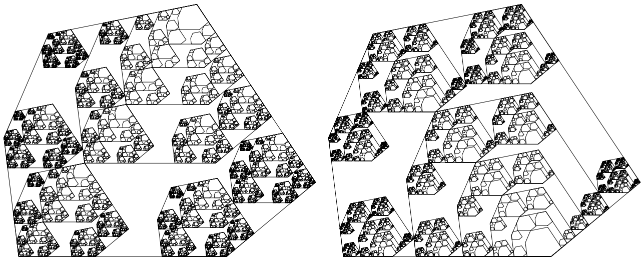

Fig. I.2. Semi-optimal Examples of the Maximal Invariant Sets of the MVID Models of Singularly Disturbed Nonlinear Control Dynamical Systems under a Non–interference Condition from [20]

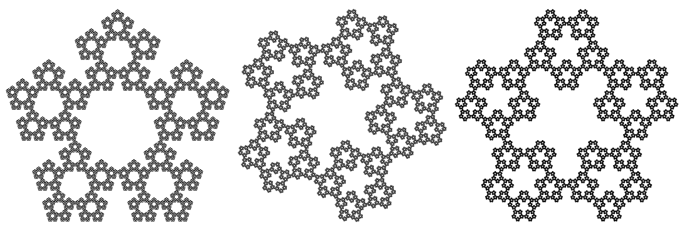

Fig. I.3. Optimal Examples of the Maximal Invariant Sets of the MVID Models of Singularly Disturbed Nonlinear Control Dynamical Systems under a Non–interference Condition from[6]

that the multiple valued iterative dynamics modeling (MVID Modeling) of disturbed control dynamical systems (DCDS) is particularly useful when the disturbance includes singularity. In this case, the steady state set, or the maximal invariant set often exhibits a fractal structure, as exemplified by Figure I.1 – Figure I.3.

The maximal invariance in the MVID modeling of DCDS was studied in a variety of viewpoints by the first author. [11], [12], [21] focused upon the controllability problems. [13], [17] dealt with the strong and weak maximal invariance. [16] was on the computational algorithm. [6] studied an optimization problem. He did not, however, take a close look at the invariant fractal itself. We plan to focus upon this aspect, through the present paper.

In another angle, this paper is closely related to the authors' pure-math papers, [19], [20], in a sense that they are all about the fractal structure. Indeed, this paper will fill in the computational aspect and compensate the applied mathematical side, left out by [19], [20]. The algorithmic aspect will have something to do with [16], but [16] was strictly on the MVID modeling and the maximal invariance, not on the fractal structure.

This paper is structured as follows. In Section II, we provide a brief overview of MVID modeling of DCDS, along with its application to control and automation theory. In Section III, we discuss how the inner and the outer Sierpinski fractals arise as the two bounding optimal solutions of a class of optimal control problems in automatic control theory. Section IV and Section V are the main sections. In these two sections, we analyze the source codes of our programs, and discuss how the MVID modeling is embedded into the programs and how effectively it works. We back up our claims with extensive numerical explorations and visualizations. For the programming, we use *Mathematica*, which can handle both the symbolic manipulation and numerical computation, with sufficient dexterity for our purpose. The complete programs are too long to be included in this paper. Interested readers are invited to contact the corresponding author.

II. Multiple Valued Iterative Dynamics Modeling

A classical model of discrete-time non-linear control dynamical system goes as follows.

$$\begin{cases} F : (x_k, u_k) \mapsto x_{k+1}, \\ G : x_k \mapsto u_k. \end{cases} \tag{II.1}$$

Here, the map G is called the *feedback control law*. In the presence of disturbance, one can extend the above model as follows by putting disturbance variables as follows, resulting in a model of a non-linear disturbed control dynamical system (DCDS).

$$\begin{cases} F : (x_k, u_k, w_k) \mapsto x_{k+1}, \\ G : (x_k, v_k) \mapsto u_k. \end{cases} \tag{II.2}$$

Here, v_k and w_k are the disturbance variables.

As explained in [13], [17], the models (II.1) and its extension (II.2) are problematic in that there are too many unknown and unknowable variables to solve the resulting functional equations. If they could have been determined, they could not have been the models of disturbance to begin with. One of the modern approaches to overcome this difficulty is the multiple valued iterative dynamics model (MVID model), which we define as follows.

Definition II.1 (Multiple Valued Iterative Dynamics Model [17])**.** *Let X, Y be non-empty sets, and $\mathscr{P}(X)$, $\mathscr{P}(Y)$ be their power sets. We say a set function $f : \mathscr{P}(X) \to \mathscr{P}(Y)$ is a* **multiple valued map (function) from** X **to** Y *if*

$$f(S) = \bigcup \{ f(x) : x \in S \}, \tag{II.3}$$

for all $S \subset X$. Here, $f(x)$ is the abbreviation of $f(\{x\})$. In particular, if $X = Y$, we call the dynamical system on X given by the iteration of f in $\mathscr{P}(X)$, the **multiple valued iterative dynamical system (MVIDS)**. *If an MVIDS was used to model a disturbed control dynamical system, we call such a model, a* **multiple valued iterative dynamics model (MVID Model)**.

Roughly speaking, the MVID modeling is a generalization of the classical model (II.1) through the iterative dynamics of $f : x_k \mapsto F(x_k, G(x_k))$, except that f is allowed to take multiple values so that it covers all possible outcomes due to the disturbance. It turns out that this approach is particularly useful, if we include sudden large disturbances, which we call, **singular disturbances**.

The bifurcation of the qualitative behavior of the dynamics due to the singularity is, in fact, common in nonlinear physics. See, for instance, [15], [14], [18] for the modeling most directly related to this paper.

One of the most important topic of a non-linear DCDS modeling is the maximal invariance. In order to get a *closed-loop system* [23] that allows an automatic control system run automatically, one must begin the dynamics in the maximal quasi-invariant set [22], [23]. On the other end of the extreme, every closed-loop automatic control dynamics ends up with the steady steady set, which happens to be the maximal full-invariant set [21]. The MVID case is somewhat more complicated. We must consider additional optimal bounds, which [13], [17] call *strong* and *weak*. See [17] for the complete characterization and classification.

III. Sierpinski Fractals as Optimal Solutions

THE previous section concluded with a comment on an optimization problem in a non-linear DCDS modeling. More specifically, the maximal invariance. Let us continue this discussion with a further optimization. As pointed out in [6], a further optimization of the strength of the controllers in the phase space must satisfy a certain symmetry condition. In 2-dimension, this optimization takes a form of Sierpinski fractals. The first and the third pictures of Figure I.3 exemplify this situation. Moreover, one can explicitly calculate the formulation of the minimal Lyapunov multipliers as the quantification of the strength of the controllers, as in the farthest right hand side of the inequality (III.1). See [6] for detail. See also, [19], [20] for the theoretical background of the computation in [6].

The minimal Lyapunov multiplier formula for the 2-dimensional case works for higher dimension too, because there are more room in higher dimension to spread out the controllers, away from the mutual interference. Even though the lower bound for the 2-dimensional case is not very good in general, and more improvement will have to be sought after for future research, this is the best we have so far [6].

$$\bar{r} = \min(\max\{r_1, \cdots, r_N\}) \geq \frac{1}{2}\left(1 - \frac{\tan\left(\frac{\pi}{N}\left\lfloor\frac{N-1}{4}\right\rfloor\right)}{\tan\left(\frac{\pi}{N} + \frac{\pi}{N}\left\lfloor\frac{N-1}{4}\right\rfloor\right)}\right). \quad \text{(III.1)}$$

IV. An Analysis of an MVID Algorithm I

IN the previous section, we saw that the solution of an optimization problem for the strength of the controllers measured by Lyapunov multiplier corresponds to the inner and the outer Sierpinski fractals. The first and the third pictures of Figure I.3 exemplify these. In this section, we will go over the definitions of the Sierpinski fractals in accordance to the MIVD modeling, and analyze the corresponding source codes of our programs. First, here is the definition of the inner Sierpinski fractal, as it appears in [19].

Definition IV.1 (Inner Sierpinski Fractal from [19])**.** *Let P_N be a convex N-gon in the Euclidean plane $\mathbb{R}^2$, with the vertices $v_1, \cdots, v_N \in \mathbb{R}^2$. Let $\mathscr{P}(\cdot)$ denote the power set. Now, let $f : \mathscr{P}(\mathbb{R}^2) \to \mathscr{P}(\mathbb{R}^2)$ be a set function in $\mathbb{R}^2$ given by*

$$f(S) = f_1(S) \cup \cdots \cup f_N(S), \quad \text{(IV.1)}$$
$$f_i(x) = r_i(x - v_i) + v_i \qquad 0 < r_i < 1.$$

Suppose further that the each distinct pair of $f_i(P_N)$ and $f_j(P_N)$ are **non-overlapping***, that is, each intersection,*

```
tilt = nn/2 - Floor[nn/2] - 1/2;
vv = Table[{N[Sin[(2 Pi (ii + tilt))/nn]],
    N[Cos[(2 Pi (ii + tilt))/nn]]}, {ii, 0, nn - 1}];
pp = {vv};
ff[xx_] :=
 Table[
  Table[
   N[rmax[nn]] (xx[[ii]][[jj]] - vv[[kk]]) +
    vv[[kk]], {kk, 1, nn}, {jj, 1, nn}],
  {ii, 1, Length[xx]}];
```

Fig. IV.1. A Screen Capture of A Part of Our Mathematica Program for Inner Sierpinski Fractals I

$f_i(P_N) \cap f_j(P_N)$, $i \neq j$, has empty interior. Then, we will call the set,

$$\mathcal{S}_{(r_1,\cdots,r_N)}(P_N) = \bigcap_{k=0}^{\infty} f^k(P_N), \quad \text{(IV.2)}$$

the **inner Sierpinski fractal** *of P_N with the* **contraction ratios** *$r_1, \cdots, r_N$. We will say a Sierpinski fractal is* **uniform** *if $r_1 = \cdots = r_N = r$, and we abbreviate it as,*

$$\mathcal{S}_r(P_N) = \mathcal{S}_{(r,\cdots,r)}(P_N).$$

We will say a uniform inner Sierpinski fractal $\mathcal{S}_r(P_N)$ is **maximal** *if the the common contraction ratio r is the largest for given P_N. Finally, an inner Sierpinski fractal is called* **regular** *if P_N is a regular polygon.*

Note that Definition IV.1 is based upon the multiple valued map f defined by the equality (IV.1) and its forward iteration given by the equality (IV.2). Using the **Table** command of *Mathematica*, one can express the multiple valued map f that consists of N contractions centered at the vertices of a convex N-gon, in a couple of simple commands as shown in Figure IV.1.

Figure IV.1 is a screen capture of a part of our Mathematica program that produced the Figure I.1, Figure I.2, Figure IV.4 and Figure IV.5. The vertices of the original polygon are defined as **vv**, using the **Table** command and the index **ii**. This particular program begin from a regular N-gon, with sine and cosine functions, but the vertices can be assigned differently to generate asymmetric cases as in Figure I.2.

The code **ff[xx_]** at the end of Figure IV.1 defines the multiple valued map, using two **Table** commands and three indices, (**kk**, **jj** and **ii**). The data-size increases exponentially as the iteration of **ff[xx]** is taken, consequently generating the intricate structure of the inner Sierpinski fractals. The iteration is controlled by the second portion of the program, the screen capture of which is presented in Figure IV.2.

Note that the part of our program captured in Figure IV.2 includes the iteration of the multiple valued map (**qq** = **ff[pp]**), and that the iteration continues as long as the number of points

```
If[Length[pp] < resolution,
  qq = ff[pp];
  Sierpinski = Graphics[{
      Table[{White, Opacity[0],
        EdgeForm[Black],
        Polygon[qq[[ii]][[kk]]]},
       {kk, 1, nn}, {ii, 1, Length[qq]}]
     }];
  pp = Flatten[qq, 1];
  iterationNumber = iterationNumber + 1;
 ];
```

Fig. IV.2. A Screen Capture of A Part of Our Mathematica Program for Inner Sierpinski Fractals II

```
Clear[nn, rr, vv, ii, jj, kk, pp, qq, ff,
 xx, iterationNumber, n]
nn = 9;
resolution = 1500;  margin = 0.125;
rmax[n_] = 1/2 (1 - Tan[π/n Floor[(n-1)/4]] / Tan[π/n + π/n Floor[(n-1)/4]]);
```

Fig. IV.3. A Screen Capture of A Part of Our Mathematica Program for Inner Sierpinski Fractals III

in consideration is under a pre-set figure that corresponds to the resolution of the resulting fractal (**Length[pp]** < **resolution**). This number and other numbers that affects the final outcome are determined in the portion of the program included in Figure IV.3.

Note that the final line of the Figure IV.3 corresponds to the optimal Lyapunov multiplier included in the inequality (III.1). It helps creating the optimal solutions exemplified in Figure IV.4 and Figure IV.5. These figures show only the optimal solutions that began from regular N-gons. Starting from more general convex polygons and setting non-uniform contraction coefficients, one can get the invariant fractals like those in Figure I.1 and Figure I.2, too.

V. An Analysis of an MVID Algorithm II

As depicted in Figure I.3, the optimal solutions of the maximal invariant sets can take different final shapes depending upon the rotation component. More over, the most optimal cases in which we can obtain the minimal Lyapunov multiplier turn out to be the rotation-free case (0° rotation) and the maximal rotation case (180° rotation). See [6] for detail. The rotation-free case yields the usual (inner) Sierpinski fractals that we discussed in the previous section. The other case turns out to be a class of the outer Sierpinski fractals, which we define as follows.

Definition V.1 (Outer Sierpinski Fractal, the First Generation). *Let P_N be a convex N-gon in the Euclidean plane $\mathbb{R}^2$, with*

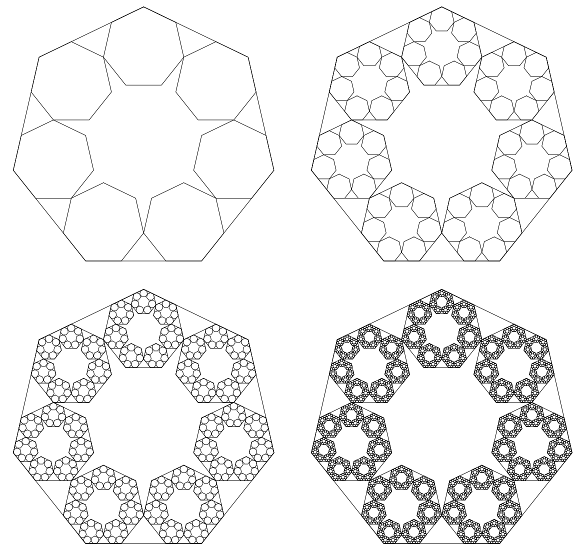

Fig. IV.4. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling I

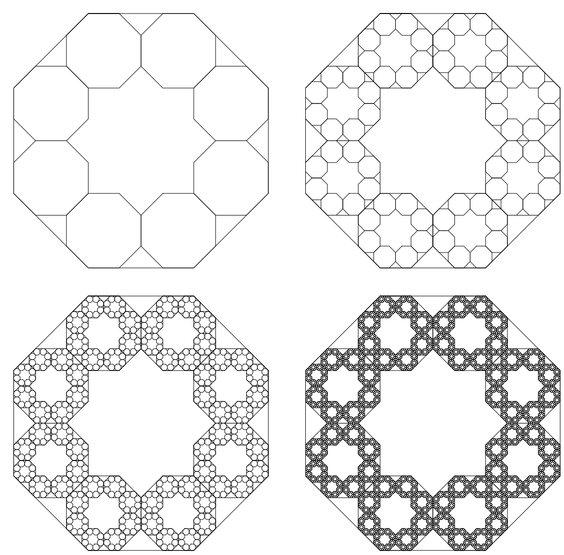

Fig. IV.5. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling II

the vertices $v_1, \cdots, v_N$, and let $r_1, \cdots, r_N$ be real numbers between 0 and 1, which we call the **contraction ratios**. *Let n^* be an integer such that $0 \leq n^* < N/2$, which we call the* **ingrowth number**. *For each $i \in \{1, \cdots, N\}$, we define the vertices, $v^1(i,1), \cdots, v^1(i,N)$, of the first generation of the fractal growth, by*

$$v^1(i,j) = -r_i(v_j - v_i) + v_i, \quad j \in \{1, \cdots, N\}. \tag{V.1}$$

We define the 1st generation outer Sierpinski polygon $P^1(i)$ and its boundary $S^1(i)$ as follows.

$$\begin{cases} P^1(0,i) = K[v^1(i,1), \cdots, v^1(i,N)], \\ S^1(0,i) = \partial P^1(0,i). \end{cases} \tag{V.2}$$

Here, $K(\cdots)$ denotes the convex hull, and $\partial(\cdot)$ denote the *boundary. Also, let us regard the original polygon as the 0th generation and denote* $P^0(0) = P_N$ *and* $S^0(0) = \partial P_N$*.*

Note that the equality (V.1) includes not only the contraction with the contraction coefficient r_i, but also the 180° rotation. In contrast to the MVID algorithm of the previous section, however, constructing the appropriate MVID for this case is not as simple as putting together all contraction maps (V.1). We need to take a couple more steps as follows.

Definition V.2 (Outer Sierpinski Fractal, the In-growth Condition)**.** *Continuing from Definition V.1, we define the vertices of* k*-th generation of the fractal growth for* $k \geq 2$*, through the following recursive process: For each* $j \in \{1, \cdots, N\}$*,*

$$\begin{aligned} &v^k(i_1, \cdots, i_{k-1}, i_k, j) \\ &\quad = -r_{i_k}\left[v^{k-1}(i_1, \cdots, i_{k-1}, j) - v^{k-1}(i_1, \cdots, i_{k-1}, i_k)\right] \\ &\qquad + v^{k-1}(i_1, \cdots, i_{k-1}, i_k), \end{aligned} \tag{V.3}$$

for all i_k *values that satisfies the following condition, which we call the* **in-growth condition***.*

$$|i_{k-1} - i_k| \geq n^*. \tag{V.4}$$

Their convex hull is denoted as follows.

$$\begin{aligned} &P^k(0, i_1, \cdots, i_k) \\ &\quad = K[v^k(i_1, \cdots, i_k, 1), \cdots, v^k(i_1, \cdots, i_k, N)]. \end{aligned}$$

Finally, their boundary is denoted as,

$$S^k(0, i_1, \cdots, i_k) = \partial P^k(0, i_1, \cdots, i_k).$$

Definition V.3 (Outer Sierpinski Fractal, the Conclusion)**.** *Continuing from Definition V.2, we define the* **outer Sierpinski fractal** *with the in-growth number* n^* *as follows.*

$$\mathcal{S}^{n^*}_{(r_1, \cdots, r_N)}(P_N) = \bigcup \{S^k(0, i_1, \cdots, i_k) : \forall (0, i_1, \cdots, i_k)\}.$$

Also, we define the **filled outer Sierpinski fractal** *as follows.*

$$\mathcal{F}^{n^*}_{(r_1, \cdots, r_N)}(P_N) = \bigcup \{P^k(0, i_1, \cdots, i_k) : \forall (0, i_1, \cdots, i_k)\}.$$

In both cases, the union is for all possible sequences of $(0, i_1, \cdots, i_k)$*. As in the inner Sierpinski case, we will say a Sierpinski fractal is* **uniform** *if* $r_1 = \cdots = r_N = r$*, and we abbreviate it as,*

$$\mathcal{S}^{n^*}_r(P_N) = \mathcal{S}^{n^*}_{(r, \cdots, r)}(P_N).$$

We will say a uniform outer Sierpinski fractal $\mathcal{S}^{n^*}_r(P_N)$ *is* **maximal** *if the the common contraction ratio* r *is the largest for given* P_N*. Finally, an outer Sierpinski fractal is called* **regular** *if* P_N *is a regular polygon. The same goes to the filled outer Sierpinski fractals.*

The equality (V.3) is our formulation of the MVID modeling associated to the visualization of the outer Sierpinski fractals. This is far tricker than the equality (IV.1) of Definition IV.1, because we cannot define a multiple valued map for arbitrary point in the phase space. This is due to the nature of the original DCDS, which is not time-invariant. Indeed, the MVID defined by the equality (V.3) depends upon the generation number k, which corresponds to a discrete unit of time.

```
tilt = nn/2 - Floor[nn/2] - 1/2;
vv =
  {Table[{N[Sin[2 Pi (ii + tilt)/nn]],
      N[Cos[2 Pi (ii + tilt)/nn]]},
     {ii, 0, nn - 1}]};
ww = vv;
Sierpinski = {White, EdgeForm[Black],
   Polygon[ww[[1]]]};
gg[xx_] :=
 Table[
  Table[
   -N[rr]
     (xx[[ii]][[Mod[jj + kk - 2, nn] + 1]] -
       xx[[ii]][[kk]]) + xx[[ii]][[kk]],
   {kk, InGrowthStart, InGrowthEnd},
   {jj, 1, nn}], {ii, 1, Length[xx]}];
g0[xx_] :=
 Table[
  Table[
   -N[rr]
     (xx[[ii]][[Mod[jj + kk - 2, nn] + 1]] -
       xx[[ii]][[kk]]) + xx[[ii]][[kk]],
   {kk, 1, nn}, {jj, 1, nn}],
  {ii, 1, Length[xx]}];
```

Fig. V.1. A Screen Capture of A Part of Our Mathematica Program for Outer Sierpinski Fractals I

The lack of time-invariance can be potentially problematic in programming. It is more difficult to program correctly, and it often results in slower speed and more memory requirement.

Figure V.1 is a screen capture of the portion of our program that resolved the difficulty discussed in the previous paragraph. As in the case with the programs of Section IV, we started with a regular N-gon (**vv** $= \cdots$). Instead of bringing in a time-dependent hybrid dynamical system, we managed to define the MVID with the iteration of only one multiple valued map **gg[xx]**, which depend upon the time-variable **kk**. We used the modular arithmetic to assure the correct starting point of the fractal growth.

Figure V.2 shows the iteration of the multiple valued map **gg[xx]**. This part is not too different from the iteration of **ff[xx]** in the programs of Section IV, captured in Figure IV.2. Bigger difference comes from the presence of a multiple valued map **g0[xx]**. This is due to the first generation of the fractal growth, defined by the equality (V.1) of Definition V.1. Indeed, **g0[xx]** applied just once, when the iteration number is 0 (**If[iterationNumber** $=$ **0**, $\cdots$).

On a side note, the in-growth condition (V.4), is put in to

```
If[iterationNumber == 0,
 vv = Flatten[g0[vv], 1];
 ww = Join[ww, vv];
 Sierpinski =
  Table[{White, EdgeForm[Black],
    Polygon[ww[[ii]]]},
   {ii, 1, Length[ww]}];
];
If[Length[vv] < resolution &&
  iterationNumber > 0,
 vv = Flatten[gg[vv], 1];
 ww = Join[ww, vv];
 Sierpinski =
  Table[{White, EdgeForm[Black],
    Polygon[ww[[ii]]]},
   {ii, 1, Length[ww]}];
];
iterationNumber = iterationNumber + 1;
```

Fig. V.2. A Screen Capture of A Part of Our Mathematica Program for Outer Sierpinski Fractals II

make sure the same program can be used to model various types of DCDS's with minimal adjustment. In many physical systems, the fractal growth should not be allowed to break into an existing solid structure, and this is where the term "in-growth" was inspired. Note that the definition of **gg[xx]** in Figure V.1 includes **InGrowthStart** and **InGrowthEnd**, which are there to accommodate this aspect.

Figure V.3 – Figure V.6 exemplify many different kinds of outer Sierpinski fractals as the visualization of the maximal invariant sets of optimized or semi-optimized MVID models of non-linear DCDS. Figure V.3 – Figure V.5 exemplify the optimal solutions of the most ideal situations. Figure V.5 exhibits a subtle difference caused by the in-growth condition, too. Figure V.6 is there to exemplify a general situation that is far from ideal, consequently not so useful in practical terms.

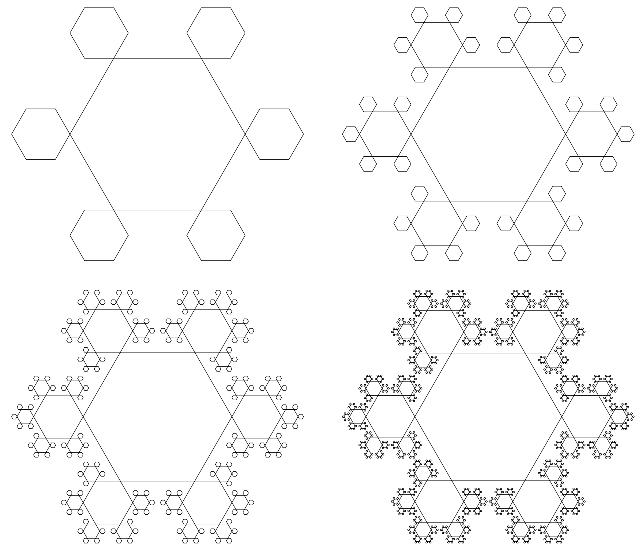

Fig. V.3. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling III

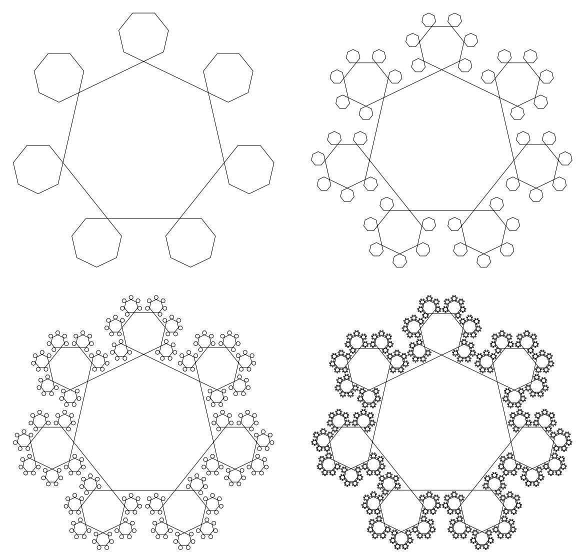

Fig. V.4. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling IV

VI. Conclusion

THROUGH this paper, we studied the algorithmic and computational aspect of MVID modeling of nonlinear disturbed control dynamical systems, particularly in relation to the optimization of Lyapunov multipliers, which quantifies the strength of the controllers. We studied the precise mathematical formulation of the resulting invariant fractals, in accordance to the MVID models that generated the fractals. We paid a particular attention to the MVID algorithms in the visualization programs, and concluded a number of selected examples of visualization.

Acknowledgments. *This paper is based partly upon the undergraduate research by the second author [4] and that of the third author [25], both of which were supervised by the first author. The authors were supported in part by National Research Experience for Undergraduates Program (NREUP), administered by Mathematical Association of America (MAA). Support for this MAA program is jointly provided by the National Science Foundation (Grant DMS-1359016) and the National Security Agency (Grant H98230-15-1-0020). Finally, the authors thank Noureen Khan, the PI of MAA NREUP project at University of North Texas at Dallas (UNT Dallas).*

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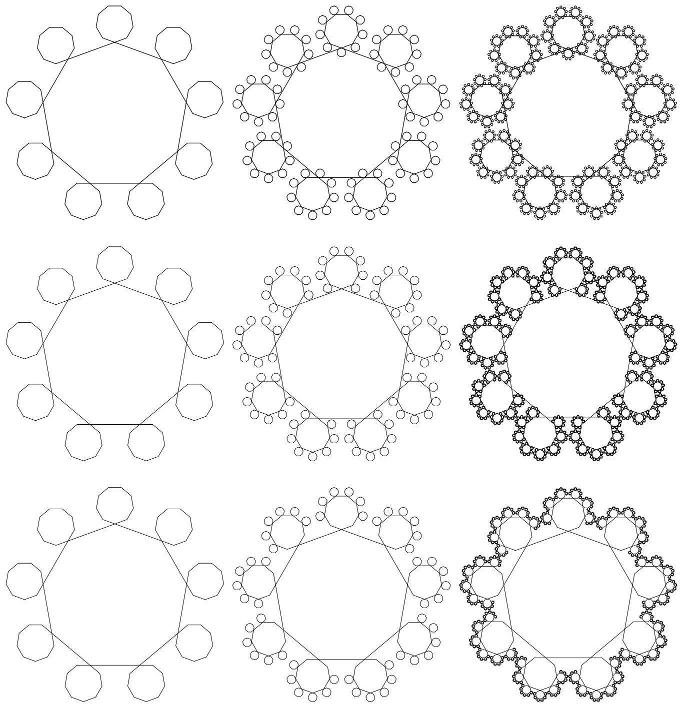

Fig. V.5. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling V

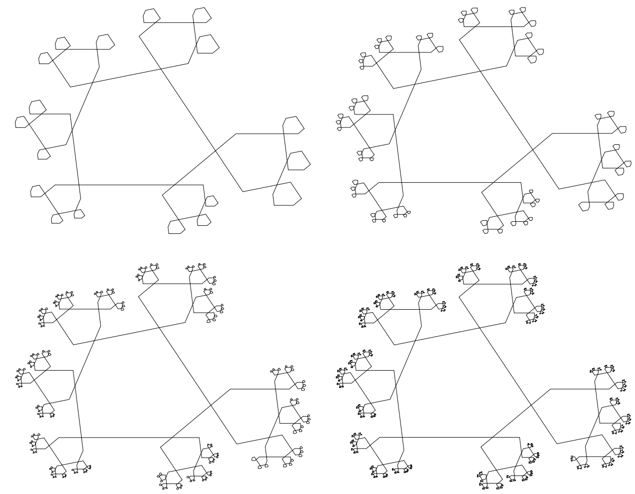

Fig. V.6. An Example of a Forward Iteration of an MVID Algorithm for a Class of Nonlinear DCDS Modeling VI

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