Lattice paths approach for transient solutions of $M/G/1$ queues using Coxian 2-phase distribution

Isnandar Slamet, Ritu Gupta, Narasimaha R. Achuthan, and Roger Collinson

Abstract—The paper aims at deriving transient solutions of non-Markovian queuing system M/G/1 starting from $(k,0)$ to $(m,n), m>n$ remaining below the barrier $Y=X$ and does not include any idle time of server through lattice path approach. The explicit form of the density and other measures of the system performance are not known. Our approach is to approximate general service time with Coxian 2-phase distribution, C_2 and represent the queuing process as a lattice path by recording the state of the system at the point of transitions. We use the lattice path combinatorics to count the feasible number of paths and corresponding probabilities. The above leads to the required density that has simple probabilistic structure and can be computed using R. The investigation of the influence of taking different values of a parameter on the behavior of the graphs of the density is also presented.

Keywords—Coxian distribution, lattice path.

I. Introduction

In recent years, research focus in queueing models has been on developing the methods to compute performance measures of non-Markovian queueing systems. We refer the readers to Takagi [1], [2], Neuts [3], [4], Gross and Harris [5], Kleinrock [6], [7] and Mohammadi and Salehi-Rad [8].

Performance measures include the length of busy period (BP), pure incomplete busy period (PIBP) and incomplete busy period (IBP). These busy periods can be explained as follows. During transient phase, suppose that the queueing system is observed continuously from start time 0 to (current) time t. During this time interval of length t, the system may include alternately several busy and idle periods of the server. This length of time t will be referred to as pure incomplete busy period (PIBP) if it does not include any idle time of server. PIBP can be represented as lattice path starting from $(k,0)$ to $(m,n), m>n$; always remaining below the barrier $Y=X$ does not include any idle time of server.

Until 1995, a vast majority of transient results available for non-Markovian queues were either in terms of Laplace-Stieltjes transforms (LSTs) or other integral transforms that are much complicated, intractable and hard to compute.

Therefore, lattice path technique has been developed to provide transient solutions in explicit form. Some authors have successfully derived the transient solutions for Markovian queueing system for $M/M/1$ (Sen and Jain [9], Bohm and Mohanty [10]), and for $M^b/M/1$ (Sen and Gupta [11]).

An innovation in tackling general service distribution is to approximate it by a Coxian phase-type distribution (Cox [12], Khosgoftar and Perros [13], Agarwal, Sen, and Borkakaty [14], [15], Harris, Marchal, and Botta [16] and Muto [17]. This approximation retains Markovian structure leading to simplistic assumption for subsequent analysis.

The approximation of general service time distribution using C_2, Coxian 2-phase distribution has been used to derive busy period density function for $M/G/1$ (Sen and Agarwal [18]), $G/G/1$, $GI_b/G/1$ queues (Agarwal, Sen, and Borkakaty [14], [15],). Busy period density of $M/G/1$ queues, approximating general distribution by C_3, Coxian 3-phase distribution has been derived by Agarwal, Sen, and Borkakaty [14]. Several authors (Sen and Jain [9], Bohm and Mohanty [10], Sen and Gupta [11]) have computed the density functions of busy period of $M/G/1$ queueing system using lattice path combinatorics (LPC) techniques. Explicit form of the density of the PIBP of $M/G/1$ queues using lattice path (LP) approach are not known.

The proposed methodology approximates a general service distribution by a Coxian 2-phase distribution and extends the LPC technique to compute the density of PIBP of $M/C_2/1$ model. Therefore, in this paper, we would mainly study and compute the closed form solution for such systems.

The rest of the paper is organized as follows. In the introduction, we recapitulate the definition of lattice path and briefly explain its application to determine the density of PIBP of $M/G/1$ queueing system. Section 2 presents the $M/C_2/1$ model. Section 3 presents pure incomplete busy period of $M/G/1$. Section 4 presents the results on counting of paths and subsequent computation of transient probabilities.

I. Slamet is with Study Program of Statistics, University of Sebelas Maret, Surakarta, 57126, Indonesia (phone: +6285 2932 44167; fax: +62 271 663375; e-mail: isnandar06@yahoo.com)

R. Gupta is with the Department of Mathematics and Statistics, Curtin University, Perth, WA, 6102, Australia (e-mail:R.Gupta@curtin.edu.au)

N. R. Achuthan is retired from the Department of Mathematics and Statistics, Curtin University, Perth, WA, 6102, Australia (e-mail: achu_tha@yahoo.com.au)

R. Collinson is with the Department of Mathematics and Statistics, Curtin University, Perth, WA, 6102, Australia (e-mail:R.Collinson@curtin.edu.au)

Finally, we present the numerical computation of density for PIBP in section 5.

II. The $M/C_2/1$ Model

The Coxian distribution as illustrated in Figure 1 is a Coxian 2-phase distribution that describes duration until an event occurs in terms of a process consisting of latent phases.

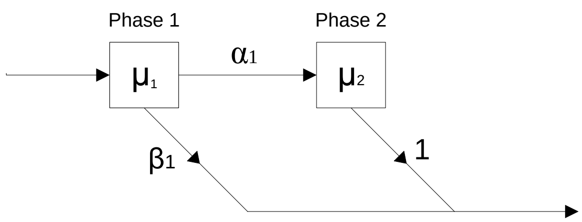


Fig. 1. Coxian 2-phase distribution

A. Transitions

For determining the transient solution, we consider time interval $(0,t)$. Observing the system at point of transitions, we let T_0 and $T_1, T_2, ...$ be the sequence of time points at which transitions take place. Let $X_0, X_1, ...$ be the number of customers at time points $T_0 = 0$, and $T_1, T_2, ...$, respectively, where $X_0 = k$ from the initial condition. Then $\{X_n\}$ is a Markov chain that has the transition probability matrix Q, where for all $n \geq 0; u = 1, 2,$

$$Q(i,j) = P\{X_{n+1} = j \mid X_n = i\}$$

$$= \begin{cases} \dfrac{\lambda}{\lambda+\mu_u}, & \text{an arrival takes place if a customer is undergoing phase u of service } (j = i+1) \\ \dfrac{\beta\mu_1}{\lambda+\mu_1}, & \text{a customer departs after completing phase 1 of service } (j = i-1) \\ \dfrac{\mu_2}{\lambda+\mu_2}, & \text{a customer departs after completing phase 2 of service } (j = i-1) \\ \dfrac{\alpha_1\mu_1}{\lambda+\mu_1}, & \text{a customer enters phase 2 of service after completing phase 1 } (j = i). \end{cases}$$

The holding time in each state is an exponential random variable with a parameter depending on the state given below: $P\{T_{n+1} - T_n > t \mid X_n = i\} = e^{-(\lambda+\mu_1)t}$ if a customer is undergoing phase u of service, $u = 1, 2.$

B. Lattice Path Representation (LPR)

Lattice path representation (LPR) starts with representing the behavior of the queueing process through a sequence of steps represented as lattice path. For example a lattice path can be constructed by representing an arrival into the system by a horizontal step, departure by vertical step and shift from phase 1 to phase 2 by a diagonal step.

Thus arrival of a customer during any phase of service, departure of a customer that can occurs at any phase of service and entry into phase 2 will be denoted by a horizontal unit step, a vertical unit step and a diagonal of $\sqrt{2}$ unit step, respectively. The vertical (horizontal) step will be denoted by a solid line or a dotted line accordingly as departure (arrival) occurs during Phase 1 or Phase 2 of the service.

C. Counting Lattice Paths

For the purpose of counting of lattice paths (LPs), we first transform lattice path to a skeleton lattice path (SLP) by removing all diagonals. For SLP we define 'Run' as follows.

Definition: Run (Agarwal, Sen and Borkakarty [14]): A sequence of consecutive horizontal (vertical) steps bounded on each side by a vertical (horizontal) step is called arrivals run denoted by AR (departures run denoted by DR).

The sequence of horizontal steps starting from the origin and preceding the first vertical step as well as the sequence of verticals at the end following the last horizontal step are called the arrivals run (AR) and departures run (DR), respectively.

Since we are approximating the service time by Coxian 2-phase, C_2, therefore, while inserting the diagonals, the following restrictions will imply on inserting runs.

• Two or more consecutive diagonal can not appear in any horizontal run.
• In a vertical, run any number of diagonals may occur.
• The first vertical step following a diagonal step has to be a dotted vertical step.
• Two or more consecutive dotted vertical steps cannot occur.
• A dotted vertical step can not immediately be preceded by a vertical step (departure after phase 2 cannot be preceded by departure after phase 1).

Finally, for a given set of horizontal and vertical runs, we have to count the number of possible LPs that can be generated keeping in view the above restrictions on the insertions of diagonals.

In a Lattice Path (LP), let k initial number of customers at the start of busy period

r	number of AR, as well as DR $(r \geq 1)$.
p	total number of diagonal inserted in AR and/or DR $(p \geq 0)$'
q	total number of diagonal inserted in AR ,
p-q	number of diagonal inserted in DR,
l_i	length of the ith AR $(i = 1, 2, ..., r)$,
L_i	length of the ith DR $(i = 1, 2, ..., r)$,
$\underset{\sim}{L}$	$(l_1, l_2, ..., l_r; L_1, L_2, ..., L_r)$,
$\underset{\sim}{L}^*$	$(l_1, l_2, ..., l_r; L_1, L_2, ..., L_{r-1})$,
$\underset{\sim}{i}$	$(i_1, i_2, ..., i_q)$, q AR in each of which a diagonal is inserted,
$l_{\underset{\sim}{i}}$	$(l_1, l_2, ..., l_q)$, lengths of AR $\underset{\sim}{i}$,
$\underset{\sim}{p}_i$	$(p_{i_1}, p_{i_2}, ..., p_{i_q})$ distances from extreme left end points where diagonals are inserted in AR runs $\underset{\sim}{i}$ including vertices at both ends of the runs).

To illustrate these notations, we refer to Fig. 2.

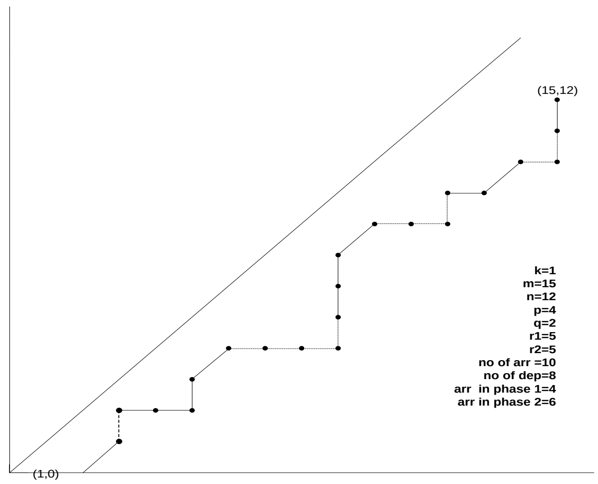


Fig. 2. $M/C_2/1$ model. Lattice path representation

III. Pure Incomplete Busy Period (PIBP) of $M/G/1$

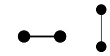
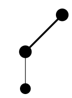
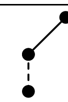
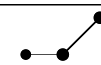
This section presents a brief description of the methodology used to determine the density of pure incomplete busy period (PIBP) of $M/G/1$ queueing system under transient state where PIBP indexed with t and k refers to a continuous period of service till time t, starting with k customers..

Let $f_k(t)$ denote the density function of PIBP of $M/C_2/1$ for $t \geq 0$, where k is the initial number of customers in the system. Such a lattice path (corresponding to a PIBP during $(0,t)$ with k initial customers in the system) will either end with a vertical step (departure), a horizontal step (arrival) or a diagonal step (shift to phase 2 of service).

In this paper, we will illustrate only case 1. All other cases can be computed similarly. Table 1 illustrates the decomposition of lattice paths into disjoint groups.

Next, the density of $f_k^2(t)$ is estimated by first counting the number of lattice paths satisfying the properties of PIBP corresponding to the ending structure of the lattice paths (See Table 1, column 4). Next, the probabilities corresponding to such paths are computed using transition probabilities corresponding to M/C$_2$ /1 model to arrive at $f_k^2(t)$.

Table 1. Structural Properties Events

Phase	Case	Events	Representation
Phase 1	A	No customer enters phase 2	
	B	Departure after phase 1 of service	
	C	Entry into phase 2 of service from phase 1 following departure after phase 1 of service	
	D	Entry into phase 2 of service from phase 1 following departure after phase 2 of service	
	E	Arrival during phase 1 of service	
	F	Entry into phase 2 of service from phase 1	
Phase 2	G	Departure after phase 2 of service	
	H	Arrival during phase 2 of service	

IV. Results

A. Density of No Customer Enters into Phase 2 of Service

Theorem 4.1. Let $f_k^1(t)$ denote the density that the system $M/C_2/1$ starting initially with k customers still in service of length t and no customer enters into phase 2 of service. Then we have

$$f_k^1(t) = \sum_{m=k}^{\infty} \sum_{n=0}^{m-1} \left\{ \binom{m+n-k}{n} - \binom{m+n-k}{m} \right\} \times \lambda^{m-k} (\beta\mu_1)^n t^{m+n-k-1} \frac{e^{-(\lambda+\mu_1)t}}{\Gamma(m+n-k)}, t > 0, \quad (1)$$

Proof. This term corresponds to the case when no customer enters phase 2 service. The number of arrivals is $m-k$, and the number of departures is n. Therefore total number of transition during busy period is $m+n-k$. Call this as N_1.

Let T_0 and $T_1, T_2, \ldots$ be the sequence of times at which the transitions occur. Let at time T_0, Poisson process starts with rate $(\lambda+\mu_1)$. The probability of an arrival occur is $\left(\frac{\lambda}{\lambda+\mu_1}\right)$ and the probability of a departure occur is $\left(\frac{\beta\mu_1}{\lambda+\mu_1}\right)$.

The probability density function of t is N_1-Erlang with parameter $(\lambda+\mu_1)$ given by

$$f_{N_1}(t)=\frac{e^{-(\lambda+\mu_1)t}\left(\lambda+\mu_1\right)^{N_1}t^{N_1-1}}{\Gamma\left(N_1\right)}.$$

The density for this case becomes

$$f_k^1(t)=\sum_{m=k+1}^{\infty}\sum_{n=0}^{m-1}\left\{\binom{m+n-k}{n}-\binom{m+n-k}{m}\right\}\times\left(\frac{\lambda}{(\lambda+\mu_1)}\right)^{m-k}\left(\frac{\beta\mu_1}{(\lambda+\mu_1)}\right)^{n}\times\frac{e^{-(\lambda+\mu_1)t}\left(\lambda+\mu_1\right)^{m+n-k}t^{m+n-k-1}}{\Gamma(m+n-k)}$$

where $\binom{m+n-k}{n}-\binom{m+n-k}{m}$ is the number of lattice paths starting from (k,0)) to (m,n) always remaining below the line $Y=X$. (Sen and Jain, 1993). Hence, we get (1).

B. Density of the Last Event is Departure after A Customer Completes Phase 1 of Service

In this paper we discuss the case where the incomplete busy period ends with a departure. Other cases of incomplete busy period that end with an arrival or diagonal can be obtained in a similar manner. For this aim, first, we develop a complete set of structural property for last departure event. Next, we develop the technique to enumerate the number of lattice paths satisfying specified structural property. Finally, the expression for the case of incomplete busy period of $M/G/1$ is derived.

Theorem 4.2. For non-negative integers k, m, n; p, q; $r(r\geq 1)$, $l_1,l_2,...,l_r$; $L_1,L_2,...,L_r$; let $\left(LP_{(k,n;l,m;r,\underset{\sim}{L})}\right)$, where $\underset{\sim}{L}=\left(l_1,l_2,...,l_r;L_1,L_2,...,L_r\right)$, denote the number of LPs from $(k,0)$ to $(m,n), m>n$ remaining below the line $Y=X$, each comprising of $m-p$ horizontal steps (including those from $(0,0)$ to $(m,n), m>n$), $n-p$ vertical steps and p diagonals, such that

(a) $m-p$ horizontal steps form r runs of lengths $l_1,l_2,...,l_r$, respectively, satisfying $l_1\geq i_0, l_2,...,l_r>0$ and $\sum_{i=1}^{r}l_i=m-p$

(b) $n-p$ vertical steps form r runs of lengths $L_1,L_2,...,L_r$, respectively, satisfying $L_1,L_2,...,L_r>0$ and $\sum_{i=1}^{r}L_j=n-p$

(c) $l_1\geq Max\left(i_0,L_1+1\right),\sum_{i=1}^{u}l_i>\sum_{i=1}^{u}L_i, u=1,2,...,r-1,$ $\sum_{i=1}^{r}l_i=m-p$ $\sum_{i=1}^{r}L_i=n-p$

(d) q diagonals representing into phase 2 are inserted each in any q out of r horizontal runs (including the vertices at both ends of the runs),

(e) The remaining *p-q* diagonals representing into phase 2 are inserted each at any *p-q-r* vertices available along the vertical runs,

Then, for $r\geq 1$, and $m>k$,

$$\left(LP_{(k,m,n;p,q;r,\underset{\sim}{L})}\right)=\sum_{R_7}\sum_{R_8}\binom{n-p-r}{p-q} \tag{2}$$

where

$$R_7=\left\{\left(i_1,i_2,...,i_q\right):1\leq i_1<i_2<...<i_q\leq r\right\}$$
$$R_8=\left\{p_i=\left(p_{i_1},p_{i_2},...,p_{i_q}\right):\Delta\leq p_{i_s}\leq l_{i_s}, s=1,2,...,q\right\},$$
$$\Delta=\begin{cases}0, \text{ if } i_s>1\\ k, \text{ if } i_s=1\end{cases}$$

Proof. Consider the skeleton path from *(k,0)* to *(m-p,n-p).* Suppose this skeleton consists of r horizontal runs and r vertical runs of lengths l_i (i=1,2,…, r), and L_j (j=1,2,…, r), respectively. One unique path will be produced by this scenario. For the purpose of insertion, suppose q diagonals are inserted into runs numbered i_1, i_2,…, i_q, respectively with lengths of l_i, l_2,…, l_q at distances $p_{i_1},p_{i_2},...,p_{i_q}$ from the extreme left end points. The remaining $p-q$ diagonals will be inserted into any $p-q$ vertices out of $n-p-r$. The number to do this is $\binom{n-p-r}{p-q}$.

Now summing $\binom{n-p-r}{p-q}$ over all possible q-tuples, (i_1, i_2,…, i_q) and $p_{i_1},p_{i_2},...,p_{i_q}$, we get (2).

Lemma 2. Let $LP_{(k,m,n;p)}$ the number of $\mathrm{LP_s}$ from $(k,0)$ to $(m,n), m>n$, remaining below the line $Y=X$, each comprising of $m-p$ horizontal steps (including those from $(0,0)$ to $(k,0)$, $n-p$ vertical steps and p diagonals, then summing (2) over r,q and L we find

$$LP_{(k,m,n;p)} = \sum_{(R_4,R_5,R_6)} LP_{(k,m,n;p,q;r,\underset{\sim}{L})} \tag{3}$$

where

$$R_4 = \{r : 1 \le r \le \max(n-p-k+1,1)\}$$

$$R_5 = \{q : \max(0, 2p-n+r) \le q \le \min(r,p)\}$$

$$R_6 = \Big\{\underset{\sim}{L} : l_1 \ge \max(k, L_1+1), \\ l_1+l_2 > L_1+L_2, \ldots, l_1+l_2+\ldots+l_r > L_1+L_2+\ldots+L_r, \\ \textstyle\sum_{i=1}^{r} l_i = m-p, \sum_{i=1}^{r} L_i = n-p\Big\}$$

For the case $r \ge 1,$ and $p=0$, we get

$$\left(LP_{(k,m,n;0,0)}\right) = \binom{m+n-k}{n} - \binom{m+n-k}{m}$$

Theorema 4.3. Let $f_k^2(t)$ denote the density that the system $M/C_2/1$ starting initially with k customers still in service of length t and the last event is departure after a customer completes phase 1 of service. Then we have

$$f_k^2(t) = \sum_{(R_1R_2R_3R_4R_5R_6R_7R_8)} \binom{n-p-r}{p-q} \alpha_1^{\,p} \beta^{n-2p} \lambda^{m-p-k} \mu_1^{\,n-p} \mu_2^{\,p} e^{-(\lambda+\mu_2)t} \\ \times \sum_{x=0}^{\infty} \frac{(\mu_2-\mu_1)^x}{x!} \frac{t^{m+n-p-k+x-1}}{\Gamma(m+n-p-k+x)} \\ \times \frac{\Gamma\left(m+n-2p-k+x-\sum_s^q \left(l_{i_s}-p_{i_s}\right)\right)}{\Gamma\left(m+n-2p-k-\sum_s^q \left(l_{i_s}-p_{i_s}\right)\right)} \tag{4}$$

where

$R_1 = \{m : k+1 \le m \le \infty\}$

$R_2 = \{n : 2 \le n \le m-1\}$

$R_3 = \left\{p : 1 \le p \le \min\left(\left[\frac{n}{2}\right], m-k\right)\right\}, [x]$ denotes the largest integer contained in x.

Proof. To prove this term, let t_1 be the total time spent in phase 1 of service and $t-t_1$ the total time spent in phase 2 of service out of the total time t spent in the system.

For fixed $\underset{\sim}{L}$, total number of transitions during the busy period of length t is given by $m+n-p-k$ as described below:

Number of arrivals: $m-p-k$

Number of departures after phase 1: $n-2p$

Number entries into phase 2: p

Number of departures after phase 2: p

The total number of transition in t_1, the time spent in phase 1 of service, consists of arrivals and departures while customers are in phase 1 of service as well as entries into phase 2 of service as explained below:

Number of arrivals: $m-p-k-\sum_s^q \left(l_{i_s}-p_{i_s}\right)$ since $\sum_s^q \left(l_{i_s}-p_{i_s}\right)$ is the number of arrivals while customers are in phase 2 of service.

Number of entries into phase 2: p.

Number of departures: $n-2p$.

Hence, we obtain N_1, total number of transition during t_1 as below.

$$N_1 = m+n-2p-k-\sum_s^q \left(l_{i_s}-p_{i_s}\right).$$

The probability density function of t_1 is $N_1-Erlang$ with parameter $(\lambda+\mu_1)$ given by

$$f_{N_1}(t) = \frac{e^{-(\lambda+\mu_1)t} (\lambda+\mu_1)^{N_1} t^{N_1-1}}{\Gamma(N_1)}.$$

The total number of transition in $t_2 = t-t_1$, the time spent in phase 2 of service, consists of arrivals and departures while customers are in phase 2 of service as explained below:

Number of arrivals: $\sum_s^q \left(l_{i_s}-k_{i_s}\right)$.

Number of departures: p.

Therefore, N_2, total number of transition during $t_2 = t-t_1$ will be

$$N_2 = \sum_s^q \left(l_{i_s} - k_{i_s} \right) + p .$$

The probability density function of t_2 is $N_2 - Erlang$ with parameter $(\lambda + \mu_2)$ given by

$$f_{N_2}(t) = \frac{e^{-(\lambda+\mu_2)t_2} \left(\lambda + \mu_1\right)^{N_2} t^{N_2-1}}{\Gamma(N_2)} .$$

For the second case, for combining the duration of phase 1 and phase 2 of service, we get

$$\begin{aligned} f_k^2(t) = & \sum_{(R_1,R_2,R_3,R_4,R_5,R_6,R_7)} \binom{n-p-r}{p-q} \\ & \times \int_0^t f_{N_1}(t_1) \left(\frac{\lambda}{\lambda+\mu_1}\right)^{N_1-n+p} \left(\frac{\alpha_1\mu_1}{\lambda+\mu_1}\right)^p \left(\frac{\beta\mu_1}{\lambda+\mu_1}\right)^{n-2p} \\ & \times f_{N_2}(t-t_1) \left(\frac{\lambda}{\lambda+\mu_2}\right)^{N_2-p} \left(\frac{\mu_2}{\lambda+\mu_2}\right)^p dt_1. \end{aligned} \tag{5}$$

Simplifying equation (5), we get

$$\begin{aligned} f_k^2(t) = & \sum_{(R_1,R_2,R_3,R_4,R_5,R_6,R_7)} \binom{n-p-r}{p-q} \alpha_1{}^p \beta^{n-2p} \lambda^{N_1+N_2-n} \mu_1{}^{n-p} \mu_2{}^p \\ & \times \frac{1}{\Gamma(N_1)} \frac{1}{\Gamma(N_2)} e^{-(\lambda+\mu_2)t} \int_0^t e^{(\mu_2-\mu_1)t_1} t_1{}^{N_1-1} (t-t_1)^{N_2-1} dt_1. \end{aligned}$$

Solving the integral part, we get (4).

V. Numerical Computations And Comments

The number of lattice paths when the system ended with departure can be counted using (2). Furthermore the explicit form of density when no customer enters phase 2 service is given by $f_k^1(t)$. We also obtained the explicit form of density when the system ending in departure after a customer completes phase 1 of service i.e. $f_k^2(t)$. An R program has been developed for numerical computation of equation (1) and (4).

The program starts with generating all possible lattice paths using the library AlgDesign, next only the paths satisfying conditions (a) to (e) presented in section 2.c are filtered. These paths form the set L and finally equation (4) is computed for the selected paths.

For different value of λ we notice from Fig. 3 that as t increases, the density function $f_k^2(t)$. increases, then after attains maximum point, it decreases. The maximum value of density increases as the value of λ increases. The rate of decrease in the value of density for smaller value of λ is less than that for the larger value of λ .

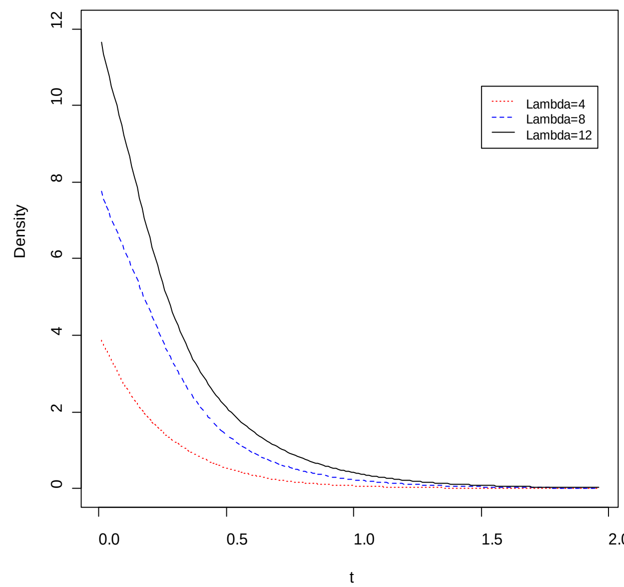


Fig. 4 Density of the incomplete busy period system of $M/C_2/1$ queue for different value of λ taking $k=1, \mu_1=8, \mu_2=10, \alpha=0.4, \beta_1=0.6$

For different value of λ we notice from Fig. 4 that as t increases, the density function of incomplete busy period decreases. The maximum value of density increases as the value of λ increases. The rate of decrease in the value of density for smaller value of λ is less than that for the larger value of λ .

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AUTHORS

Isnandar Slamet was born in Sragen, Indonesia. He completed his BA in statistics from Gadjah Mada University, Yogyakarta, Indonesia in 1989. His Master and PhD all in the area of statistics from Curtin University, Australia has been completed in 2000 and 2013, respectively.

Ritu Gupta was born in India. She has BA (Hons), Master and PhD all in the area of statistics from Delhi University, India.

Narasimaha R. Achuthan has a BA (Hons) in Mathematics from Delhi university, a Master of Statistics, a Diploma in OR and a PhD from Indian Statistical Institute (Kolkata, India).

Isnandar has been a Lecturer in the Department of Mathematics, at Sebelas Maret University, Indonesia since 1992.He became a Senior Lecturer from 2009. Ritu is a Senior Lecturer in the Department of Mathematics and Statistics, at Curtin University, Perth Australia.. She has more than 20 years of experience in consulting, teaching and research She has written more than 30 consulting reports, has presented 20 training courses in Statistics, developed a software for reserves estimation and has published more than 25 papers in referred journals. Her other research interest include combinatorics, queueing systems, design of experiments, soft computing methods and evidence based practice. Achuthan is a Senior Lecturer in the Department of Mathematics and Statistics at Curtin University Perth, Australia. He He has around 44 years of experience in consulting, teaching and research. His current research interests include areas of Operations Research, Applied statistics, Data Mining, Graph theory and their applications to Management Problems. He has published more than 50 papers in international journals.

Dr. Slamet is a member of Statistical Society of Australia Inc. (SSAI) and Indonesian Mathematical Society (IndoMS). Dr. Gupta is an accredited member of the Statistical Society of Australia.