Global Practical Tracking by Output Feedback for Uncertain Nonlinear Systems Under A Weaker Condition

Keylan Alimhan, Naohisa Otsuka, and Orken J. Mamyrbayev

Abstract— This paper considers the problem of global practical tracking via output feedback control for a class of more general uncertain high-order nonlinear systems. Under a weaker growth condition, by introducing sign function and necessarily modifying the homogeneous domination approach, this paper proposes a new control scheme to achieve the global practical tracking. It is shown that the designed controller guarantees that the state of the resulting closed-loop system is globally bounded and the tracking error converges to a prescribed arbitrarily small neighborhood of the origin after a finite time.

Keywords—output feedback, practical tracking, nonlinear system, homogeneous domination.

I. Introduction

THIS paper deals with the problem of global practical output tracking by output feedback for a class of more general high-order nonlinear systems described by

$$\begin{aligned} \dot{x}_i &= x_{i+1}^{p_i} + \phi_i(t,x,u),\ i=1,\dots,n-1, \\ \dot{x}_n &= u + \phi_n(t,x,u), \\ y &= x_1 - y_r \end{aligned} \quad (1)$$

where $x=(x_1,\dots,x_n)^{\mathrm{T}} \in R^n$ and $u \in R$ are the system state and the control input, respectively. For $i=1,\ \dots,\ n,\ \phi_i(t,x,u)$ are unknown continuous functions and $p_i \in R_{odd}^{\geq 1} := \{ p/q \in [0,\infty) : p$ and q are odd integers, $p \geq q\ \}$ $(i=1,\dots,n-1)$ are said to be the high orders of the system, with p_n obviously equal to one (which is not a limitation since we can easily set $v := u^{p_n}$ in the case of non-unity p_n) and y_r is a reference signal to be tracked. Although in the usual tracking problem the reference signal $y_r(t),\ t\in[0,\infty)$ as well as its derivates are assumed to be known, but in our problem only the error $y = x_1 - y_r$ between the output x_1 and the reference signal y_r are assumed to be measureable. Hence only y is allowed to use in the design of the control. There are two reasons to restriction the only measurement to be the error signal. One, in some practice control applications, is inevitable that the error signal is the one to be directly measured. For example, in a missile guidance system, instead of measuring the absolute position of the moving target, that is signal y_r , the onboard radar keeps measuring the distance/error between the missale and the target [1]. The other one is assuming only error signal also makes the actuator design simple, as the controller does not depend on the signal to be tracked explicitly. In this way, the controller is more adaptive to different reference signals[2].

The problem of global output tracking control of nonlinear systems is one of the most important and challenging problems in the field of nonlinear control and lots of efforts have been made during the last decades, see [1-11], as well as the references therein. With the help of the nonlinear output regulator theory [3], [4] and the method of adding a power integrator [12-14], series of research results have been obtained [5-7]. For details, in [8], practical output tracking via smooth state feedback for nonlinear systems was considered. Compared with state feedback control, the theory of output control developed slower, because there is no general and effective method to design a nonlinear observer. Recently, in [9-11] and [2], the practical output feedback tracking problem was also investigated for a class of nonlinear systems with higher-order growing unmeasurable states, extending the results on stabilization in [15-18].

In [9-11] and [2], the following condition on the uncertain term $\phi_i(\cdot)$ is assumed:

$$\left|\phi_i(t,z,u)\right| \leq C\left(\left|x_1\right|^{(r_i+\tau)/r_1} + \cdots + \left|x_i\right|^{(r_i+\tau)/r_i}\right) + C \quad (2)$$

where $C>0$, $\tau>0$ or $-2/(p_1 p_2 \cdots p_{n-1}(2n+1)) < \tau < 0$ are constants and r_i 's are defined as $r_1=1, r_{i+1}p_i = r_i+\tau>0$, $i=1,\dots,n$. Nevertheless, from both practical and theoretical points of view, it is still somewhat restrictive to require system (1) satisfying such restriction. To illustrate the limitation, let us consider the following simple system:

$$\dot{x}_1 = x_2 + x_1^{3/5},\ \ \dot{x}_2 = u, \quad y = x_1 - y_r$$

This work was supported in part by JSPS KAKENHI Grant Number 26420428.

Keylan Alimhan is with the Department of Mathematical and Computer Modeling, L.N. Gumilyov Eurasian National University, Astana, Kazakhstan and he is also with the School of Science and Engineering, Tokyo Denki University, Tokyo, Japan (e-mail: keylan@live.jp).

Naohisa Otsuka is with the Division of Science, School of Science and Engineering, Tokyo Denki University, Tokyo, Japan (Tel: +81-49-296-5143; Fax: +81-49-296-6403; e-mail: otsuka@mail.dendai.ac.jp).

Orken J. Mamyrbayev is with Institute of Information and Computational Technologies, Almaty, Kazakhstan (e-mail: morkenj@mail.ru).

where $p_1 = p_2 = 1$, $\phi_1 = x_1^{3/5}$ and $\phi_2 = 0$. For the simple system, it is easily verified that the works [9-11] and [2] cannot lead to any output feedback tracking controller because of the presence of low-order term $x_1^{3/5}$ dissatisfied the growth condition. Naturally, an interesting problem may be proposed:

(i) *Is it possible to further relax the nonlinear growth condition on* τ *in* (2)?

(ii) *Under the weaker assumption, how can one design an output tracking controller for the nonlinear system* (1) *by output feedback*?

In this paper, by introducing a combined homogeneous domination and sign function approach, we shall solve the above problems.

II. Mathematical Preliminaries

At first, we give the following notations which will be used in this study.

Notations: R^n denotes the real n-dimensional space and $R^+ := [0, \infty)$. For any vector $x = (x_1, \ldots, x_n)^T \in R^n$, denote $\overline{x}_i := (x_1, \ldots, x_i)^T \in R^i$, $i = 1, \ldots, n$, $\|x\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. A sign function $\operatorname{sgn}(x)$ is defined as: $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = 0$ if $x = 0$, and $\operatorname{sgn}(x) = -1$ if $x < 0$. For any $\alpha \in R^+$ and $x \in R$, the function $\lceil x \rfloor^\alpha$ is defined as $\lceil x \rfloor^\alpha = \operatorname{sgn}(x)|x|^\alpha$. A function $f : R^n \to R$ is said to be C^k-function, if its partial derivatives exist and are continuous up to order k, $1 \le k < \infty$.

A C^0 function means it is continuous. A C^∞ function means it is *smooth*, that is, it has continuous partial derivatives of any order. Besides, the arguments of functions (or functionals) are sometimes omitted or simplified, whenever no confusion can arise from the context. For instance, we sometimes denote a function $f(x(t))$ by $f(x)$, $f(\cdot)$, or f.

First, we recall some important definitions regarding to homogeneous systems (For more details, see, e.g,. [19], [20], [22] and [21]). Now, let $x = (x_1, \ldots, x_n) \in R^n$ be a fixed coordinate, and $s > 0$, $r_i > 0$ $(i = 1, \ldots, n)$ be real numbers. Then:

(i) *A dilation* $\Delta_s(x)$ is a mapping defined by

$$\Delta_s(x) = \left(s^{r_1} x_1, \cdots, s^{r_n} x_n\right), \quad \forall s > 0$$

where r_i are called *the weights of the coordinate*. For simplicity of notation, the dilation weight is denoted by $\Delta = (r_1, \ldots, r_n)$.

(ii) A function $V \in C(R^n, R)$ is said to be *homogeneous of degree* τ if there is a real number $\tau \in R$ such that

$$V(\Delta_s(x)) = s^\tau V(x_1, \cdots, x_n), \quad \forall x \in R^n - \{0\}.$$

(iii) A vector field $f \in C(R^n, R^n)$ is said to be *homogeneous of degree* τ if there is a real number $\tau \in R$ such that for $i = 1, \ldots, n$

$$f_i(\Delta_x(x)) = s^{\tau + r_i} f_i(x_1, \cdots, x_n), \quad \forall x \in R^n - \{0\}.$$

(iv) *A homogeneous* p*-norm* is defined as

$$\|x\|_{\Delta, p} = \left(\sum_{i=1}^n |x_i|^{p/r_i}\right)^{1/p}, \ \forall x \in \mathbb{R}^n, \ p \ge 1.$$

For the simplicity, write $\|x\|_\Delta$ for $\|x\|_{\Delta,2}$.

Next, in what follows, some useful lemmas are cited, which are used in the main body.

Lemma1[20]. Given a dilation weight $\Delta = (r_1, \ldots, r_n)$, suppose $V_1(x)$ and $V_2(x)$ are homogeneous of degree τ_1 and τ_2, respectively. Then, $V_1(x)V_2(x)$ is also homogeneous with respect to the same dilation Δ. Moreover, the homogeneous degree of $V_1(x)V_2(x)$ is $\tau_1 + \tau_2$.

Lemma2[20]. Suppose $V : R^n \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ. Then, the following holds:

(i) $\partial V / \partial x_i$ is also homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i.

(ii) There is a constant $\sigma > 0$ such that $V(x) \le \sigma \|x\|_\Delta^\tau$. Moreover, if $V(x)$ is positive-definite, there is a constant $\rho > 0$ such that $\rho \|x\|_\Delta^\tau \le V(x)$.

Now, we introduce several technical lemmas which will play an important role and be frequently used in the later control design.

Lemma3[5]. For any real numbers $x \ge 0$, $y > 0$ and $m \ge 1$, the following inequality holds:

$$x \le y + (x/m)^m \left((m-1)/y\right)^{m-1}.$$

Lemma4[23]. For all $x, y \in R$ and a constant $p \ge 1$ the following inequalities holds:

(i) $|x + y|^p \le 2^{p-1} |x^p + y^p|$,

$$\left(|x| + |y|\right)^{1/p} \le |x|^{1/p} + |y|^{1/p} \le 2^{(p-1)/p} \left(|x| + |y|\right)^{1/p}$$

If $p \in R_{odd}^{\ge 1}$, then

(ii) $|x - y|^p \le 2^{p-1} |x^p - y^p|$ and

$$\left|x^{1/p} - y^{1/p}\right| \le 2^{(p-1)/p} |x - y|^{1/p}.$$

Lemma5[23]. Let c, d be positive constants. Then, for any real-valued function $\gamma(x, y) > 0$, the following inequality holds:

$$|x|^c |y|^d \le \frac{c}{c+d} \gamma(x, y) |x|^{c+d} + \frac{d}{c+d} \gamma^{-c/d}(x, y) |y|^{c+d}.$$

Lemma6[24]. For $x, y \in R$ and $0 < p \le 1$ the following inequality holds:

$$\left(|x| + |y|\right)^p \le |x|^p + |y|^p.$$

When $p = a/b \le 1$, where $a > 0$ and $b > 0$ are *odd* integers

$$\left|x^p + y^p\right| \le 2^{1-p} |x + y|^p.$$

Lemma7[25]. If $p = a/b \in R_{\text{odd}}^{\ge 1}$ with $a \ge b \ge 1$ being some real numbers, then for any $x, y \in R$

$$\left|x^p - y^p\right| \le 2^{1-1/b}\left|\operatorname{sgn}(x)|x|^a - \operatorname{sgn}(y)|y|^a\right|^{1/b}.$$

Lemma8[8]. If $f:\ [a,b] \to R\ \ (a \le b)$ is monotone continuous and satisfies $f(a)=0$, then

$$\left|\int_a^b f(x)dx\right| \le |f(b)|\cdot|b-a|.$$

This paper deals with the practical output tracking problem by output feedback for nonlinear systems (1). Here, we first give a precise definition of our practical tracking problem [9], [11].

The problem of global practical tracking by an output feedback: Consider system (1) and assume that the reference signal $y_r(t)$ is a time-varying C^1- bounded function on $[0,\infty)$. For any given $\varepsilon > 0$, design an output controller having the following structure

$$\begin{cases} \dot{\zeta} = \alpha(\zeta, y),\ \ \zeta(0) \in R^{n-1} \\ u = \beta(\zeta, y), \end{cases} \tag{3}$$

where α, β are some smooth functions, such that

i) All the state $[x(t), \zeta(t)] \in R^{2n-1}$ of the closed-loop system (1) with output controller (3) is well-defined on $[0,+\infty)$ and globally bounded.

ii) For any initial state $[x(0), \zeta(0)]$, there is a finite time $T := T(\varepsilon, x(0), \zeta(0)) > 0$, such that

$$|y(t)| = |x_1(t) - y_r(t)| < \varepsilon, \quad \forall t \ge T > 0. \tag{4}$$

In order to solve the global practical output tracking problem, we made the following assumption:

Remark1. Assumption1, which gives the nonlinear growth condition on the system drift terms, encompasses the assumptions in existing results [9-11] and [2]. Specifically, when $\tau \ge 0$, it reduces to Assumptions in [9-10] and [1-2]. When τ is some ratios of odd integers in $\tau \in [-1/\sum_{l=1}^{n} p_1 \cdots p_{l-1},\ 0]$, it encompasses the condition used in [11]. This means that the system studied in this paper is less restrictive and allows for a much broader class of systems.

Assumption2. The reference signal $y_r(t)$ is continuously differentiable. Moreover, there is a known constant D >0, such that

$$|y_r(t)| + |\dot{y}_r(t)| \le D, \quad \forall t \in [0,\infty).$$

Now, we state the main result of this paper as follows:

Theorem1. Under assumptions1-2 on system (1), the global practical output tracking problem stated above is solvable by output controller of the form (3).

III. Output Feedback Tracking Controller Design

In this section, we will ingeniously combine homogeneous domination theory and sign function approach to solve this output tracking problem. Before designing the controller, introduce the following new coordinate transformation: Letting

$\kappa_1 = 0, \quad \kappa_i = (\kappa_{i-1}+1)/p_{i-1}$ for $i = 2,\ldots,n$, define

$$z_1 := y, \quad z_i := x_i / M^{\kappa_i}, \quad i = 2,\ldots,n,\ \ v := u / M^{\kappa_n + 1} \tag{7}$$

where $M \ge 1$ is a rescaling gain to be determined later. Then, the system (1) can be described in the new variables z_i as

$$\begin{aligned} \dot{z}_i &= M z_{i+1}^{p_i} + \psi_i(t,z,v), \quad i = 1,\ldots,n-1, \\ \dot{z}_n &= Mv + \psi_n(t,z,v), \quad y = z_1 \end{aligned} \tag{8}$$

where

$$\psi_1(t,z,v) := \phi_1(t,x,u) - \dot{y}_r,$$
$$\psi_i(t,z,v) := \frac{\phi_i(t,x,u)}{M^{\kappa_i}}, \quad i = 2,\ldots,n.$$

Now, using the relation $r_j = \tau\kappa_j + 1/(p_1 \ldots p_{j-1})$, one can obtain the inequalities for $j = 2,\ldots,i,\ i = 1,\ldots,n$

$$\kappa_j \frac{r_{i+1}p_i}{r_j} - \kappa_i \begin{cases} \le \dfrac{\tau\kappa_j}{\tau\kappa_j + 1/p_1 \ldots p_{j-1}}, & \tau \ge 0 \\ \le \dfrac{\tau\kappa_j + \kappa_j 1/p_1 \ldots p_{i-1} - \kappa_i 1/p_1 \ldots p_{j-1}}{\left(\tau \sum_{l=1}^{j} p_1 \ldots p_{l-2} + 1\right)/p_1 \ldots p_{j-1}}, & 0 > \tau > -1\Big/\sum_{l=1}^{n} p_1 \ldots p_{l-1} \end{cases}$$
$$\begin{cases} \le 1, & \tau \ge 0 \\ \le 0, & 0 > \tau > -1\Big/\sum_{l=1}^{n} p_1 \ldots p_{l-1} \end{cases} \tag{9}$$

which implies $M^{\kappa_i(r_i+\tau)/r_i - \kappa_i} \le M^{1-\nu_i}$ for some $0 < \nu_i < 1$.

Now, using Assumption1, Lemmas3-8, $M \ge 1$ and above the fact that, the following inequalities can be obtained:

$$|\psi_1(t,z,v)| \le \bar{C}_1 |z_1|^{(r_1+\tau)/r_1} + \bar{C}_2$$
$$\begin{aligned} |\psi_i(t,z.v)| &\le \bar{C}_1 \sum_{j=1}^{i} M^{\kappa_j(r_i+\tau)/r_j - \kappa_i} |z_j(t)|^{(r_i+\tau)/r_j} + \frac{C_2}{M^{\kappa_i}} \\ &\le \bar{C}_1 M^{1-\nu_i} \sum_{j=1}^{i} |z_j(t)|^{(r_i+\tau)/r_j} + \frac{\bar{C}_2}{M^{\kappa_i}}, \quad i = 2,\ldots,n \end{aligned} \tag{10}$$

where $\bar{C}_1,\ \bar{C}_2 \ge 0,\ \nu_i > 0$ are some constants.

Step1. We first construct a state feedback controller for the nominal nonlinear system of (8)

$$\dot{z}_i = M z_2^{p_i}, \quad i = 1,\ldots,n-1,\ \ \dot{z}_n = M v^{p_n},\ \ y = z_1 \tag{11}$$

where $p_i \in R_{\text{odd}}^{\ge 1}$ for $i = 1,\ldots,n-1$. Following [26], one can construct a state feedback controller globally stabilizing system (11) in the following form:

$$v(z) = -\beta_n^{r_{n+1}/\mu} [\xi_n]^{r_{n+1}/\mu} = -\left[\sum_{i=1}^{n} \bar{\beta}_i [z_i]^{\mu/r_i}\right]^{r_{n+1}/\mu} \tag{12}$$

where $\mu \in R_{odd}$ is such that $\mu \geq \max_{1\leq i\leq n}\{1, \tau + r_i\}$ $\overline{\beta}_i = \beta_n \cdots \beta_i$, $\beta_i > 0\ (i = 1,\dots,n)$ are the appropriately determined controller gains and ξ_n is determined recursively via

$$\begin{aligned} z_1^* &= 0, & \xi_1 &= [z_1]^{\mu/r_1} - [z_1^*]^{\mu/r_1} \\ z_2^* &= -\beta_1^{r_2/\mu}[\xi_1]^{r_2/\mu}, & \xi_2 &= [z_2]^{\mu/r_2} - [z_2^*]^{\mu/r_2} \\ &\vdots & &\vdots \\ z_i^* &= -\beta_{i-1}^{r_i/\mu}[\xi_{i-1}]^{r_i/\mu}, & \xi_i &= [z_i]^{\mu/r_i} - [z_i^*]^{\mu/r_i}. \end{aligned} \tag{13}$$

Further, one can construct a homogeneous observer $\hat{z} = [\hat{z}_1, \hat{z}_2, \dots, \hat{z}_n]^T \in R^n$ for system (11) in the form

$$\begin{aligned} \dot{\eta}_2 &= -ML_1\hat{z}_2^{p_1}, & \hat{z}_2 &= [\eta_2 + L_1 z_1]^{r_2/r_1} \\ \dot{\eta}_i &= -ML_{i-1}\hat{z}_i^{p_{i-1}}, & \hat{z}_i &= [\eta_i + L_{i-1}\hat{z}_{i-1}]^{r_i/r_{i-1}}, \quad i = 3,\dots,n \end{aligned} \tag{14}$$

where $\hat{z}_1 = z_1 = y$ and $L_s > 0\ (s = 1,\dots,n-1)$ are the observer gains to be determined later. Then, by certainty equivalence principle, we can replace z_i with $\hat{z}_i$ in (12) and obtain an output feedback controller

$$v(\hat{z}) = -\beta_n^{r_{n+1}/\mu}[\xi_n]^{r_{n+1}/\mu} = -\left[\sum_{i=1}^{n}\overline{\beta}_i[\hat{z}_i]^{\mu/r_i}\right]^{r_{n+1}/\mu} \tag{15}$$

where $\hat{z}_1 = z_1$ and $\hat{z} = [\hat{z}_1, \hat{z}_2, \dots, \hat{z}_n]^T$.

Now, define

$$\begin{aligned} Z &:= [z_1, \dots, z_n, \eta_2, \dots, \eta_n]^{\mathrm{T}} \\ F(Z) &:= \left[z_2^{p_1}, \dots, z_n^{p_{n-1}}, v(z_1, \eta_2, \dots, \eta_n), f_{n+1}, \dots, f_{2n-1}\right]^{\mathrm{T}} \end{aligned} \tag{16}$$

where $f_{n+1} := \dot{\eta}_2, f_{n+2} := \dot{\eta}_3, \dots, f_{2n-1} := \dot{\eta}_n$. Then, the closed-loop system (11) with the output feedback controller (15) can be rewritten in a compact form as

$$\dot{Z} = F(Z) = \left[z_2^{p_1}, \dots, z_n^{p_{n-1}}, v(z_1, \eta_2, \dots, \eta_n), f_{n+1}, \dots, f_{2n-1}\right]^{\mathrm{T}}. \tag{17}$$

Moreover, it can be verified that $F(Z)$ is homogeneous of degree τ with dilation weight

$$\Delta = [R_1, R_2, \dots, R_{2n-1}] = [\underbrace{r_1, r_2, \dots, r_n}_{\text{for } z_1,\dots,z_n}, \underbrace{r_1, r_2, \dots, r_{n-1}}_{\text{for } \eta_2,\dots,\eta_n}]. \tag{18}$$

Then, with these results and notations, the following proposition can be obtained [26].

Proposition1. The observer gains $L_i > 0,\ i = 1,\dots,n-1$ and the controller gains $\beta_1,\dots,\beta_n > 0$ can be recursively determined so that the closed-loop system (11) with (15) admits a Lyapunov function $V(Z)$ for system (16) such that

(i) $V(Z)$ is positive definitive and proper with respect to Z

(ii) $V(Z)$ is homogeneous of degree $2\mu - \tau$ with dilation (18)

(iii) the derivative of $V(Z)$ along (11)-(14)-(15) satisfies

$$\frac{\partial V(Z)}{\partial Z}F(Z) \leq -\gamma\|Z\|_{\Delta}^{2\mu} \tag{19}$$

where $\gamma > 0$ is a constant, $\|Z\|_{\Delta} = \sqrt{\sum_{i=1}^{2n-1}|Z_i|^{2/r_i}}$ and

$$\begin{aligned} V &= V_n + \sum_{i=2}^{n}U_i = \sum_{i=1}^{n}\int_{z_i^*}^{z_i}\left[[s]^{\mu/r_i} - [z_i^*]^{\mu/r_i}\right]^{(2\mu-\tau-r_i)/\mu}ds \\ &+ \sum_{i=2}^{n}\int_{[\gamma_i]^{(2\mu-\tau-r_{i-1})/r_{i-1}}}^{[z_i]^{(2\mu-\tau-r_{i-1})/r_i}}\left[[s]^{r_{i-1}/(2\mu-\tau-r_{i-1})} - \gamma_i\right]ds, \quad \gamma_i = \eta_i + L_{i-1}z_{i-1}. \end{aligned}$$

Step2. Next, an output controller archiving the *practical* tracking for the entire system (1) will be constructed using the coupled controller-observer design method [26] with the result in the first step.

Using the notations (7) and (8), it is easy to see that the closed-loop system (8) with (15) can be written in a compact form as

$$\dot{Z} = MF(Z) + [\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot), \dots, \psi_n(\cdot), 0, \dots, 0]^{\mathrm{T}}. \tag{20}$$

Now, it follows from *Proposition1* that there exist suitable observer gains $L_i > 0\ (i = 1,\dots,n-1)$ and controller gains $\beta_i > 0\ (i = 1,\dots,n)$ which ensure the existence of a positive definitive and proper Lyapunov function $V(Z)$ with the homogeneous degree $2\mu - \tau$ satisfying

$$\frac{\partial V(Z)}{\partial Z}F(Z) \leq -\gamma\|Z\|_{\Delta}^{2\mu} \text{ for some } \gamma > 0. \tag{21}$$

Hence, the time derivative of $V(Z)$ along the trajectory of (20) satisfies

$$\dot{V}(Z) \leq -M\gamma\|Z\|_{\Delta}^{2\mu} + \sum_{i=1}^{n}\frac{\partial V(Z)}{\partial Z_i}\psi_i(\cdot). \tag{22}$$

Further, using (10), one obtains

$$\begin{aligned} \dot{V}(Z) \leq& -M\gamma\|Z\|_{\Delta}^{2\mu} \\ &+ \overline{C}_1\sum_{i=1}^{n}M^{1-\nu_i}\left|\frac{\partial V(Z)}{\partial Z_i}\right|\Big[|z_1|^{(r_i+\tau)/r_1} + |z_2|^{(r_i+\tau)/r_2} \\ &+ \cdots + |z_i|^{(r_i+\tau)/r_i}\Big] + \overline{C}_2\sum_{i=1}^{n}\frac{1}{M^{\kappa_i}}\left|\frac{\partial V(Z)}{\partial Z_i}\right|. \end{aligned} \tag{23}$$

Since, by Lemma2 and *Proposition1*, $\partial V(Z)/\partial Z_i$ is homogeneous of degree $2\mu - \tau - r_i$, the term

$$\left|\frac{\partial V(Z)}{\partial Z_i}\right|\left(|z_1|^{(r_i+\tau)/r_1} + |z_2|^{(r_i+\tau)/r_2} + \cdots + |z_i|^{(r_i+\tau)/r_i}\right) \tag{24}$$

is homogeneous of degree 2μ, and hence it follows from Lemma1 and Lemma2 that for each $i = 1,\dots,n$ there exists a constants $\lambda_i > 0$ such that

$$\left|\frac{\partial V(Z)}{\partial Z_i}\right|\left(|z_1|^{(r_i+\tau)/r_1} + |z_2|^{(r_i+\tau)/r_2} + \cdots + |z_i|^{(r_i+\tau)/r_i}\right) \leq \lambda_i\|Z\|_{\Delta}^{2\mu}. \tag{25}$$

Furthermore, it follows from Lemma1 and Lemma5 that there are positive constants $a_1, \overline{a}_2, \tilde{a}_2$ such that

$$\bar{C}_2\left|\frac{\partial V(Z)}{\partial Z_1}\right| \le a_1\left(M^{1/2\mu}\|Z\|_\Delta\right)^{2\mu-\tau-r_1}\left(M^{-(2\mu-\tau-r_1)/(2\mu(\tau+r_1))}\right)^{\tau+r_1}$$
$$\le \frac{\gamma}{2}M\|Z\|_\Delta^{2\mu}+\bar{a}_2 M^{-(2\mu-\tau-r_1)/(\tau+r_1)},$$
$$\bar{C}_2\left|\frac{\partial V(Z)}{\partial Z_i}\right| \le a_1 M^{\kappa_i}\|Z\|_\Delta^{2\mu-\tau-r_i}\left(M^{-\kappa_i/(\tau+r_i)}\right)^{\tau+r_i}$$
$$\le M^{\kappa_i}\|Z\|_\Delta^{2\mu}+\tilde{a}_2 M^{-2\mu\kappa_i/(\tau+r_i)},\quad i=2,\dots,n$$

Now, substituting (25) and the above into (24) leads to

$$\begin{aligned}\dot{V}(Z) &\le -M\left(\frac{\gamma}{2}-\bar{C}_1\sum_{i=1}^{n}\lambda_i M^{-\nu_i}-(n-1)M^{-1}\right)\|Z\|_\Delta^{2\mu}\\ &\quad + a_2\left(M^{-(2\mu-\tau-r_1)/(\tau+r_1)}+\sum_{i=2}^{n}M^{-2\mu\kappa_i/(\tau+r_i)}\right)\\ &= -M\left(\frac{\gamma}{2}-G_1(M)\right)\|Z\|_\Delta^{2\mu}+G_2(M)\end{aligned}\tag{26}$$

where $a_2=\max(\bar{a}_2,\tilde{a}_2)$,

$$\begin{aligned}G_1(M)&=\bar{C}_1\sum_{i=1}^{n}\lambda_i M^{-\nu_i}+(n-1)M^{-1}\quad\text{and}\\ G_2(M)&=a_2\left(M^{-(2\mu-\tau-r_1)/(\tau+r_1)}+\sum_{i=2}^{n}M^{-2\mu\kappa_i/(\tau+r_i)}\right)\end{aligned}\tag{27}$$

both of which are positive and monotonically decreasing to zero as M increases indefinitely.

Next, it will be shown that (26) implies the existence of a gain $M\ge 1$ which achieves the robust practical tracking for system (1). Since $V(Z)$ is homogeneous of degree $2\mu-\tau$ and positive definite, it follows from Lemma2 that there are two constants $\sigma_2\ge\sigma_1>0$ satisfying

$$\sigma_1\|Z\|_\Delta^{2\mu-\tau}\le V(Z)\le\sigma_2\|Z\|_\Delta^{2\mu-\tau}.\tag{28}$$

Now, define

$$\mathfrak{M}=\left\{M\ge 1\,\middle|\,\frac{\gamma}{2}-G_1(M)>0\right\},\tag{29}$$

and take an arbitrary $M\in\mathfrak{M}$. Then, (26) together with (28) leads to the inequality

$$\dot{V}(Z)\le-\kappa(M)V(Z)^{2\mu/(2\mu-\tau)}+G_2(M)\tag{30}$$

where

$$\kappa(M)=\left(\frac{\gamma}{2}-G_1(M)\right)\sigma_2^{-2\mu/(2\mu-\tau)}>0.\tag{31}$$

First, it will be shown similarly as in [5] that the state $Z(t)$ of closed-loop system (21) is well-defined on $[0,+\infty)$ and globally bounded. Since $\kappa(M)>0$ is strictly monotonically increasing to $\gamma\sigma_2^{-2\mu/(2\mu-\tau)}>0$ as $M\to\infty$ and $G_2(M)$ is positive and strictly monotonically decreasing to zero as $M\to\infty$, it is easily seen that, for any given $\varepsilon>0$, one can choose a sufficiently large $M\in\mathfrak{M}$ so as to satisfy

$$\sigma_1^{-1/(2\mu-\tau)}\left(2G_2(M)/\kappa(M)\right)^{1/2\mu}<\varepsilon.\tag{32}$$

Next, introduce a subset by

$$\Omega=\left\{Z\in\mathbb{R}^{2n-1}\,\middle|\,V(Z)\ge\left(2G_2(M)/\kappa(M)\right)^{(2\mu-\tau)/2\mu}\right\}\subset R^{2n-1}\tag{33}$$

and let $Z(t)$ be the trajectory of (20) with an initial state $Z(0)$. Suppose $Z(t)\in\Omega$ for some $t\in[0,\infty)$. Then it follows from (30) that

$$\begin{aligned}\dot{V}(Z(t))&\le-\kappa(M)V(Z(t))^{2\mu/(2\mu-\tau)}+G_2(M)\\ &\le -G_2(M)<0\end{aligned}\tag{34}$$

This implies that, as long as $Z(t)\in\Omega$, $V(Z(t))$ is strictly decreasing with time t, and hence $Z(t)$ must enter the complement set $R^{2n-1}-\Omega$ in a finite time $T\ge 0$ and stay there forever. Therefore, one can obtain the following relations:

$$\begin{aligned}&V(Z(t))-V(Z(0))=\int_0^t\dot{V}(Z(t))dt<0,\quad t\in[0,T)\\ &V(Z(t))<\left(2G_2(M)/\kappa(M)\right)^{(2\mu-\tau)/2\mu},\quad t\in[T,\infty)\end{aligned}\tag{35}$$

which together with (27) lead to

$$\begin{aligned}|Z_i(t)|&\le\|Z(t)\|_\Delta^{r_i}\\ &\le\left(V(Z(t))/\sigma_1\right)^{r_i/(2\mu-\tau)}\\ &\le\sigma_1^{-r_i/(2\mu-\tau)}V(Z(0))^{r_i/(2\mu-\tau)},\quad t\in[0,T)\\ |Z_i(t)|&\le\|Z(t)\|_\Delta^{r_i}\\ &\le\left(V(Z(t))/\sigma_1\right)^{r_i/(2\mu-\tau)}\\ &\le\sigma_1^{-r_i/(2\mu-\tau)}\left(2G_2(M)/\kappa(M)\right)^{r_i/2\mu},\quad t\in[T,\infty)\end{aligned}\tag{36}$$

for $i=1,\dots,2n-1$. Thus, the solution $Z(t)$ of system (20) is well-defined and globally bounded on $[0,+\infty)$.

Next, it will be shown that

$$|y(t)|=|x_1(t)-y_r(t)|<\varepsilon,\quad\forall t\ge T>0.\tag{37}$$

This is easily shown from (27), (34) and (31) as follows:

$$\begin{aligned}|y(t)|&=|x_1(t)-y_r(t)|\\ &=|z_1(t)|\le\|Z(t)\|_\Delta\\ &\le\left(V(Z(t))/\sigma_1\right)^{1/(2\mu-\tau)}\\ &\le\sigma_1^{-1/(2\mu-\tau)}\left(2G_2(M)/\kappa(M)\right)^{1/2\mu}<\varepsilon\end{aligned}\tag{38}$$

Finally, since the choice of $M\in\mathfrak{M}$ depends on $\varepsilon>0$, the finite time $T>0$ depends on $\varepsilon>0$. Further, it is obvious that $T>0$ is dependent on each trajectory of (20), or equivalently, on each initial state $Z(0)$ of (20). Therefore, the finite time $T>0$ satisfying (36) is dependent on both $\varepsilon>0$ and $Z(0)$, i.e., $T:=T(\varepsilon,x(0),\zeta(0))$. This completes the proof of Therorem1.

IV. CONCLUSIONS

In this paper, an output feedback tracking controller for a class of high-order uncertain nonlinear systems was presented under weaker condition. It was shown that the global practical tracking problem is solvable using the homogenous observer and controller, which can be explicitly constructed. First, we designed an output feedback controller for the nominal system without the perturbing nonlinearties. Then, we utilized the homogeneous domination approach by introducing an adjustable scaling gain into the output feedback controller obtained for the nominal system. Further, it was also shown that an appropriate choice of gain will enable us to globally track for a class of uncertain nonlinear systems in finite time. Finally, the proposed approach can also widen the applicability to a broader class of systems.

REFERENCES

[1] Q. Gong and C. Qian, C. 'Global practical output regulation of a class of nonlinear systems by output feedback', *Proc. the 44th IEEE conference on decision and control, and the European control conference*, Seville, Spain, pp. 7278-7283, 2005.

[2] J, Zhai and S. Fei, 'Global practical tracking control for a class of uncertain nonlinear systems, *IET Control Theory and Applications*, vol 5, Issue 11, pp. 1343 – 1351, 2011.

[3] A. Isidori, C.I. Byrnes, "Output regulation of nonlinear system", *IEEE Trans. on Automatic Control*, 35 , pp. 31-140, 1990.

[4] C.I. Byrnes,, F.D. Psicoli, A. Isidori, Output Regulation of Uncertain Nonlinear Systems. Boston: Birkhäuser, 1997.

[5] C.J. Qian, W. Lin, "Practical output tracking of nonlinear systems with uncontrollable unstable linearization", *IEEE Trans. on Automatic Control*, 47, pp. 21−36, 2002.

[6] L. Marconi, A. Isidori, "Mixed internal model-based and feedforward control for robust tracking in nonlinear systems", Automatica, 36, pp. 993−1000, 2000.

[7] W. Lin, R. Pongvuthithum, "Adaptive output tracking of inherently nonlinear systems with nonlinear parameterization", *IEEE Trans. on Automatic Control*, 48, pp. 1737−1749, 2003.

[8] Z.Y. Sun, Y.G. Liu, "Adaptive practical output tracking control for high-order nonlinear uncertain systems", *Acta Automatica sinica*, 34, pp. 984-989, 2008.

[9] K. Alimhan, H. Inaba, "Practical output tracking by smooth output compensator for uncertain nonlinear systems with unstabilisable and undetectable linearization", *Int. J. of Modelling, Identification and Control*, 5 , pp.1-13, 2008.

[10] K. Alimhan, H. Inaba, "Robust practical output tracking by output compensator for a class of uncertain inherently nonlinear systems", *Int. J. of Modelling, Identification and Control*, 4, (2008), pp.304-314.

[11] W.P. Bi, J.F. Zhang, "Global practical tracking control for high-order nonlinear uncertain systems", *Proc. of the Chinese Control and Decision Conference*, pp.1619-1622, 2010.

[12] W. Lin, C.J. Qian, "Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems", *Systems & Control Letters*, 39, pp.339−351, 2000.

[13] C.J. Qian, W. Lin, "Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization", *Systems & Control Letters*, 42, (2001), pp.185−200.

[14] C.J. Qian, W. Lin, "A continuous feedback approach to global strong stabilization of nonlinear systems", *IEEE Trans. on Automatic Control*, 46, pp.1061−1079, 2001.

[15] B. Yang, W. Lin, "Homogeneous observers, Iterative design, and global stabilization of high-order nonlinear systems by output feedback", *IEEE Trans. on Automatic Control*, 49, pp.1069-1080, 2004.

[16] B. Yang, W. Lin, "Robust output feedback stabilization of uncertain nonlinear systems with uncontrollable and unobservable linearization", *IEEE Trans. on Automatic Control*, 50, pp.619-630, 2005.

[17] C. Qian, W. Lin, Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems, *IEEE Trans. on Automatic Control*, 51, 1457-1471, 2006.

[18] J. Polendo, C. Qian, "A generalized homogeneous domination approach for global stabilization of inherently non-linear systems via output feedback", *Int. J. of Robust Nonlinear Control*, 17, pp.605–629, 2007.

[19] M. Kawski, 'Homogeneous stabilizing feedback laws', *Control Theory Adv. Technol*., Vol. 6, pp. 497-516, 1990.

[20] H. Hermes, 'Homogeneous coordinates and continuous asymptotically stabilizing feedback controls', *Differential equations (Colorado Springs, Co, 1989)*, *Lecture Notes in Pure and Applied Mathematics*, Vol. 127, pp. 249-260, Dekker, New York, 1991.

[21] C. Qian, 'A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems', *Proc. American control conference*, pp.4708-4715, 2005.

[22] C. Qian and W. Lin, 'Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems', *IEEE Trans. Automat. Contr.*, Vol. 51, pp. 1457-1471, 2006.

[23] J. Polendo, C. Qian, "A universal method for robust stabilization of nonlinear systems: unification and extension of smooth and non-smooth approaches", *Proc. of the American Control Conference*, pp.4285-4290, 2006.

[24] X. Huang, W. Lin and B. Yang, "Finite-time stabilization in the large for uncertain nonlinear systems", *Proc. of the American Control Conference*, pp.1073–1078, 2004.

[25] X.-H. Zhang and X.-J. Xie, "Global state feedback stabilization of nonlinear systems with high-order and low-order nonlinearitiesities," *Int. J. of Control*, 87, pp.642–652, 2014.

[26] F.-Z. Gao and Y.-U. Wu, Global stabilisation for a class of more general high-order time-delay nonlinear systems by output feedback, *Int. J. of Control*, Vol.88, No.8, 1540-1553, 2015.