Asymptotic Solutions of Integral Boundary Problem

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Abstract— This work is devoted to the asymptotic solutions of integral boundary value problem for the Inter-linear second order differential equation of Fredholm type. Studying an integral boundary value task, obtaining solution assessment of the set singular perturbed integral boundary value problem and difference estimate between the solutions of singular perturbed and unperturbed tasks; determination of singular perturbed integral boundary value problem solution behavior mode and its derivatives in discontinuity (jump) of the considered section and determination of the solution initial jumps values at discontinuity and of an integral member of the equation, as well, creation of asymptotic solution expansion assessing a residual member with any range of accuracy according to a small parameter by means of Cauchy task with an initial jump, at that selection of initial conditions due to singular perturbed boundary value problem solution behavior mode and its derivatives in the jump point. In the paper there applied methods of differential and integral equations theories, boundary function method, method of successive approximations and method of mathematical induction.

Keywords—Asymptotic solutions, differential equations, integral boundary problem, integro-partial differential equations, method substantiation, singular, small parameter. About four key words or phrases in alphabetical order, separated by commas.

I. Introduction

THEME actuality. Overall interest of mathematicians in singular perturbed equations determined with the fact, that they function as mathematical models in many applied tasks, connected with processes of diffusion, heat and mass transfer, in chemical kinetics and combustion, in the problems of heat distribution in slender bodies, in semiconductor theories, quantum mechanics, biology and biophysics and many other branches of science and engineering. Singular perturbed equations is an important class of differential equations.

Theory of singular perturbed equations developed in the 50-th, starting with fundamental works of an academician of Russian Academy of Sciences Tikhonov A.N [1-3], considering the qualitative theory of ordinary differential equations with small parameter upon derivatives. Tikhonov A.N. proved the theorem on limit transfer, establishing a connection between the solution of degenerated (unperturbed) problem, obtained from the initial task upon a small parameter similar to zero and from solution of initial singular perturbed tasks for the systems of nonlinear ordinary differential equations.

Member-Correspondent of RAS Lyusternik L.A. and Vishik M.I. [4,5] elaborated effective asymptotic technique of singular perturbed linear ordinary differential equations and the ones in partial derivatives; Vasiljyeva A.B. [6] developed asymptotic technique of solution of initial problem researched by Tikhonov A.N. RAS Member-Correspondent Imanaliyev M.I. [7] elaborated asymptotic technique of solution of singular perturbed systems of nonlinear integro-partial differential equations, which gained further development in many subsequent works and got named the boundary function method. At present, there are various modifications of the method thereof in the works of Trenogin V.A. [8], Butuzov V.F. [9] and the others. In particular, there offered an angular boundary function method for equations in partial derivatives in the areas, the boundaries of which contain angular points.

II. Asymptotic Solutions of Integral Boundary Value Problem

Setting up a problem. Let us consider the following Fredholm-type integro- partial differential equation

$$L_\varepsilon y \equiv \varepsilon y'' + A(t)y' + B(t)y = \\ = F(t) + \int_0^1 \left[K_0(t,x)y(x,\varepsilon) + K_1(t,x)y'(x,\varepsilon)\right]dx \tag{1}$$

with integral boundary conditions

$$y(0,\varepsilon) = a_0 + \int_0^1 \left[b_0(t)y(t,\varepsilon) + b_1(t)y'(t,\varepsilon)\right]dt, \\ y(1,\varepsilon) = a_1 + \int_0^1 \left[c_0(t)y(t,\varepsilon) + c_1(t)y'(t,\varepsilon)\right]dt, \tag{2}$$

where $\varepsilon > 0$ - a small parameter, $a_i, i = 0,1$ – certain acquainted permanents, $A(t), B(t), F(t), b_i(t), c_i(t)$, $K_i(t,x),\ i = 0,1$ – certain acquainted functions, defined in the domain $D = (0 \le t \le 1, 0 \le x \le 1)$ [10].

Solution of $y(t,\varepsilon)$ singular perturbed integral boundary value problem (1), (2) at ε small parameter vanishing will not tend to a solution $\bar{y}(t)$ of an ordinary unperturbed (singular) task, obtained from (1), (2) at $\varepsilon = 0$:

$$L_0\bar{y} \equiv A(t)\bar{y}' + B(t)\bar{y} = F(t) + \int_0^1 \left[K_0(t,x)\bar{y}(x) + K_1(t,x)\bar{y}'(x)\right]dx$$

with boundary condition at $t = 0$ or at $t = 1$, but tends to the solution $y_0(t)$, changed, unperturbed equation

$$L_0 y_0(t) = F(t) + \Delta(t) + \int_0^1 \left[K_0(t,x)y_0'(x) + K_1(t,x)y_0'(x)\right]dx \tag{3}$$

with changed boundary condition at the point $t = 0$;

$$y_0(0) = a_0 + \Delta_0 + \int_0^1 \left[b_0(t)y_0(t) + b_1(t)y_0(t)\right]dt \tag{4}$$

or with changed boundary condition at the point $t = 1$:

$$y_0(1) = a_1 + \Delta_1 + \int_0^1 \left[c_0(t)y_0(t) + c_1(t)y_0(t)\right]dt \tag{5}$$

Assume, that:

I. Functions $A(t), B(t), F(t), b_i(t), c_i(t)$ и $K_i(t,x)$, $i = 0,1$ are sufficiently smooth in the domain $D = (0 \le t \le 1,\ 0 \le x \le 1)$;

II. Function $A(t)$ at the segment [0,1] satisfies inequation: $A(t) \ge \gamma \equiv const > 0,\ 0 \le t \le 1$;

III. Number $\lambda = 1$ at sufficiently small ε is not a proper value of the kernel $J(t,s,\varepsilon)$:

$$J(t,s,\varepsilon) = \bar{J}(t,s) + \mathrm{O}\left(\varepsilon + e^{\left(-\frac{1}{\varepsilon}\int_s^1 A(x)dx\right)}\right) =$$

$$= \frac{1}{A(t)}\left[K_1(t,s) + \int_s^1 \left[K_0(t,x)K(x,s) + K_1(t,x)K'(x,s)\right]dx\right] + \mathrm{O}\left(\varepsilon + e^{\left(-\frac{1}{\varepsilon}\int_s^1 A(x)dx\right)}\right),$$

where the function $K(t,s)$ expressed with a formula

$$K(t,s) = \exp\left(\int_s^t -\frac{B(x)}{A(x)}dx\right).$$

IV. True an inequation:

$$\Delta_0^0 \equiv 1 - \int_0^1 \left[\left(b_0(s)K(s,0) + b_1(s)K'(s,0)\right) + \left(b_1(s) + \int_s^1 \left(b_0(t)K(t,s) + b_1(t)K'(t,s)\ dt\right)\right)\right]\sigma(s)\ ds \ne 0,$$

where function $\sigma(t)$ has a view

$$\sigma(t) = \varphi(t) + \int_0^1 R(t,s)\varphi(s)ds,$$

and function $R(t,s)$ –kernel resolvent $\bar{J}(t,s)$, and function $\varphi(t)$ may be represented as

$$\varphi(t) = \frac{1}{A(t)}\int_0^1 \left[K_0(t,x)K(x,0) + K_1(t,x)K'(x,0)\right]dx.$$

V. True an equation:

$$\Delta_1^0 \equiv N\left(K(1,0) + \int_0^1 K(1,s)\sigma(s)ds\right) + \frac{1}{A(0)}\int_0^1 K(1,s)\left(\frac{K_1(s,0)}{A(s)} + \int_0^1 \frac{R(s,x)K_1(x,0)}{A(x)}dx\right)ds - \frac{c_1(0)}{A(0)} -$$

$$-\int_0^1 c_0(t)\left[N\left(K(t,0) + \int_0^t K(t,s)\sigma(s)ds\right) + \frac{1}{A(0)}\int_0^t K(t,s)\left(\frac{K_1(s,0)}{A(s)} + \int_0^1 \frac{R(s,x)K_1(x,0)}{A(x)}dx\right)ds\right]dt -$$

$$-\int_0^1 c_1(t)\left[N\left(K'(t,0) + \sigma(t) + \int_0^t K'(t,s)\sigma(s)ds\right) + \frac{1}{A(0)}\left(\frac{K_1(t,0)}{A(t)} + \int_0^1 \frac{R(t,x)K_1(x,0)}{A(x)}dx\right) +\right.$$

$$\left. + \frac{1}{A(0)}\int_0^t K'(t,s)\left(\frac{K_1(s,0)}{A(s)} + \int_0^1 \frac{R(s,x)K_1(x,0)}{A(x)}dx\right)ds\right]dt \ne 0,$$

where

$$N = \frac{1}{\Delta_0^0}\left[\frac{1+b_1(0)}{A(0)} + \frac{1}{A(0)}\int_0^1\left(b_1(s) + \int_s^1 \left(b_0(t)K(t,s) + b_1(t)K'(t,s)\right)dt\right) \times \right.$$

$$\left. \times \left(\frac{K_1(s,0)}{A(s)} + \int_0^1 \frac{R(s,x)K_1(x,0)}{A(x)}dx\right)ds\right].$$

With reference to 1.7, section 1 it can be seen, that solution $y(t,\varepsilon)$ of integral boundary value problem (1), (2) at the point $t = 0$ is limited and its first derivative $y'(t,\varepsilon)$ at the point $t = 0$ has unlimited growth of the order $\mathrm{O}\left(\frac{1}{\varepsilon}\right)$ at $\varepsilon \to 0$. Hence, for creating the boundary value problem solution asymptotics (1), (2), let's preliminarily consider an auxiliary Cauchy problem with an initial jump, i.e., consider an equation (1) with initial conditions at the point $t = 0$:

$$y(0,\varepsilon) = a_0 + \int_0^1 \left[b_0(t)y(t,\varepsilon) + b_1(t)y'(t,\varepsilon)\right]dt, \tag{6}$$

$$y'(0,\varepsilon) = \frac{\alpha}{\varepsilon},$$

where $\alpha = \alpha(\varepsilon)$ – regularly dependent on ε permanent, represented as:

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots \tag{7}$$

Let us define $\alpha(\varepsilon)$ in such a way, that the solution $y(t,\alpha,\varepsilon)$ of the problem (1), (6) was the solution of boundary value task (1), (2), i.e., to fulfill the second condition (2):

$$y(1,\alpha,\varepsilon) = a_1 + \int_0^1 \left[c_0(t)y(t,\alpha,\varepsilon) + c_1(t)y'(t,\alpha,\varepsilon)\right]dt. \tag{8}$$

Creating asymptotic solution of Cauchy problem with an initial jump. Solution $y(t,\varepsilon) = y(t,\alpha,\varepsilon)$ of Cauchy problem (2.1), (2.6) we will search as the sum:

$$y(t,\varepsilon) = y_\varepsilon(t) + w_\varepsilon(\tau), \tag{9}$$

where $\tau = t/\varepsilon$ - boundary-layer independent invariable, $y_\varepsilon(t)$ -solution's regular part, defined at the section [0, 1] and $w_\varepsilon(\tau)$ -boundary-layer part of the solution, defined at $\tau \ge 0$.

Preliminarily multiply equations (1) by ε and further insert formula (9) into equation (1). Thereupon we obtained:

$$\varepsilon^2 y''_\varepsilon(t) + \varepsilon A(t) y'_\varepsilon(t) + \varepsilon B(t) y_\varepsilon(t) + \ddot{w}_\varepsilon(\tau) + A(\varepsilon\tau)\dot{w}_\varepsilon(\tau) +$$
$$+ \varepsilon B(\varepsilon\tau) w_\varepsilon(\tau) = \varepsilon F(t) + \varepsilon \int_0^1 \left[K_0(t,x) y_\varepsilon(x) + K_1(t,x) y'_\varepsilon(x) \right] dx +$$
$$+ \varepsilon \int_0^1 \left[K_0(t,x) w_\varepsilon\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} K_1(t,x) \dot{w}_\varepsilon\left(\frac{x}{\varepsilon}\right) \right] dx,$$

where the point $(\cdot)$– a derivative per τ beyond integral members and a derivative per x in integral members. If we make replacement $x = \varepsilon s, \quad dx = \varepsilon ds, \quad 0 \le s \le \frac{1}{\varepsilon} = \infty$, then from here it follows that:

$$\varepsilon^2 y''_\varepsilon(t) + \varepsilon A(t) y'_\varepsilon(t) + \varepsilon B(t) y_\varepsilon(t) + \ddot{w}_\varepsilon(\tau) + A(\varepsilon\tau)\dot{w}_\varepsilon(\tau) + \varepsilon B(\varepsilon\tau) w_\varepsilon(\tau) =$$
$$= \varepsilon F(t) + \varepsilon \int_0^1 \left[K_0(t,x) y_\varepsilon(x) + K_1(t,x) y'_\varepsilon(x) \right] dx +$$
$$+ \varepsilon \int_0^\infty \left[\varepsilon K_0(t,\varepsilon s) w_\varepsilon(s) + K_1(t,\varepsilon s) \dot{w}_\varepsilon(s) \right] ds. \qquad (10)$$

And now let's write out separately the equations with factors, dependent on t, and separately equation with factors dependent on τ [11]. Then from (10) we obtain following equations separately for $y_\varepsilon(t)$ and separately for $w_\varepsilon(t)$:

$$\varepsilon y''_\varepsilon(t) + A(t) y'_\varepsilon(t) + B(t) y_\varepsilon(t) = F(t) +$$
$$+ \int_0^1 \left[K_0(t,x) y_\varepsilon(x) + K_1(t,x) y'_\varepsilon(x) \right] dx + \qquad (11)$$
$$+ \int_0^\infty \left[\varepsilon K_0(t,\varepsilon s) w_\varepsilon(s) + K_1(t,\varepsilon s) \dot{w}_\varepsilon(s) \right] ds;$$

$$\ddot{w}_\varepsilon(\tau) + A(\varepsilon\tau)\dot{w}_\varepsilon(\tau) + \varepsilon B(\varepsilon\tau) w_\varepsilon(\tau) = 0. \qquad (12)$$

Let's insert (9) into initial conditions (6), (7):

$$y_\varepsilon(0) + w_\varepsilon(0) = a_0 + \int_0^1 \left[b_0(t) y_\varepsilon(t) + b_1(t) y'_\varepsilon(t) + b_0(t) w_\varepsilon\left(\frac{t}{\varepsilon}\right) + \frac{1}{\varepsilon} b_1(\varepsilon\tau) \dot{w}_\varepsilon\left(\frac{t}{\varepsilon}\right) \right] dt =$$
$$= \left(\tau = \frac{t}{\varepsilon}, \quad dt = \varepsilon\, d\tau, \quad 0 \le \tau \le \frac{1}{\varepsilon} = \infty \right) = a_0 + \int_0^1 \left[b_0(t) y_\varepsilon(t) + b_1(t) y'_\varepsilon(t) \right] dt + \qquad (13)$$
$$+ \int_0^\infty \left[\varepsilon b_0(\varepsilon\tau) w_\varepsilon(\tau) + b_1(\varepsilon\tau) \dot{w}_\varepsilon(\tau) \right] d\tau,$$
$$y'_\varepsilon(0) + \frac{1}{\varepsilon} \dot{w}_\varepsilon(0) = \frac{1}{\varepsilon} \left[\alpha_0 + \varepsilon\alpha_1 + \varepsilon^2 \alpha_2 + \cdots \right].$$

Solutions $y_\varepsilon(t)$ and $w_\varepsilon(\tau)$ of equations (11) and (12) we will search as following series in terms of a small parameter ε:

$$y_\varepsilon(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots,$$
$$w_\varepsilon(t) = w_0(t) + \varepsilon w_1(t) + \varepsilon^2 w_2(t) + \cdots. \qquad (14)$$

Functions $A(\varepsilon\tau), \quad B(\varepsilon\tau), \quad b_i(\varepsilon\tau)$, and $K_i(t,\varepsilon s)$ expand in series of Taylor development:

$$A(\varepsilon\tau) = A(0) + \frac{A'(0)}{1!}\varepsilon\tau + \ldots + \frac{A''(0)}{2!}(\varepsilon\tau)^2 + \ldots,$$
$$B(\varepsilon\tau) = B(0) + \frac{B'(0)}{1!}\varepsilon\tau + \ldots + \frac{B''(0)}{2!}(\varepsilon\tau)^2 + \ldots,$$
$$b_i(\varepsilon\tau) = b_i(0) + \frac{b'_i(0)}{1!}\varepsilon\tau + \ldots + \frac{b''_i(0)(\varepsilon\tau)^2}{2!} + \ldots, \quad i = 0,1, \qquad (15)$$
$$K_i(t,\varepsilon s) = K_i(t,0) + \frac{K'_i(t,0)\varepsilon s}{1!} + \ldots + \frac{K''_i(t,0)}{2!}(\varepsilon s)^2 + \ldots \quad i = 0,1.$$

Inserting transformations in due form (14), (15) in the equations (11), (12) we obtain

$$\varepsilon(y''_0(t) + \varepsilon y''_1(t) + \varepsilon^2 y''_2(t) + \ldots) + A(t)(y'_0(t) + \varepsilon y'_1(t) + \varepsilon^2 y'_2(t) + \cdots) + B(t)(y_0(t) + \varepsilon y_1(t) +$$
$$+ \varepsilon^2 y_2(t) + \cdots) = F(t) + \int_0^1 \Big[K_0(t,x)(y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots) + K_1(t,x)(y'_0(x) + \qquad (16)$$
$$+ \varepsilon y'_1(x) + \varepsilon^2 y'_2(x) + \cdots) \Big] dx + \int_0^\infty \left[\varepsilon \left(K_0(t,0) + \varepsilon s K'_0(t,0) + \frac{(\varepsilon s)^2 K''_0(t,0)}{2!} + \cdots \right) \times \right.$$
$$\times \left(w_0(s) + w_1(s)\varepsilon + w_2(s)\varepsilon^2 + \cdots \right) + \left(K_1(t,0) + \varepsilon s K'_1(t,0) + \frac{(\varepsilon s)^2 K''_1(t,0)}{2!} + \cdots \right) \times$$
$$\left. \times \left(\dot{w}_0(s) + \dot{w}_1(s)\varepsilon + \dot{w}_2(s)\varepsilon^2 + \cdots \right) \right] ds;$$

$$\ddot{w}_0(\tau) + \varepsilon \ddot{w}(\tau) + \varepsilon^2 \ddot{w}_2(\tau) + \cdots + (A(0) + \frac{A'(0)}{1!}\tau\varepsilon + \frac{A''(0)}{2!}(\tau\varepsilon)^2 + \ldots) \times$$
$$\times (\dot{w}_0(\tau) + \varepsilon \dot{w}_1(\tau) + \varepsilon^2 \dot{w}_2(\tau) + \cdots) + \varepsilon (B(0) + \frac{B'(0)}{1!}\tau\varepsilon + \frac{B''(0)}{2!}(\tau\varepsilon)^2 + \ldots) \times \qquad (17)$$
$$\times (w_0(\tau) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau) + \cdots) = 0.$$

Similarly insert transformations (14), (15) into initial conditions (13). Thereupon we obtain:

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots + w_0(0) + \varepsilon w_1(0) + \varepsilon^2 w_2(0) + \cdots =$$
$$= a_0 + \int_0^1 \left[b_0(t)(y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots) + b_1(t)(y'_0(t) + \varepsilon y'_1(t) + \varepsilon^2 y'_2(t) + \cdots) \right] dt + \qquad (18)$$
$$+ \int_0^\infty \left[\varepsilon \left(\left(b_0(0) + b'_0(0)\varepsilon\tau + \frac{b''_0(0)}{2!}(\varepsilon\tau)^2 + \cdots \right)(w_0(\tau) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau) + \cdots) \right) + \right.$$
$$\left. + \left(b_1(0) + b'_1(0)\varepsilon\tau + \frac{b''_1(0)}{2!}(\varepsilon\tau)^2 + \cdots \right)(\dot{w}_0(\tau) + \varepsilon \dot{w}_1(\tau) + \varepsilon^2 w'_2(\tau) + \cdots) \right] d\tau,$$
$$y'_0(0) + \varepsilon y'_1(0) + \varepsilon^2 y'_2(t) + \cdots + \frac{1}{\varepsilon}(\dot{w}_0(0) + \varepsilon \dot{w}_1(0) + \varepsilon^2 w'_2(0) + \cdots) =$$
$$= \frac{1}{\varepsilon}(\alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots).$$

We compare factors at like powers ε. Thereupon from (16) we obtain

$$A(t) y'_0(t) + B(t) y_0(t) = F(t) + \int_0^1 \left[K_0(t,x) y_0(x) + K_1(t,x) y'_0(x) \right] dx + \Delta_0(t), \qquad (19_0)$$

$$A(t) y'_k(t) + B(t) y_k(t) = F_k(t) +$$
$$+ \int_0^1 \left[K_0(t,x) y_k(x) + K_1(t,x) y'_k(x) \right] dx + \Delta_k(t), \quad k \ge 1 \quad , \qquad (19_k)$$

where

$$\Delta_0(t) = \int_0^\infty K_1(t,0) \dot{w}_0(s)\, ds, \quad \Delta_k(t) = \int_0^\infty K_1(t,0) \dot{w}_k(s)\, ds,$$

$$F_k(t) = -y''_{k-1}(t) + \int_0^\infty \left[\left(K_0(t,0) w_{k-1}(s) + s K'_0(t,0) w_{k-2}(s) + \frac{s^2}{2!} K''_0(t,0) w_{k-1}(s) + \right. \right. \qquad (20)$$
$$\left. \left. + \cdots + \frac{s^{k-1}}{(k-1)!} K_0^{(k-1)}(t,0) w_0(s) \right) + \left(s K'_1(t,0) \dot{w}_{k-1}(s) + \frac{s^2}{2!} K''_1(t,0) \dot{w}_{k-2}(s) + \cdots + \frac{s^k}{k!} K_1^{(k)}(t,0) \dot{w}_0(s) \right) \right] ds,$$
$$k \ge 1.$$

Similarly, dealing with (2.17) we will have:

$$\ddot{w}_0(\tau) + A(0)\dot{w}_0(\tau) = 0, \qquad (21_0)$$
$$\ddot{w}_k(\tau) + A(0)\dot{w}_k(\tau) = \Phi_k(\tau), \qquad (21_k)$$

where a function $\Phi_k(\tau)$ is expressed through $w_i(\tau), \quad i < k$:

$$\Phi_k(\tau) = -\left[\tau A'(0)\dot{w}_{k-1}(\tau) + \cdots + \frac{\tau^k}{k!}A^{(k)}(0)\dot{w}_0(\tau)\right] - \tag{22}$$
$$-\left[B(0)w_{k-1}(\tau) + \tau B'(0)w_{k-2}(\tau) + \cdots + \frac{\tau^{k-1}}{(k-1)!}B^{(k-1)}(0)w_0(\tau)\right], \quad k \ge 1.$$

Now we compare in (2.18) the factors at like powers ε :

$$y_0(0) + w_0(0) = a_0 + \int_0^1 \left[b_0(t)y_0(t) + b_1(t)y_0'(t)\right]dt + b_1(0)\int_0^\infty \dot{w}_0(\tau)\,d\tau, \quad \dot{w}_0(0) = \alpha_0; \tag{23_0}$$

$$y_k(0) + w_k(0) = \int_0^1 \left[b_0(t)y_k(t) + b_1(t)y_k'(t)\right]dt + b_1(0)\int_0^\infty \dot{w}_k(\tau)d\tau + \int_0^\infty \left[\left(b_0(0)w_{k-1}(\tau) + \right.\right. \tag{23_k}$$
$$\left.\left. + \tau b_0'(0)w_{k-2}(\tau) + \cdots + \frac{\tau^{k-1}b_0^{(k-1)}(0)}{(k-1)!}w_0(\tau)\right) + \left(\tau b_1'(0)\dot{w}_{k-1}(\tau) + \cdots + \frac{\tau^k}{k!}b_k^{(k)}(0)\dot{w}_0(\tau)\right)\right]d\tau,$$
$$y_{k-1}'(0) + \dot{w}_k(0) = \alpha_k, \qquad k \ge 1.$$

Let us consider the problems (19_0), (21_0) and (23_0), defining zero-order approximations $y_0(t)$ and $w_0(\tau)$. It follows from here, that supplementary condition (23_0) is insufficient unambiguous definition $y_0(t)$ and $w_0(\tau)$ from equations (19_0), (21_0). For zero-order approximation unambiguous definition there is needed three initial conditions, and initial conditions (23_0) consist of two conditions. For one missing initial condition we use a condition of boundary-layer solution $w_0(\tau)$, i.e.,

$$w_0(\infty) = 0, \qquad \dot{w}_0(\infty) = 0.$$

For this purpose, we integrate an equation (21_0) according to τ from 0 to ∞. Thereafter, owing to boundary layer rating $w_0(\tau)$ we obtain

$$\dot{w}_0(0) + A(0)w_0(0) = 0. \tag{24}$$

From that expression with account of initial condition (23_0) for $w_0(\tau)$ we obtain: $\alpha_0 + A(0)w_0(0) = 0$.

From here we will find the missing initial condition for $w_0(\tau)$:

$$w_0(0) = -\frac{\alpha_0}{A(0)}. \tag{25}$$

Let's refer to the equation (19_0) with an initial condition (23_0), (25). As the function $\Delta_0(t)$, entering into an equation (19_0), is defined with the formula (20), then in virtue of boundary layer rating $w_0(\tau)$ and (25) we have:

$$\Delta_0(t) = -K_1(t,0)w_0(0) = \frac{\alpha_0}{A(0)}K_1(t,0) \tag{26}$$

Thus, equation (19_0) with an initial condition (23_0) due to (25), (26) will be as follows:

$$A(t)y_0'(t) + B(t)y_0(t) = F(t) + \Delta_0(t) + \int_0^1 \left[K_0(t,x)y_0(x) + K_1(t,x)y_0'(x)\right]dx,$$
$$y_0(0) = a_0 + \Delta_0 + \int_0^1 \left[b_0(t)y_0(t) + b_1(t)y_0'(t)\right]dt, \tag{27}$$

where

$$\Delta_0 = \frac{(1+b_1(0))}{A(0)}\alpha_0, \qquad \Delta_0(t) = \frac{\alpha_0}{A(0)}K_1(t,0). \tag{28}$$

By this means, zero-order approximation $y_0(t)$ of the solution regular part $y_\varepsilon(t)$, $0 \le t \le 1$ is defined from integro-differential equation of Fredholm-type first order with an integral initial condition (27), and values of initial jumps at the point $t = 0$ and an integral member of the equation $\Delta_0, \Delta_0(t)$, entering into the problem (27), are defined from the formula (28), and zero-order approximation $w_0(\tau)$ of boundary layer of the solution $w_\varepsilon(\tau)$, $\tau \ge 0$ is defined from the equation (21_0) with initial conditions (23_0), (25) [12]:

$$\ddot{w}_0(\tau) + A(0)\dot{w}_0(\tau) = 0, \qquad w_0(0) = -\frac{\alpha_0}{A(0)}, \qquad \dot{w}_0(0) = \alpha_0. \tag{29}$$

Now, let us consider the problems (19_k), (21_k), (23_k), defining k-approximation $y_k(t), w_k(\tau)$ of the solution $y(t,\varepsilon)$ of Cauchy task with an initial jump (1), (6). From here, it follows that supplementary condition (23_k) is not enough for unambiguous definition of $y_k(t)$ and $w_k(\tau)$. To define the missing initial condition we use the condition of boundary layer rating solution $w_k(\tau)$:

$$w_k(\infty) = \dot{w}_k(\infty) = 0, \qquad k \ge 1.$$

We integrate an equation (21_k) according to τ from 0 to ∞ and use the boundary-layer rating condition. Then we obtain [13]

$$\dot{w}_k(0) + A(0)w_k(0) = -\int_0^\infty \Phi_k(\tau)d\tau, \qquad k \ge 1. \tag{30}$$

If to take into account an initial condition (23_k) for $w_k(0)$, then from here we have an initial condition for $w_k(\tau)$:

$$w_k(0) = -\frac{1}{A(0)}\left[\alpha_k - y_{k-1}'(0) + \int_0^\infty \Phi_k(\tau)d\tau\right], \qquad k \ge 1. \tag{31}$$

Let us point out, that improper integrals, entering into (2.30), (31), converge (see below). The problem (19_k), (23_k) due to the function boundary layer rating $w_k(\tau)$ has a view:

$$A(t)y_k'(t) + B(t)y_k(t) = F_k(t) + \Delta_k(t) + \tag{32}$$
$$+ \int_0^1 \left[K_0(t,x)y_k(x) + K_1(t,x)y_k'(x)\right]dx, \quad k \ge 1.$$

$$y_k(0) = -(1 + b_1(0))w_k(0) + \int_0^1 \left[b_0(t)y_k(t) + b_1(t)y_k'(t)\right]dt + a_k,$$

where $F_k(t)$, $\Delta_k(t)$ expressed by formulae (2.20) and a_k is presented as:

$$a_k = \int_0^\infty \left[\left(b_0(0)w_{k-1}(\tau) + \tau b_0'(0)w_{k-2}(\tau) + \cdots + \frac{\tau^{k-1}b_0^{(k-1)}(0)}{(k-1)!}w_0(\tau)\right) + \right. \tag{33}$$
$$\left. + \left(\tau b_1'(0)\dot{w}_{k-1}(\tau) + \frac{\tau^2}{2!}b_1''(0)\dot{w}_{k-2}(\tau) + \cdots + \frac{\tau^k}{k!}b_1^{(k)}(0)\dot{w}_0(\tau)\right)\right]d\tau.$$

Therefore, the task (21_k), (23_k) for $w_k(\tau)$ with account of (31) receives a view:

$$\ddot{w}_k(\tau) + A(0)\dot{w}_k(\tau) = \Phi_k(\tau),$$
$$w_k(0) = -\frac{1}{A(0)}\left[\alpha_k - y'_{k-1}(0) + \int_0^\infty \Phi_k(\tau)\,d\tau\right], \qquad (34)$$
$$\dot{w}_k(0) = \alpha_k - y'_{k-1}(0), \qquad k \ge 1,$$

where $\Phi_k(\tau)$ expressed by the formula (22) [14].

Thus, k-approximation $y_k(t),\ \ k \ge 1$ of the solution regular part $y_\varepsilon(t),\ \ 0 \le t \le 1$ defined proceeding from the problem (32), k-approximation $w_k(\tau),\ \ k \ge 1$ of the solution boundary-layer part $w_\varepsilon(\tau),\ \ \tau \ge 0$ is defined from the problem (34).

Determination of solution asymptotic coefficients of Cauchy problem with an initial jump

Let us consider the problem (27) to determine zero approximation $y_0(t)$ of the solution regular part $y_\varepsilon(t)$. From here, we have the following task:

$$y'_0(t) + \frac{B(t)}{A(t)}y_0(t) = F_0(t), \qquad (35)$$

$$y_0(0) = a_0 + \Delta_0 + \int_0^1 \left[b_0(t)y_0(t) + b_1(t)y'_0(t)\right]dt,$$

where the function $F_0(t)$ has a view

$$F_0(t) = \frac{1}{A(t)}\left[F(t) + \Delta_0(t)\right] + \frac{1}{A(t)}\int_0^\infty \left[K_0(t,x)y_0(x) + K_1(t,x)y'_0(x)\right]dx, \qquad (36)$$

and $\Delta_0, \Delta_0(t)$ expressed by formulae (28). Solving the equation (35) according to the known formula for the first-order linear equation, we obtain,

$$y_0(t) = y_0(0)K(t,0) + \int_0^t K(t,s)F_0(s)\,ds, \qquad (37)$$

where the function $K(t,s)$ has a view from the condition III.

We insert formula (37) into (36):

$$F_0(t) = \frac{1}{A(t)}\left(F(t) + \Delta_0(t)\right) + \frac{1}{A(t)}\int_0^1 K_0(t,x)\left[y_0(0)K(x,0) + \int_0^x K(x,s)F_0(s)ds\right]dx +$$
$$+\frac{1}{A(t)}\int_0^1 K_1(t,x)\left[y_0(0)K'(x,0) + F_0(x) + \int_0^x K'(x,s)F_0(s)ds\right]dx = \frac{1}{A(t)}\left(F(t) + \Delta_0(t)\right) +$$
$$+\frac{y_0(0)}{A(t)}\int_0^1 \left[K_0(t,x)K(x,0) + K_1(t,x)K'(x,o)\right]dx + \frac{1}{A(t)}\left[\int_0^1 K_0(t,x)\,dx\int_0^x K(x,s)F_0(s)ds + \right.$$
$$\left. + \int_0^1 K_1(t,s)F_0(s)ds + \int_0^1 K_1(t,x)dx\int_0^x K'(x,s)F_0(s)ds\right].$$

Applying Dirichlet formula for double integral of the formula (38) and introducing indication:

$$f_0(t) = \frac{1}{A(t)}(F(t) + \Delta_0(t)) + \frac{y_0(0)}{A(t)}\int_0^1 \left[K_0(t,x)K(x,0) + K_1(t,x)K'(x,0)\right]dx, \qquad (39)$$

we obtain the following integral equation of Fredholm

$$F_0(t) = f_0(t) + \int_0^1 \bar{J}(t,s)F_0(s)ds, \qquad (40)$$

where the kernel $\bar{J}(t,s)$ has a view from the condition III. As the kernel $J(t,s)$ according to the condition III is not located on the integral equation spectrum (40), then integral equation (40) at the section [0,1] has the only solution $F_0(t)$ and is expressed by the formula

$$F_0(t) = f_0(t) + \int_0^1 R(t,s)f_0(s)\,ds, \qquad (41)$$

where $R(t,s)$ - kernel resolvent $\bar{J}(t,s)$. Let us insert now (39) into (41):

$$F_0(t) = \frac{1}{A(t)}(F(t) + \Delta(t)) + \frac{y_0(0)}{A(t)}\int_0^1 \left[K_0(t,x)K(x,0) + K_1(t,x)K'(x,0)\right]dx + \int_0^1 R(t,s) \times$$
$$\times\left[\frac{1}{A(s)}(F(s) + \Delta(s)) + \frac{y_0(0)}{A(s)}\int_0^1 (K_0(s,x)K(x,0) + K_1(s,x)K'(x,0))dx\right]ds = \frac{1}{A(t)}(F(t) + \Delta(t)) +$$
$$+\int_0^1 \frac{R(t,s)}{A(s)}(F(s) + \Delta(s))\,ds + \frac{y_0(0)}{A(t)}\int_0^1 \left[K_0(t,x)K(x,0) + K_1(t,x)K'(x,0)\right]dx + y_0(0)\int_0^1 \frac{R(t,s)}{A(s)}\,ds \times$$
$$\times\int_0^1 (K_0(s,x)K(x,0) + K_1(s,x)K'(x,0))\,dx = \frac{1}{A(t)}(F(t) + \Delta(t)) + \int_0^1 \frac{R(t,s)}{A(s)}(F(s) + \Delta(s))\,ds +$$
$$+ y_0(0)\int_0^1 \left[\frac{1}{A(t)}(K_0(t,x)K(x,0) + K_1(t,x)K'(x,0))\right]dx +$$
$$\left. + \int_0^1 \frac{R(t,s)}{A(s)}(K_0(s,x)K(x,0) + K_1(s,x)K'(x,0))ds\right]dx.$$

Subsequently we have

$$F_0(t) = \omega(t) + y_0(0)\sigma(t), \qquad (42)$$

where the function $\omega(t)$ is expressed by the formula

$$\omega(t) = \frac{1}{A(t)}(F(t) + \Delta_0(t)) + \int_0^1 \frac{R(t,s)}{A(s)}(F(s) + \Delta_0(s))\,ds, \qquad (43)$$

and the function $\sigma(t)$ is presented from the condition IV.

Now the formula (37) in virtue of (2.42) is presented in the form of the formula

$$y_0(t) = y_0(0)\left[K(t,0) + \int_0^t K(t,s)\sigma(s)ds\right] + \int_0^t K(t,s)\omega(s)ds. \qquad (44)$$

Let's define unknown initial condition $y_0(0)$. For that purpose, the formula (44) is inserted into initial condition (35). Then we will have:

$$y_0(0) = a_0 + \Delta_0 + y_0(0)\int_0^1 \left[(b_0(s)K(s,0) + b_1(s)K'(s,0)) + \right.$$
$$\left. + \left(b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s))\,dt\right)\sigma(s)\right]ds +$$
$$+ \int_0^1 \left[b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s))\,dt\right]\omega(s)\,ds$$

Or

$$y_0(0)\left[1 - \int_0^1 \left((b_0(s)K(s,0) + b_1(s)K'(s,0)) + \left(b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s))dt\right)\right)\sigma(s)ds\right] =$$
$$= a_0 + \Delta_0 + \int_0^1 \left[b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s))dt\right]\omega(s)ds.$$

Thereupon we obtain the initial condition $y_0(0)$:

$$y_0(0) = \frac{1}{\Delta_0^0}\left[a_0 + \Delta_0 + \int_0^1 \left(b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s))\,dt\right)\omega(s)\,ds\right], \qquad (45)$$

where $\Delta_0^0 \ne 0$ is expressed by the formula from the condition IV. If an expression (45) is inserted into (37), we obtain the solution $y_0(t)$ of regular part zero approximation of $y_\varepsilon(t)$ solution:

$$y_0(t)=\frac{1}{\Delta_0^0}\left[a_0+\Delta_0+\int_0^1\left(b_1(s)+\int_s^1(b_0(t)K(t,s)+b_1(s)K'(t,s))\,dt\right)\omega(s)\,ds\right]\times$$
$$\times K(t,0)+\int_0^t K(t,s)F_0(s)\,ds \qquad (46)$$

or with account of (42), (45) we have

$$y_0(t)=\frac{1}{\Delta_0^0}\left[a_0+\Delta_0+\int_0^1\left(b_1(s)+\int_s^1(b_0(t)K(t,s)+b_1(s)K'(t,s))\,dt\right)\omega(s)\,ds\right]\times$$
$$\times\left(K(t,0)+\int_0^t K(t,s)\sigma(s)\,ds\right)+\int_0^t K(t,s)\omega(s)\,ds\,, \qquad (46')$$

where the function $\sigma(t)$ is defined by the formula from the condition IV, the function $\omega(t)$ - from the formula (43), which depends on $\Delta_0(t)$ and $\Delta_0,\Delta_0(t)$ - from the formula (28).

Let us consider the problem (29) to determine zero approximation $w_0(\tau),\tau\geq 0$ of boundary layer part of $w_\varepsilon(\tau)$ solution:

$$\ddot{w}_0(\tau)+A(0)\ddot{w}_0(\tau)=0\,,$$
$$\dot{w}_0(0)=-\frac{\alpha_0}{A(0)},\qquad \dot{w}_0(0)=\alpha_0\,. \qquad (47)$$

Hence, directly follows

$$\dot{w}_0(\tau)=\alpha_0 e^{-A(0)\tau}\,. \qquad (48)$$

Integrating the equation (2.48) with an initial condition (47), we obtain

$$w_0(\tau)=-\frac{\alpha_0}{A(0)}e^{-A(0)\tau},\ \tau\geq 0\,. \qquad (49)$$

Therefore, zero approximations $y_0(t),w_0(\tau)$ of solving Cauchy problem with an initial jump (1), (6) are fully defined and expressed by formulae (46), (49).

To create any approximations $w_\kappa(\tau),\kappa\geq 1$ of the solution boundary layer part we turn to the problem (34). Let us consider the problem (34) for κ=1:

$$\ddot{w}_1(\tau)+A(0)\dot{w}_1(\tau)=\Phi_1(\tau)\,,$$
$$w_1(0)=-\frac{1}{A(0)}\left[\alpha_1-y_0'(0)+\int_0^\infty\Phi_1(\tau)\,d\tau\right],$$
$$\dot{w}_1(0)=\alpha_1-y_0'(0)\,,$$

where the function $\Phi_1(\tau)$ due to (22) and (49) has a view

$$\Phi_1(\tau)=-[\tau A'(0)\dot{w}_0(\tau)+B(0)w_0(\tau)]=-\left[\tau A'(0)\alpha_0+B(0)\frac{\alpha_0}{A(0)}\right]e^{-A(0)\tau}\,.$$

From here, it can be seen that the function $\Phi_1(\tau)$ is the first degree boundary-layer multinomial of Lyusternik-Vishik [15]. It follows that, improper integral, entering into an initial condition, converged. It is possible to make sure that the solution $w_1(\tau)$ of the problem thereof has a view:

$$w_1(\tau)=L_1(\tau)e^{-A(0)\tau},\ \tau\geq 0\,,$$

where $L_1(\tau)$ –a certain multinomial as regard to τ . Therefore, the solution $w_1(\tau)$ is a boundary-layer multinomial of Lyusternik-Vishik. Let us assume that, solution of the problem (34) to (κ-1)- approximation is a boundary-layer multinomial of Lyusternik-Vishik

$$w_i(\tau)=L_i(\tau)e^{-A(0)\tau},\ i=\overline{1,\kappa-1},\ \tau\geq 0\,, \qquad (50)$$

where $L_i(\tau)$ –a certain known multinomial according to τ .then it is possible to prove by an induction method that the solution $w_\kappa(\tau)$ to the problem (34) is Lyusternik-Vishik multinomial:

$$w_\kappa(\tau)=L_\kappa(\tau)e^{-A(0)\tau},\ \tau\geq 0\,, \qquad (51)$$

where $L_\kappa(\tau)$ –a known multinomial as regard to τ .

Now let us turn to the problem (33) with account of (50) to define $y_\kappa(t),\kappa\geq 1$ of a regular of the solution $y_\varepsilon(t)$ of Cauchy problem with initial (1), (6):

$$A(t)y_\kappa'(t)+B(t)y_\kappa(t)=F_\kappa(t)+\Delta_\kappa(t)+\int_0^1[K_0(t,x)y_\kappa(x)+K_1(t,x)y_\kappa'(x)]dx$$
$$y_\kappa(0)=a_\kappa+\Delta_\kappa+\int_0^1[b_0(t)y_\kappa(t)+b_1(t,x)y_\kappa'(t)]dt,\ \kappa\geq 1 \qquad (52)$$

where functions $F_\kappa(t),\Delta_\kappa(t)$ are expressed by the formulae (20) and a_κ,Δ_κ has a view:

$$a_\kappa=\int_0^\infty\left[\left(b_0(0)w_{\kappa-1}(\tau)+\tau b_0'(0)w_{\kappa-1}(\tau)+\ldots+\frac{\tau^{(\kappa-1)}}{(\kappa-1)!}b_0^{(\kappa-1)}(0)w_0(\tau)\right)+\right.$$
$$\left.+\left(\tau b_1'(0)\dot{w}_{\kappa-1}(\tau)+\frac{\tau^2}{2!}b_1''(0)\dot{w}_{\kappa-2}(\tau)+\ldots+\frac{\tau^\kappa}{\kappa!}b_1^{(\kappa)}(0)\dot{w}_0(\tau)\right)\right]d\tau\,, \qquad (53)$$
$$\Delta_\kappa=-\frac{(1+b_1(0))}{A(0)}\left[\alpha_\kappa-y_{\kappa-1}'(0)+\int_0^\infty\Phi_\kappa(\tau)d\tau\right],\ \kappa\geq 1\,.$$

Having divided an equation (52) to $A(t)$ and solved it we obtained:

$$y_\kappa(t)=y_\kappa(0)K(t,0)+\int_0^t K(t,s)\overline{F}_\kappa(s)ds\,, \qquad (54)$$

Where the function $\overline{F}_\kappa(t)$ has a view:

$$\overline{F}_\kappa(t)=\frac{1}{A(t)}\left[F_\kappa(t)+\Delta_\kappa(t)+\int_0^1(K_0(t,x)y_\kappa(x)+K_1(t,x)y_\kappa'(x))dx\right]. \qquad (55)$$

We insert (54) into (55). Then for $\overline{F}_\kappa(t)$ we obtain integral equation of Fredholm type (40):

$$\overline{F}_\kappa(t)=f_\kappa(t)+\int_0^1 J(t,s)\overline{F}_\kappa(s)ds\,, \qquad (56)$$

where a constant term $f_\kappa(t)$ has a view:

$$f_\kappa(t)=\frac{1}{A(t)}\left[(F_\kappa(t)+\Delta_\kappa(t))+y_\kappa(0)\int_0^1(K_0(t,x)K(x,0)+K_1(t,x)K'(x,0))dx\right] \qquad (57)$$

An integral equation (56) due to the condition III at the section [0, 1] has the only solution by means of resolvent $R(t,s)$ of the kernel $J(t,s)$:

$$\overline{F}_\kappa(t)=f_\kappa(t)+\int_0^1 R(t,s)f_\kappa(s)ds\,. \qquad (58)$$

Inserting (57) into (58), we obtain

$$\overline{F}_\kappa(t)=\omega_\kappa(t)+y_\kappa(0)\sigma(t)\,, \qquad (59)$$

where $\sigma(t)$ has a view from the condition IV and the function $\omega_\kappa(t)$ is expressed by the formula

$$\omega_\kappa(t) = \frac{1}{A(t)}(F_\kappa(t)+\Delta_\kappa(t)) + \int_0^1 \frac{R(t,s)}{A(s)}(F_\kappa(s)+\Delta_\kappa(s))ds \cdot \qquad (60)$$

Now the formula (54) by virtue of (59) presented as

$$y_\kappa(t) = y_\kappa(0)\left[K(t,0)+\int_0^t K(t,s)\sigma(s)ds\right] + \int_0^t K(t,s)\omega_\kappa(s)ds \cdot \qquad (61)$$

Inserting (61) into an integral initial condition (52), we obtain an equation for determining $y_\kappa(0)$:

$$\Delta_0^0 y_\kappa(0) = a_\kappa + \Delta_\kappa + \int_0^1\left[b_1(s) + \int_s^1 (b_0(t)K(t,s) + b_1(t)K'(t,s)dt)\right]\omega_\kappa(s)ds \cdot$$

As in compliance with the condition IV, the value $\Delta_0^0 \neq 0$, subsequently defining an initial condition $y_\kappa(0)$ and inserting it into the formula (54), we will have the solution $y_\kappa(t)$:

$$y_\kappa(t) = \frac{1}{\Delta_0^0}\left[a_\kappa + \Delta_\kappa + \int_0^1\left(b_1(s) + \int_s^1 (b_0(t)K(t,s)+b_1(t)K'(t,s))\,dt\right)\omega_\kappa(s)\,ds\right]K(t,0) + \int_0^t K(t,s)\overline{F}_\kappa(s)\,ds \qquad (62)$$

or due to (2.59):

$$y_\kappa(t) = \frac{1}{\Delta_0^0}\left[a_\kappa + \Delta_\kappa + \int_0^1\left(b_1(s) + \int_s^1 (b_0(t)K(t,s)+b_1(t)K'(t,s))\,dt\right)\omega_\kappa(s)\,ds\right] \times \left[K(t,0)+\int_0^t K(t,s)\sigma(s)ds\right] + \int_0^t K(t,s)\omega_\kappa(s)ds, \qquad \kappa \geq 1, \qquad (62')$$

where a_κ, Δ_κ and $\omega_\kappa(t)$ are expressed by the formulae (53) and (60), and functions $K(t,s), \sigma(t)$ have a view from the conditions III, IV.

Therefore, any regular part asymptotic approximations $y_\kappa(t), 0 \leq t \leq 1, \kappa \geq 0$ and boundary layer part $w_\kappa(\tau), \tau \geq 0, \kappa \geq 0$ of the solution $y(t,\varepsilon)$ of Cauchy problem with an initial jump (1), (6) are constructed and expressed by formulae (46), (49), (51) and (62) [16].

Lemma 2.1. Coefficients $y_\kappa(t), w_\kappa(\tau), \kappa \geq 0$ of regular part splitting at $0 \leq t \leq 1$ $y_\varepsilon(t)$ and boundary layer at $\tau \geq 0$ of the function part $w_\varepsilon(\tau)$ have following assessments:

$$\left|y_\kappa^{(i)}(t)\right| \leq C, \qquad 0 \leq t \leq 1, \quad i = 0,1,$$

$$\left|\frac{d^i w_\kappa(\tau)}{d\tau^i}\right| \leq CL(\tau)e^{-A(0)\tau}, \qquad \tau \geq 0,\ \kappa \geq 0, \qquad (63)$$

where $A(0) > 0, C > 0$ a certain permanent, independent on ε and $L(\tau)e^{-A(0)\tau}$ –Lyusternik-Vishik boundary-layer multinomial.

Proof. Let us prove the assessments (63) by the method of mathematical induction. First we will prove assessments (63) for zero approximation $y_0(t)$ and $w_0(\tau)$. Let us consider the solution $w_0(\tau)$ of the problem (47), which is distinctly expressed by the formula (49):

$$w_0(\tau) = -\frac{\alpha_0}{A(0)}e^{-A(0)\tau}, \quad \tau \geq 0.$$

The function thereof is Lyusternik-Vishik boundary-layer multinomial of zero degree, i.e., $C = \frac{\alpha_0}{A(0)}, \quad L(\tau) = 1$.

From here there directly follow assessments (63) for $w_0(\tau)$ and $\dot{w}_0(\tau)$ [17].

Let us consider now the formula (46) and its derivative with account of the function $K(t,s)$ from the condition III:

$$y_0'(t) = \frac{1}{\Delta_0^0}\left[a_0 + \Delta_0 + \int_0^1\left(b_1(s) + \int_z^1 (b_0(t)K(t,s)+b_1(t)K'(t,s))\,dt\right)\omega(s)\,ds\right] \times K'(t,0) + F_0(t) + \int_0^t K'(t,s)F_0(s)\,ds \cdot \qquad (64)$$

Herein $\Delta_0^0 \neq 0$ and Δ_0 are accordingly defined from the condition IV and (28), functions $F_0(t), \omega(t)$ are accordingly expressed by the formulae (42), (43), the function $K(t,s)$ takes the form from the condition III, and functions $b_0(t), b_1(t)$ and permanent a_0 - derived boundary conditions (2). From (46), (64) due to the condition I we directly obtain values (63) for $y_0(t)$ and $y_0'(t)$. Therefore, values (63) have been proved for zero approximation $y_0(t)$ and $w_0(\tau)$.

Let us assume, that assessed values (63) are true till (к-1)-approximation inclusively. Let us prove assessed values (63) for к-approximation $y_\kappa(t)$ and $w_\kappa(\tau)$. Proceeding from the formula (51) we directly receive assessed values (63) for $w_\kappa(\tau)$ and $\dot{w}_\kappa(\tau)$. Let us consider the formula (62) and its derivative

$$y_\kappa'(t) = \frac{1}{\Delta_0^0}\left[a_\kappa + \Delta_\kappa + \int_0^1\left(b_1(s) + \int_s^1 (b_0(t)K(t,s)+b_1(t)K'(t,s))\,dt\right)\omega_\kappa(s)\,ds\right]K'(t,0) + \overline{F}_\kappa(t) + \int_0^t K'(t,s)\overline{F}_\kappa(s)\,ds, \qquad (65)$$

where a_κ, Δ_κ and $\omega_\kappa(t)$ are accordingly defined from (53) and (60), and $\overline{F}_\kappa(t)$ is expressed by the formula (59). Evaluating (62), (65) and taking into account the conditions I-IV, (53), (59), (60), we obtain assessed values (63) for $y_\kappa(t)$ and $y_\kappa'(t)$. Lemma has been proved [18].

III. RESULTS

1.There have been created asymptotic Cauchy function representations and boundary functions of an integral boundary value problems by means of fundamental system of singular perturbed linear like differential equation solutions. There obtained Green functions, expressed with Cauchy functions and boundary functions.

2.We obtained analytical solution representation of an integral boundary problem.

3. We received in the space of continuous functions the asymptotic per small parameter solution assessments of integral boundary problem and it is stated that integral

boundary value problem at discontinuity possesses the phenomenon of zero order initial jump.

IV. Conclusion

Work's outcomes represent a theoretical value. We obtained asymptotic formulae for end problem solution Thesis's results can be applied in scientific researches on the singular perturbed equation theory. Obtained asymptotic decomposition solutions are applicable as initial approximations for numerical techniques implementation.

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