Planar Swarming Motion under Single Leader as Nash Equilibrium

Aykut Yıldız and A. Bülent Özgüler

Abstract**—**Two dimensional foraging swarms are modeled as a dynamic noncooperative game played by swarm members, each one of which minimizes its total effort during the journey by controlling its velocity. It is assumed that each member monitors its distance only to the member that starts the journey up front, called the leader. The leader is only concerned with minimizing its total control effort. The foraging location is assumed to be known by all members. It is shown that a unique Nash equilibrium exists under certain assumptions on the nature of relative weighing between the motions along the two coordinates in the plane. The Nash equilibrium displays a number of observed characteristics of biological swarms; for instance, a V-shape formation is preserved during the whole journey.

Keywords**—**Differential game, dynamic multi-agent system, Nash equilibrium, rendezvous problem.

I. INTRODUCTION

A foraging swarm is a collective movement in search of food and is typically represented by migrating birds. The motion of birds takes place in space and is hence three dimensional. The resulting swarm is usually V shaped so that there is a leading bird that is at the very front. It is a leader, not necessarily because it coordinates or commands, but by its geographical position in the swarm. In order to capture mechanism of swarm formation, we have modeled foraging swarms in one dimension as a dynamic noncooperative game in a sequence of articles, [1], [2], and [3], we have shown the existence of Nash equilibria under a number of different assumptions concerning information exchange structure among the group members. We now continue our investigation of swarm formation and extend the results obtained in one of the games in [3] to two dimensions(2-D). The main feature of the game investigated here is that there is a positional (geographical) leader and all the other members in the group interact only with the leader. The nature of the interaction is monitoring their distance to the leader throughout the foraging activity by controlling their velocities. We will limit, for simplicity, the exposition of the results obtained here to 2-D motion.

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A. Yıldız is with Department Of Electrical and Electronics Engineering, TED University, Ankara, 06420, Turkey, (Tel no: +90 312 5850278, e-mail: aykut.yildiz@tedu.edu.tr)

A. B. Özgüler is with Department Of Electrical and Electronics Engineering, İ.D. Bilkent University, Ankara, 06800, Turkey, (e-mail: ozguler@ee.bilkent.edu.tr)

In this modeling exercise our main concern is to capture and explain certain features observed in biological swarms but 2-D swarm models also have many practical applications in robotics [4] and in automated vehicles [5], [6]. The energy minimization idea has been employed in [7] in their study of stability in multi-dimensional swarms. Their idea of ”artificial potential energy” is also used in cost functions used in this study. Research involving game theory for robots also exist such as the investigation of formation control of [8]. Also in [9], game theory is utilized for interpreting dual agent prey-and-predator games. In [10] flocks are modeled as a graph with nodes of agents and our “directed star” information assumption here may be viewed as a particular graph structure among many that are possible. All our assumptions on the information or graph structure are oriented towards obtaining explicit expressions for the trajectories of motion in a Nash equilibrium.

A fundamental difference between 1-D and 2-D (or higher dimensions) is that in 1-D a total ordering relation exists. Among the many possible partial orderings in 2-D, we choose one that allows us to obtain a game with a Nash equilibrium that can explicitly be described. We say that an agent is close to the foraging location if and only if it is closer to it in both horizontal and in vertical directions. (Note that one could use an infinite variety of “notions of closeness” obtained by different norms in 2-D, including the perhaps most natural Euclidean distance.)

The 2-D noncooperative, dynamic, N-person swarm game is defined in the next Section II. Section III contains a concise summary of the main result which describes and illustrates the Nash equilibrium. Section IV is on conclusions. Appendix covers a thorough derivation of optimal paths of agents as well as the proof of existence of a Nash equilibrium.

II. PROBLEM DEFINITION

Our problem of interest is the motion in the x_1x_2-plane of N agents that have a foraging location in mind or in sense. The foraging location is normalized to be the origin $(x_1, x_2) = (0, 0)$ and each agent moves to reach this location with minimum “effort” using their velocities in x_1, x_2 directions as control inputs in a finite time interval. Let “prime” denote“transpose.” If

$$\mathbf{x}^i(t) = \begin{bmatrix} x_1^i(t) & x_2^i(t) \end{bmatrix}', \ \mathbf{u}^i(t) = \begin{bmatrix} u_1^i(t) & u_2^i(t) \end{bmatrix}',$$

is the position and the input vectors of agent-i in the plane, then this agent minimizes

$$L^i = \int_0^T [\frac{(\mathbf{x}^i - \mathbf{x}^1)'Q(\mathbf{x}^i - \mathbf{x}^1)}{2} - \sum_{k=1}^{2} r_k|x_k^i - x_k^1| + \frac{(\mathbf{u}^i)'\mathbf{u}^i}{2}]dt,$$

(1)

where the dependence on t of $\mathbf{x}^i, \mathbf{u}^i$ is suppressed, subject to

$\mathbf{u}^i = \dot{\mathbf{x}}^i$,

for all $i = 1, ..., N$. Together with the boundary conditions of specified $\mathbf{x}^i(0)$ and fixed $\mathbf{x}^i(T) = \mathbf{0}$ for all $i \in \{1, 2, ..., N\}$, this defines a non-cooperative differential game of N players. Note that the agent-1 is distinguished among all as it is only concerned with minimizing its kinetic energy in reaching the foraging location. Each agent-$2, ..., N$ keeps track of its distance to agent-1 and minimizes its total effort which is composed of three components in the time interval $[0, T]$. The attraction component is the integral of the first term, the repulsion component is the integral of the second term, and the kinetic energy is the integral of the last term in the integrand. The attraction and repulsion components penalize proximity to agent-1 and separation from agent-1 and together they can be viewed as an "artificial potential energy" term in the cost, [7]. The weights $Q \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix and $r_1, r_2 \in \mathbb{R}$ are positive constants, which are, *for simplicity*, assumed to be the same for all $i = 2, ..., N$. The assumption that Q is positive definite ensures that each cost L^i is convex without the repulsion term. The repulsion term is the one that makes the existence of a Nash equilibrium considerably more difficult to establish and makes the game more interesting.

III. MAIN RESULTS

Let us assume, without loss of generality, that the leader is in the first quadrant of the x_1x_2-plane at $t = 0$. Then, by the definition of a leader, we have that

$$0 < x_i^1(0) < \min_j\{x_i^j(0)\},\ i = 1, 2,\ j = 2, ..., N. \tag{2}$$

Let

$$Q = \begin{bmatrix} a & \epsilon \\ \epsilon & b \end{bmatrix}, \tag{3}$$

where a, b and $ab - \epsilon^2$ are positive so that the attraction term is a positive definite quadratic form of distances to the leader in horizontal (x_1) and vertical (x_2) directions. It turns out that if $\epsilon = 0$, then the attraction term is decoupled in x_1 and x_2 coordinates so that the agents play a game in two directions simultaneously but independently. If $\epsilon < 0$, then, as long as all agents remain in the first quadrant of the plane, the cross term $\epsilon(x_1^i - x_1^1)(x_2^i - x_2^1)$ acts as a repulsion between agent-i and the leader. If, on the other hand, $\epsilon > 0$, then it has the attraction effect. Let us define two functions of $x \in \mathbb{R}$ by

$$\begin{aligned} f(x) &= \tfrac{sinh[(T-t)\sqrt{x}]}{sinh(T\sqrt{x})}, \\ g(x) &= \tfrac{1}{x}\{1 - \tfrac{sinh[\sqrt{x}(T-t)]}{sinh(\sqrt{x}T)} - \tfrac{sinh(\sqrt{x}t)}{sinh(\sqrt{x}T)}\}. \end{aligned} \tag{4}$$

Theorem 1. Suppose Q is such that $|a - b|$ is sufficiently large. There exists $\epsilon_0 < 0$ such that for every $\epsilon \in (\epsilon_0, 0)$, a Nash equilibrium for the game defined by (1) exists. The Nash equilibrium has the following properties.

P1. Agent-1 remains the leader along both spatial directions throughout the journey.

P2. The leader trajectory and distances of the followers to the leader are given by

$$\begin{aligned} &\mathbf{x}^1(t) = (1 - \tfrac{t}{T})\mathbf{x}^1(0), \\ &\mathbf{x}^i(t) - \mathbf{x}^1(t) = f(Q)[\mathbf{x}^i(0) - \mathbf{x}^1(0)] + g(Q)\mathbf{r} \ \ , 2 \le i \le N, \end{aligned} \tag{5}$$

where $\mathbf{r} = [r_1 \quad r_2]'$ and

$$\begin{aligned} &f(Q) := sinh[\sqrt{Q}(T-t)]sinh(\sqrt{Q}T)^{-1}, \\ &g(Q) := Q^{-1}[I - f(Q) - sinh(\sqrt{Q}t)sinh(\sqrt{Q}T)^{-1}]. \end{aligned}$$

P3. The swarm center $\mathbf{x}^c = (1/N)(\mathbf{x}^1 + ... + \mathbf{x}^N)$ follows the trajectory

$$\begin{aligned} \mathbf{x}^c(t) = \ & [(1 - \tfrac{t}{T})I - f(Q)]\mathbf{x}^1(0) \\ & + \tfrac{1}{N}f(Q)\textstyle\sum_{i=1}^N \mathbf{x}^i(0) \\ & + \tfrac{1}{N}g(Q)\textstyle\sum_{i=2}^N \mathbf{s}^i(0), \end{aligned}$$

where $\mathbf{s}^i(0) = [r_1 sgn(x_1^i - x_1^1) \quad r_2 sgn(x_2^i - x_2^1)]'$.

It follows that, under the assumption (2), whenever the cross terms in the quadratic form has a repulsive effect, then a Nash equilibrium exists. In this Nash equilibrium, the leader follows a straight line trajectory since its optimal speed is zero at all times. The distance of agent-i to the leader has two components. The first component relates the initial distance and the second, the vector $\mathbf{r}$ of weights for repulsion in both directions. The relationships are established through hyperbolic matrix functions f and g of the attraction weight matrix Q. In this Nash solution, the leader remains the leader throughout the journey. However, this is a consequence of the assumption that the repulsion weights r_1 and r_2 are assumed to be uniformly the same for all agents. In the 1-D version of the same game in [3], the weights are allowed to be non-uniform and, in a Nash equilibrium attained for some repulsion weights, a rank change among the followers do occur for some initial conditions.

The necessity of assumptions on the matrix Q is confirmed by simulations and can be supported as follows. In the $(x_1^i - x_1^1)(x_2^i - x_2^1)$-plane the assumptions put a constraint on the shape and the orientation of the level curves $(\mathbf{x}^i - \mathbf{x}^1)'Q(\mathbf{x}^i - \mathbf{x}^1) =$ *constant*, which are ellipses. The assumption $\epsilon < 0$ holds if and only if the major axis of the ellipse is in the first and third quadrants. The assumptions that $|a - b|$ is large and $|\epsilon|$ is small together ensure that the major axis is not in the vicinity of the line of angle $\pi/4$. Both assumptions are thus, intuitively, slowing down the speed of approach of agent-i to the leader and preventing a change of rank, which is of course necessary for an admissible equilibrium.

We also mention that the Nash solution of Theorem 1 can be shown to be unique with respect to strategies (choice of inputs) that are continuous functions of initial positions.

Examples. Let us choose $T = 1$, $r_1 = 2$, $r_2 = 4$, $a = 20$, $b = 5$, $N = 21$, $\mathbf{x}_1(0) = [10, 12, ..., 30, 14, 18, ..., 50]$, $\mathbf{x}_2(0) = [10, 14, ..., 50, 12, 14, ..., 30]$ This choice of initial positions places the swarm members in a V-formation and agent-1 as the leader. In Fig. 1, $\epsilon = -0.5$ and the Nash equilibrium obtained is such that there is no change of order. In Fig. 2, the choice $\epsilon = 30$ results in trajectories $\mathbf{x}^i(t)$ of (**P2**) above, which violate the postulate of "no order change" through which those expressions are obtained. Note that the

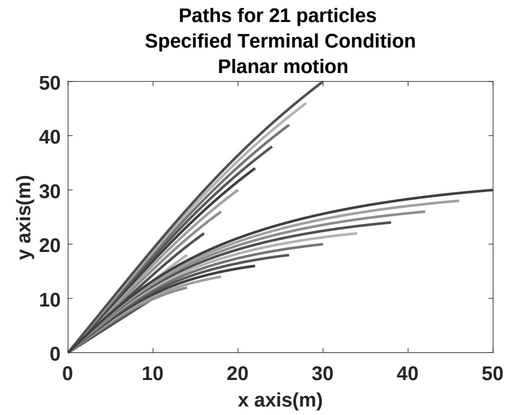

Fig. 1: Admissible paths due to no change of leader for $\epsilon = -0.5$

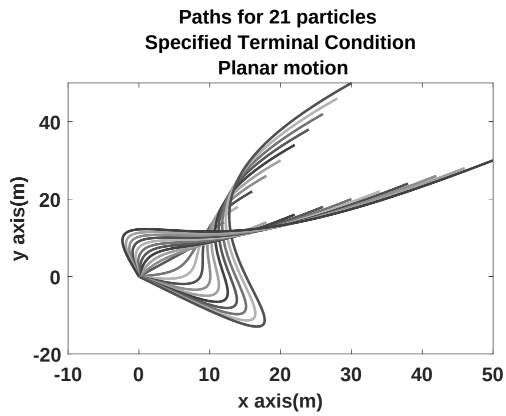

Fig. 2: Non admissible paths due to change of leader for $\epsilon = 30$

fact that paths of two agents intersect implies that there is a change of order between these two agents. The resulting swarming motion is not a Nash equilibrium (although it may be optimal in some sense).

IV. CONCLUSIONS

It is reassuring that a noncooperative dynamic game results in a Nash equilibrium and a swarming behavior in 2-D. Although, we have here extended only one of the results that have been obtained earlier for 1-D in this work, all other games (with different assumptions of information structure as well as foraging location) of [1]-[3] that do yield a Nash equilibrium should also be amenable to extension. It would be challenging to investigate whether other notions of closeness also result in a Nash equilibrium and, if so, what type of swarm formations in the plane and space they would yield.

APPENDIX

The existence proof of the Nash equilibrium of Theorem 1 and the derivation of trajectory expressions are given below.

We first employ the necessary conditions of optimality for each cost function, see e.g., [11], [12]. Consider the Hamiltonian $H^i =$

$$(\mathbf{p}^i)'\mathbf{u}^i + \frac{(\mathbf{u}^i)'\mathbf{u}^i + (\mathbf{x}^i - \mathbf{x}^1)'Q(\mathbf{x}^i - \mathbf{x}^1)}{2} - \sum_{k=1}^{2} r_k |x_k^i - x_k^1|,$$

where $\mathbf{p}^i$ is the co-state associated with agent-i. Necessary conditions of optimality

$$\frac{\partial H^i}{\partial \mathbf{u}^i} = \mathbf{0}, \dot{\mathbf{p}^i} = \frac{-\partial H^i}{\partial \mathbf{x}^i},$$

yield

$$\begin{aligned} &\mathbf{u}^i = -\mathbf{p}^i, \\ &\dot{\mathbf{p}}^i = -Q(\mathbf{x}^i - \mathbf{x}^1) + Rsgn(\mathbf{x}^i - \mathbf{x}^1), \end{aligned}$$

for our Hamiltonian, where $R = diag[r_1, r_2]$. These differential equations coupled with $\mathbf{u}^i = \dot{\mathbf{x}}^i$, $i \in \{1, 2, ..., N\}$, result in a nonlinear state equation having signum type nonlinearity [13]. Let $A := W \otimes Q$, the Kronecker product, where $W \in \mathbb{R}^{N \times N}$,

$$W = \begin{bmatrix} 0 & \mathbf{0}' \\ -\mathbf{e} & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & ... & 0 \\ -1 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & ... & 1 \end{bmatrix}, \tag{6}$$

where $\mathbf{e}$ is a column vector of all ones with length $N-1$ and $\mathbf{0}$, of all zeros.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \mathbf{s}(t), \tag{7}$$

where the identity matrices have sizes $2N$ and

$$\begin{aligned} &\mathbf{x} = [\mathbf{x}^1 \ \mathbf{x}^2 \ ... \ \mathbf{x}^N]', \\ &\mathbf{p} = [\mathbf{p}^1 \ \mathbf{p}^2 \ ... \ \mathbf{p}^N]', \\ &\mathbf{s} = [\mathbf{s}^1 \ \mathbf{s}^2 \ ... \ \mathbf{s}^N]', \end{aligned}$$

are all $2N$-vectors with $\mathbf{s}^i(t) := R\, sgn(\mathbf{x}^i - \mathbf{x}^1)$. Note that W is diagonalizable with $W = USU^{-1}$, where U is a matrix with entries in first column equal to 1, diagonal entries equal to 1, and all other entries equal to 0 and S is a matrix given by $S = diag\{0, I\}$. The matrix Q is also diagonalizable with $Q = \bar{T}D\bar{T}^{-1}$, where

$$\bar{T} = \begin{bmatrix} \epsilon & \epsilon \\ v_1 & v_2 \end{bmatrix},$$

and $D = diag[d_1, d_2]$, with v_1, v_2, d_1 and d_2 given by

$$\begin{aligned} v_1 &= -0.5[a - b - \sqrt{(a-b)^2 + 4\epsilon^2}], \\ v_2 &= -0.5[a - b + \sqrt{(a-b)^2 + 4\epsilon^2}], \\ d_1 &= 0.5[a + b + \sqrt{(a-b)^2 + 4\epsilon^2}], \\ d_2 &= 0.5[a + b - \sqrt{(a-b)^2 + 4\epsilon^2}]. \end{aligned} \tag{8}$$

It follows by, e.g., [14], that $A = V\Lambda V^{-1}$, where $\Lambda = S \otimes D$ and $V = U \otimes \bar{T}$.

We now postulate that there is a solution to the game in which agent-1 is always ahead (closer to the origin) in both x_1 and x_2 directions, that is, for $t \in [0, T]$,

$$x_i^1(t) < x_i^2(t), x_i^1(t) < x_i^3(t), ..., x_i^1(t) < x_i^N(t), \ i = 1, 2.$$

Then, $sgn\{\mathbf{x}^j(t) - \mathbf{x}^1(t)\} = [1 \ \ 1]'$ for all $t \in [0, T]$ and $1 < j \leq N$, which fixes $\mathbf{s}(t) = \mathbf{e} \otimes R$ in (7). The linear system, then, has the solution that can be expressed as

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} = \phi(t) \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{p}(0) \end{bmatrix} + \psi(t, 0)\mathbf{s}(0), \tag{9}$$

where $\phi(t)$ is the state transition matrix of (7) and $\psi(t,0)$ is the matrix related to its input component. They can be computed using e.g., Laplace transform, as

$$\phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix},$$
$$\psi(t,0) = \int_0^t \begin{bmatrix} \phi_{12}(t-\tau) \\ \phi_{22}(t-\tau) \end{bmatrix} d\tau,$$

where the blocks of $\phi(t)$ and $\psi(t,0)$ are given by

$$\begin{aligned} \phi_{11}(t) &= \phi_{22}(t) = V\,\Gamma_{11}(t)V^{-1}, \\ \phi_{12}(t) &= -V\,\Gamma_{12}(t)V^{-1}, \\ \phi_{21}(t) &= -V\,\Gamma_{21}(t)V^{-1}, \end{aligned} \tag{10}$$

$$\begin{aligned} \psi_1(t,0) &= V\,\Omega_1(t,0)V^{-1}, \\ \psi_2(t,0) &= V\Omega_2(t,0)\,V^{-1}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\Gamma_{11}(t) = \text{diag}\left[1, 1, \gamma_{11}(t), ..., \gamma_{11}(t)\right], \\ &\Gamma_{12}(t) = \text{diag}\left[t, t, \gamma_{12}(t), ..., \gamma_{12}(t)\right], \\ &\Gamma_{21}(t) = \text{diag}\left[0, 0, \gamma_{21}(t), ..., \gamma_{21}(t)\right], \\ &\Omega_1(t,0) = \text{diag}\left[-\tfrac{t^2}{2}, -\tfrac{t^2}{2}, \omega_1(t), ..., \omega_1(t)\right], \\ &\Omega_2(t,0) = \text{diag}\left[t, t, \omega_2(t), ..., \omega_2(t)\right]. \end{aligned}$$

Here,

$$\begin{aligned} &\gamma_{11}(t) = \text{diag}[cosh(d_1t), cosh(d_2t)], \\ &\gamma_{12}(t) = \text{diag}[\tfrac{sinh(d_1t)}{d_1}, \tfrac{sinh(d_2t)}{d_2}], \\ &\gamma_{21}(t) = \text{diag}[d_1sinh(d_1t), d_2sinh(d_2t)], \\ &\omega_1(t) = \text{diag}[\tfrac{1-cosh(d_1t)}{d_1^2}, \tfrac{1-cosh(d_2t)}{d_2^2}], \\ &\omega_2(t) = \text{diag}[\tfrac{sinh(d_1t)}{d_1}, \tfrac{sinh(d_2t)}{d_2}]. \end{aligned}$$

Substituting the terminal condition $\mathbf{x}^i(T) = \mathbf{0}$ for $1 \le i \le N$ after evaluating (9) at T, we get

$$\phi_{11}(T)\mathbf{x}(0) + \phi_{12}(T)\mathbf{p}(0) + [\psi_1(T,0) - \psi_2(T,0)]\mathbf{s}(0) = \mathbf{0}.$$

Since $\phi_{12}(T)$ is clearly nonsingular for $T > 0$, $\mathbf{p}(0)$ can be obtained from this equation and substituted into (9) to obtain

$$\begin{aligned} \mathbf{x}(t) \;&= \{\phi_{11}(t) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\phi_{11}(T)\}\mathbf{x}(0) \\ &+\{\psi_1(t,0) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\psi_1(T,0)\}\mathbf{s}(0). \end{aligned} \tag{12}$$

As was done in [3], it is easy to see that the leader remains the same in both directions if and only if $f(Q)$ and $g(Q)$ are positive matrices. In view of simulations in Fig. 1, we are encouraged to investigate existence of Nash equilibrium for small and negative ϵ.

We now need to verify that the postulate of "no change of leader" is satisfied by the solution.Consider

$$\mathbf{y}(t) = K(t)\mathbf{y}(0) + L(t)\hat{\mathbf{r}}(0), \tag{13}$$

where

$$\mathbf{y}(t) = [x_1^1(t)\ x_2^1(t)\ x_1^2(t) - x_1^1(t)\ x_2^2(t) - x_2^1(t)\ \dots\ x_1^N(t) - x_1^1(t)\ x_2^N(t) - x_2^1(t)]',$$

is a vector of pairwise distances in both directions and $\hat{\mathbf{r}}(0) = [0 \ \ 0 \ \ \mathbf{e} \otimes \mathbf{r}]$ where $\mathbf{e}$ is defined in (6).

The transformation $M := U^{-1} \otimes I \in \mathbb{R}^{2N\times 2N}$ converts the optimal trajectories in (12) to distances from leader yielding

$$K(t) = MVf(\Lambda)V^{-1}M^{-1}, L(t) = MVg(\Lambda)V^{-1}M^{-1},$$

where f and g are as defined in (4). It is now easy to see that the postulate of no change of reader holds under the assumptions of Theorem 1 if and only if $K(t)$ and $L(t)$ are positive matrices for all $t \in [0, T]$ under those assumptions. We will establish this in by following sequence of four lemmas below. Here, we note in passing that the expressions obtained for $K(t)$ and $L(t)$ below in Lemma 1 are used in order to obtain the expressions for pairwise distances of Theorem 1 from (13).

Lemma 1: i) K(t) is a positive matrix if and only if f(Q) is a positive matrix. ii) L(t) is a positive matrix if and only if g(Q) is a positive matrix.

Proof: i)

$$\begin{aligned} K(t) \;&= \begin{bmatrix} \bar{T} & 0 \\ 0 & I\otimes\bar{T} \end{bmatrix} \begin{bmatrix} \frac{T-t}{T}I & 0 \\ 0 & f(I\otimes D) \end{bmatrix} \begin{bmatrix} \bar{T}^{-1} & 0 \\ 0 & I\otimes\bar{T}^{-1} \end{bmatrix}, \\ &= \begin{bmatrix} \frac{T-t}{T}I & 0 \\ 0 & (I\otimes\bar{T})(I\otimes f(D))(I\otimes\bar{T}^{-1}) \end{bmatrix}, \\ &= \begin{bmatrix} \frac{T-t}{T}I & 0 \\ 0 & I\otimes[\bar{T}f(D)\bar{T}^{-1}] \end{bmatrix}, \\ &= \begin{bmatrix} \frac{T-t}{T}I & 0 \\ 0 & I\otimes f(Q) \end{bmatrix}. \end{aligned}$$

Here, it can be observed that $K(t)$ is a positive matrix if and only if $f(Q)$ is a positive matrix. ii) The proof is similar to that above:

$$\begin{aligned} L(t) \;&= MVg(\Lambda)V^{-1}M^{-1}, \\ &= \begin{bmatrix} \frac{(T-t)t}{2}I & 0 \\ 0 & I\otimes(\bar{T}g(D)\bar{T}^{-1}) \end{bmatrix}. \end{aligned} \tag{14}$$

□

Lemma 2: $f(x)$ is a decreasing function of positive x.

Proof: Let $\tilde{f}(x) = f(x^2)$. Computing $\dot{\tilde{f}}(x)$, we have,

$$\dot{\tilde{f}}(x) = \frac{(T-t)cosh[x(T-t)]sinh(xT) - sinh[x(T-t)]Tcosh(xT)}{(.)^2}.$$

Using hyperbolic identities and arranging the result, we obtain,

$$\dot{\tilde{f}}(x) = \frac{(2T-t)sinh(xt) - tsinh[x(2T-t)]}{(.)^2}.$$

Now, we compute the Taylor series of both components of the numerator of this equations as,

$$(2T-t)sinh(xt) = (2T-t)xt + \frac{(2T-t)(xt)^3}{3!} + \frac{(2T-t)(xt)^5}{5!} + ...$$

$$tsinh[x(2T-t)] = tx(2T-t) + \frac{t[x(2T-t)]^3}{3!} + \frac{t[x(2T-t)]^5}{5!} + ...$$

Subtracting term by term, we deduce that $(2T-t)sinh(xt) < tsinh[x(2T-t)]$, which ensures that $f(x)$ is a decreasing function of x. □

We now use (8) to write $f(Q)$ and $g(Q)$ more explicitly as

$$\begin{aligned} &f(Q) \\ &= \tfrac{1}{\epsilon(v_2-v_1)} \begin{bmatrix} \epsilon[f(d_1)v_2 - f(d_2)v_1] & \epsilon^2[f(d_2) - f(d_1)] \\ v_1v_2[f(d_1) - f(d_2)] & \epsilon[f(d_2)v_2 - f(d_1)v_1], \end{bmatrix}. \end{aligned} \tag{15}$$

$$\begin{aligned} &g(Q) \\ &= \tfrac{1}{\epsilon(v_2-v_1)} \begin{bmatrix} \epsilon[g(d_1)v_2 - g(d_2)v_1] & \epsilon^2[g(d_2) - g(d_1)] \\ v_1v_2[g(d_1) - g(d_2)] & \epsilon[g(d_2)v_2 - g(d_1)v_1] \end{bmatrix}. \end{aligned} \tag{16}$$

Lemma 3: $f(Q)$ is a positive matrix if and only if $\epsilon < 0$.

Proof: By Lemma 2, $f(x)$ is a decreasing function of positive x. Now since $v_1 > v_2$, $f_{11} = \frac{f(d_1)v_2 - f(d_2)v_1}{(v_2-v_1)}$ is positive if and only if the numerator of f_{11} is negative.

It holds that $f(d_1)v_2 - f(d_2)v_1 < f(d_1)v_1 - f(d_2)v_1 < [f(d_1) - f(d_2)]v_1 < 0$, hence f_{11} is positive if and only if $f(d_1) < f(d_2)$ which holds by Lemma 2. Since $v_1 > v_2$, $f_{12} = \frac{\epsilon[f(d_2)-f(d_1)]}{(v_2-v_1)}$ is positive if and only if $f(d_2) > f(d_1)$ by Lemma 2 and $\epsilon < 0$. Since $v_1 > v_2$, $v_1v_2 = -\epsilon^2 < 0$, $f_{21} = \frac{v_1v_2[f(d_1)-f(d_2)]}{\epsilon(v_2-v_1)}$, is positive if and only if $f(d_1) < f(d_2)$ by Lemma 2 and $\epsilon < 0$. Finally, since $v_1 > v_2$, $v_2 < 0$, and $v_1 > 0$; $f_{22} = \frac{f(d_2)v_2-f(d_1)v_1}{(v_2-v_1)}$ is positive if and only if $f(d_1) > 0$ and $f(d_2) > 0$. □

Lemma 4: g(Q) is a positive matrix if $|a-b| >> 0 \gtrsim \epsilon$, i.e., if $|a-b|$ is sufficiently large and ϵ is negative with $|\epsilon|$ sufficiently small .

Proof: Let us first suppose that $a > b$ and consider the Maclaurin series expressions for $g[d_1(\epsilon)]$ and $g[d_2(\epsilon)]$.

$$G_1(\epsilon) := g[d_1(\epsilon)], G_2(\epsilon) := g[d_2(\epsilon)],$$

where $G_1(\epsilon)$ and $G_2(\epsilon)$ are composite functions of ϵ. Suppose that $\epsilon \simeq 0$ and consider

$$\tilde{G}_1(\epsilon) \simeq g[d_1(\epsilon)], \tilde{G}_2(\epsilon) \simeq g[d_2(\epsilon)],$$

where $\tilde{G}_1(\epsilon)$ and $\tilde{G}_2(\epsilon)$ are truncated polynomials of second order with respect to ϵ. The explicit expressions for those truncated polynomials can be derived using chain rule and expanding $d_1(\epsilon)$ and $d_2(\epsilon)$ about the point $\epsilon = 0$ as

$$\tilde{G}_1(\epsilon) = g(a) + \frac{2g'(a)}{|a-b|}\epsilon^2, \tilde{G}_2(\epsilon) = g(b) - \frac{2g'(b)}{|a-b|}\epsilon^2. \quad (17)$$

The condition

$$g[d_1(\epsilon)] - g[d_2(\epsilon)] < 0 \text{ where } d_1 > d_2,$$

implies that $g(Q)$ in (16) is a positive matrix and will be implied by

$$\begin{aligned} &g[d_1(\epsilon)] - g[d_2(\epsilon)] \\ &\simeq g(a) - g(b) + \tfrac{2[g'(a)+g'(b)]}{|a-b|}\epsilon^2 < 0, \end{aligned}$$

whenever $d_1 > d_2$ for small $|\epsilon|$. Now if $a >> b$, then since $e^{-\sqrt{x}(T-t)} \simeq 0$ and $e^{-\sqrt{x}t} \simeq 0$ as $x >> 0$,

$$g(a) - g(b) \simeq \frac{1}{a} - \frac{1}{b} < 0,$$

$$g'(a) + g'(b) \simeq -\frac{1}{a^2} - \frac{1}{b^2} < 0.$$

It follows that $g(Q)$ indeed has positive entries in case $a > b$. When $b > a$, this time the expressions obtained in place of (17) are

$$\tilde{G}_1(\epsilon) = g(b) + \frac{2g'(a)}{|a-b|}\epsilon^2, \tilde{G}_2(\epsilon) = g(a) - \frac{2g'(b)}{|a-b|}\epsilon^2.$$

and the same conclusion is reached. □

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Aykut Yıldız received the B.S, M.S, and PhD degrees in Electrical and Electronics Engineering Department from Bilkent University, Ankara, Turkey, in 2007, 2010, and 2016 respectively. Currenty, he is working in TED University as an assistant professor.

From July 2007 to August 2011, he was a research assistant in MİLDAR project funded by The Scientific and Technical Research Council of Turkey (TÜBİTAK). From 2011 to 2012, he has been supported by the Servo Control Project by Military Electronics Industry Co. (ASELSAN). He was a Post doc researcher at the TÜBİTAK project entitled "Game Theoretical Modeling of Swarms" at Bilkent University. His current research interests are in Control Theory and Swarm Theory.

A. Bülent Özgüler received his PhD at the Electrical Engineering Department of the University of Florida, Gainesville in 1982. He was a researcher at the Marmara Research Institute of TÜBİTAK during 1983-1986. He spent one year at the Institut für Dynamische Systeme, Bremen Universität, Germany, on Alexander von Humboldt Scholarship during 1994-1995. He has been with the Electrical and Electronics Engineering Department of Bilkent University, Ankara since 1986. He was at Bahçeşehir University in 2008-2009 academic year, on leave from Bilkent University. Prof. Özgüler's research interests are in the areas of decentralized control, stability robustness, realization theory, linear matrix equations, and application of system theory to social sciences. He has about 60 research papers in the field and is the author of two books Linear Multichannel Control: A System Matrix Approach, Prentice Hall, 1994 and, with K. Saadaoui, Fixed order controller design: A parametric approach, LAP Lambert Academic Publishing, 2010.