Capitulation of the 2-ideal class group of the fields $K = Q(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ where p, q_1, q_2 and q_3 are distinct primes such that $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$

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***Abstract*—Let $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$, be real biquadratic number field where, p, q_1, q_2 and q_3 be distinct prime numbers with $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$. Let $K_2^{(1)}$ be the Hilbert 2-class field of K. Let $K_2^{(2)}$ be the Hilbert 2-class field of $K_2^{(1)}$ and $K^{(*)}$ the genus field of K. We suppose that $K_2^{(1)} \neq K^{(*)}$ and $Gal(K_2^{(1)}/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We study the capitulation problem of the 2-ideal classes of K in the sub-extensions of $K_2^{(1)}/K$ and we determine the structure of $Gal(K_2^{(2)}/K)$.**

Keywords– fundamental unit, Hilbert 2-class field, ideal class group, class number,genus field.

1. Introduction

Let K be an algebraic number field and C_K its ideal class group in the ordinary sence. Suppose L is a finite algebraic number extension of K. Then there is a canonical homomorphism

$$j : C_K \to C_L$$

induced by extension of ideals. Then ker(j) consists of those ideal classes in K which capitulate in L. One of the main goals in capitulation theory is to determine ker(j).

If $K_2^{(1)}$ is the Hilbert 2-class field of K, then by class field theory the Galois group $Gal(K_2^{(1)}/K)$ and the 2-class group $C_{2,K}$ of K are canonically isomorphic. Let $K_2^{(n)}$ be the Hilbert 2-class field of $K_2^{(n-1)}$, then $K_2^{(n)}/K$ is a Galois extension for each non negative integer n and

$$K \subset K_2^{(1)} \subset K_2^{(2)} \subset .... \subset K_2^{(n)} \subset ....$$

is the 2-Hilbert class tower of field K. terminates at $K_2^{(1)}$ or $K_2^{(2)}$ [10].

In the following, we give some known results about the structure of Galois group $Gal(K_2^{(2)}/K)$ where $C_{2,K}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see, for instance, [4], Section 1). Let K be an algebraic number field such that $C_{2,K} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and let G be the Galois group of $K_2^{(2)}/K$. Then if G' is the commutator subgroup of G, we have $G' = Gal(K_2^{(2)}/K_2^{(1)})$, and

$$G/G' \simeq Gal(K_2^{(1)}/K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

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Let Q_m, D_m, S_m be the quaternion, dihedral and semidihedral groups of order 2^m, So that in term of generators and relations,

$$Q_m = \langle x, y \ |x^{2^{m-2}} = y^2 = a, a^2 = 1, y^{-1}xy = x^{-1}\rangle;$$
$$D_m = \langle x, y \ |x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{-1}\rangle;$$
$$S_m = \langle x, y \ |x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{2^{m-2}-1}\rangle.$$

By [2, Theorem 4.5, Chap 5] we have G is isomorphic to $D_m, Q_m \ or \ S_m$. The commutator subgroup G' of G is always cyclic: $G' = \langle x^2\rangle$. The group G has exactly three sub-groups of index 2. Namely, $\langle x\rangle$; $\langle x^2, y\rangle$ and $\langle x^2, xy\rangle$. When G is not the quaternion group of order 8, only one of the three maximal sub-groups of G is cyclic. When $m \geq 4$ the other two maximal sub-groups of G are not abelian and their maximal abelian factor groups are again isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Of course, when G is the quaternion group of order 8 its three maximal subgroups are cyclic and when G is the dihedral group of order 8, its three sub-groups are abelian. None of the proper factor groups of G is of quaternion type. According to what we just said, the Hilbert 2-class field tower of K terminates in at most two steps. If $K_2^{(1)} \neq K_2^{(2)}$, then the Galois group $Gal(K_2^{(2)}/K_2^{(1)})$ is cyclic and $Gal(K_2^{(2)}/K)$ is a quaternion, dihedral or semidihedral group.

Let $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ be a biquadratic number field where $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$. In this paper, we first give a rank for some reel biquadratic number fields. Then, in section 3 we give the Hasse unit index for some real biquadratic number fields and we give the list of real biquadratic number field K such that its 2- ideal class group of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$(Theoreme 3.4). In the last section we give the 2-ideal classes of K, which capitulate in the genus field of K. Consequently we prove the following:

Theorem 1.1. *Let p, q_1, q_2, q_3 be distinct primes with $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$ and $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$. Assume that the 2-ideal class group of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then:*
1. If $K_2^{(2)} = K_2^{(1)}$ we have $Gal(K_2^{(2)}/K)$ is abelien.
2. If $K_2^{(1)} \neq K_2^{(2)}$ we have $Gal(K_2^{(2)}/K)$ is dihedral.

2. Rank of 2-ideal class group of some real biquadratic number fields

The following notations will be used throughout the paper:

K a real biquadratic number field
k=$\mathbb{Q}(\sqrt{m})$ a quadratic subfield of K with odd class number
E the group of units of k
N the norm map
r the number of primes of k which are ramified in K e a positive integer defined by $2^e = [E : E \cap N(K^*)]$
ε_m the fundamental unit of $\mathbb{Q}(\sqrt{m})$
h(K) the class number of K
$h_2(K)$ the 2-part of h(K)
h(m) the class number for the quadratic number field $\mathbb{Q}(\sqrt{m})$
$C_{2,K}$ the 2-ideal class group of K
$K^{(*)}$ the genus field of K
$K_2^{(1)}$ the Hilbert 2-class field of K
$K_2^{(2)}$ the Hilbert 2-class field of $K_2^{(1)}$
Q_K the hasse unit index of the biquadratic number field K
$(\frac{a,d}{\mathcal{P}})$ the Hilbert's 2-th power norm residue symbol mod $\mathcal{P}$

Lemma 2.1. *We keep the same notation as above, the rank of $C_{2,K}$ is equal to r-e-1.*

Proof. See [1] □

Remark 2.2. We have:
1) e=0 if and only if -1 and ε_m are norms in the extension K/k.
2) e=1 if and only if -1 is a norm and ε_m is not a norm, or -1 is not a norm and ε_m or $-\varepsilon_m$ is a norm in the extension K/k.
2) e=2 if and only if -1, ε_m and $-\varepsilon_m$ are not norms in the extension K/k.

Lemma 2.3. *Let F be a real quadratic number field with fundamental unit ε and discriminant D. Suppose that $N_{F/\mathbb{Q}}(\varepsilon) = 1$. Then there exists a positive square free integer m dividing D such that m ε is a square in F.*

Proof. See [5] □

Remark 2.4. In the proof of lemma 2.3 [see 5], the integer m is norm in the extension $F/\mathbb{Q}$.

Lemma 2.5. *Let p, q_1, q_2, q_3 be distinct primes with $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$, and $K = \mathbb{Q}(\sqrt{q_1 q_2}, \sqrt{p q_1 q_3})$. Then, we have:*
1) e=0 if and only if one of the following conditions is satisfied:
(i) $(\frac{q_1 q_2}{p}) = (\frac{q_1 q_2}{q_3}) = -1.$
(ii) $(\frac{q_1}{p}) = (\frac{q_2}{p}) = -(\frac{q_1 q_2}{q_3}) = 1.$
1) e=1 if and only if one of the following conditions is satisfied:
(i) $(\frac{q_1 q_2}{p}) = -(\frac{q_1 q_2}{q_3}) = -1.$
(ii) $(\frac{q_1}{p}) = (\frac{q_2}{p}) = (\frac{q_1 q_2}{q_3}) = -1.$
(iii) $(\frac{q_1}{p}) = (\frac{q_2}{p}) = (\frac{q_1 q_2}{q_3}) = 1.$

Proof. The discriminant of $\mathbb{Q}(\sqrt{q_1 q_2})$ is equal to $q_1 q_2$, by lemma 2.3 there exists an integer $m | q_1 q_2$ such that m is a norm in the extension $\mathbb{Q}(\sqrt{q_1 q_2})/\mathbb{Q}$ [see remark 2.4] and $\sqrt{m \varepsilon_{q_1 q_2}} \in \mathbb{Q}(\sqrt{q_1 q_2})$. Since $\varepsilon_{q_1 q_2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{q_1 q_2})$ then m must be contained in $\{q_1, q_2\}$. Either way, we can conclude that:

$$\sqrt{q_1 \varepsilon_{q_1 q_2}} \in \mathbb{Q}(\sqrt{q_1 q_2}) \ or \sqrt{q_2 \varepsilon_{q_1 q_2}} \in \mathbb{Q}(\sqrt{q_1 q_2}) \ (1)$$

Consequently $\varepsilon_{q_1 q_2} = q_1 u^2$ or $\varepsilon_{q_1 q_2} = q_1 v^2$ with u and v are in $\mathbb{Q}(\sqrt{q_1 q_2})$.
It is easy to see that the primes of $\mathbb{Q}(\sqrt{q_1 q_2})$ ramified in K are exactly those lying above p and q_3. Denote $S = \{p, q_3\}$, and $\mathcal{P}$ a prime ideal of $\mathbb{Q}(\sqrt{q_1 q_2})$ which is ramified in K lying above $\ell \in$ S,

-if ℓ remain inert in $\mathbb{Q}(\sqrt{q_1 q_2})$, then we have:

$$(\frac{-1, p q_1 q_3}{\mathcal{P}}) = (\frac{N_{\mathbb{Q}(\sqrt{q_1 q_2})/\mathbb{Q}}(-1), p q_1 q_3}{\ell}) = 1$$

$$(\frac{\varepsilon_{q_1 q_2}, p q_1 q_3}{\mathcal{P}}) = (\frac{N_{\mathbb{Q}(\sqrt{q_1 q_2})/\mathbb{Q}}(\varepsilon_{q_1 q_2}), p q_1 q_3}{\ell}) = 1$$

-if ℓ is decomposed in $\mathbb{Q}(\sqrt{q_1 q_2})$, then we have:

$$(\frac{\varepsilon_{q_1 q_2}, p q_1 q_3}{\mathcal{P}}) = (\frac{q_1 u^2, p q_1 q_3}{\mathcal{P}}) = (\frac{q_1, p q_1 q_3}{\mathcal{P}}) = (\frac{q_1}{\ell}).$$

Using remark 2.2, the lemma 2.5 follows immediately. □

Lemma 2.6. *Let p, q_1, q_2, q_3 be distinct prime numbers with $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$ and $K = \mathbb{Q}(\sqrt{q_1 q_2}, \sqrt{p q_1 q_3})$. Then the 2-ideal class group of K is of rank equal to 2 if and only if the following condition is satisfied:*

$$(\frac{p}{q_1}) = (\frac{p}{q_2}) = 1,$$

Proof. By lemma 2.1 the rank of $C_{2,K}$ is equal to r-e-1. The positive integer e is given by lemma 2.5. One can compute the positive integer r and the lemma follows. □

Lemma 2.7. *Let p, q_1, q_2, q_3 be distinct prime numbers with $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 \ (mod\ 4)$ and $L = \mathbb{Q}(\sqrt{q_1 q_3}, \sqrt{p q_1 q_2})$. Then the 2-ideal class group of L is cyclic if and only if one of the following conditions is satisfied:*
1) $(\frac{q_1 q_3}{p}) = -1.$
2) $(\frac{q_1}{p}) = (\frac{q_3}{p}) = (\frac{q_1 q_2}{q_3}) = -1.$
3) $(\frac{q_1}{p}) = (\frac{q_3}{p}) = -(\frac{q_1}{q_2}) = -(\frac{q_3}{q_2}) = -1.$
4) $(\frac{q_1}{p}) = (\frac{q_3}{p}) = (\frac{q_1}{q_2}) = (\frac{q_3}{q_2}) = -1.$

Proof. With the same technique used in proof for lemma 2.5, one can compute a positive integer e for biquadratic field L, and using lemma 2.1 we verify that the 2-ideal class group of L is of rank equal to 1, if and only if one of condition 1), 2), 3), 4) of lemma 2.7 is satisfied. □

3. The Hasse unit index for some real biquadratic fields

Lemma 3.1. *Let $p,\ q_1,\ q_2$ and q_3 be a distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1(mod\ 4)$ and $(\frac{p}{q_1}) = (\frac{p}{q_2}) = -(\frac{p}{q_3}) = 1$. Then the biquadratic number field, $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ contains the following units:*

$$\sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_3}},\ \sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_2q_3}}$$

Consequently $Q_K = 4$.

Proof. The discriminant of $\mathbb{Q}(\sqrt{pq_1q_3})$ is equal to pq_1q_3, then there exists an integer $m|pq_1q_3$ such that $\sqrt{m\varepsilon_{pq_1q_3}} \in \mathbb{Q}(\sqrt{pq_1q_3})$. Since $\varepsilon_{pq_1q_3}$ is the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_3})$ then $m \notin \{1, pq_1q_3\}$. On other hand since $(\frac{p}{q_3}) = -1$ then p, q_3, q_1q_3, pq_3 are not a norms in the extension $\mathbb{Q}(\sqrt{pq_1q_3})/\mathbb{Q}$ so $m \notin \{p, q_3, pq_3, q_1q_3\}$ and we have $\sqrt{m\varepsilon_{pq_1q_3}} \in \mathbb{Q}(\sqrt{pq_1q_3})$ such that $m|pq_1q_3$ and $m \notin \{1, p, q_3, pq_3, q_1q_3, pq_1, pq_1q_3\}$.
Either way we can conclude that:

$$\sqrt{q_1\varepsilon_{pq_1q_3}} \in \mathbb{Q}(\sqrt{pq_1q_3}) \qquad (2)$$

With the same reason we have:

$$\sqrt{q_2\varepsilon_{pq_2q_3}} \in \mathbb{Q}(\sqrt{pq_2q_3}) \qquad (3)$$

Consequently,using (1), (2) and (3), we obtain that the unit $\sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_3}}$, $\sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_2q_3}}$ are contained in K consequently $Q_K = 4$. □

Lemma 3.2. *Let $p,\ q_1,\ q_2$ and q_3 be distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1(mod4)$, and $(\frac{p}{q_1}) = (\frac{p}{q_1}) = (\frac{p}{q_3}) = 1$. Then the biquadratic number field, $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ contains exactly one of the following units:*

$$\sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_2}},\ \sqrt{\varepsilon_{pq_2q_3}\varepsilon_{pq_2q_3}},\ \sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}}$$

Consequently, $Q_K = 2$.

Proof. The discriminant of $\mathbb{Q}(\sqrt{pq_1q_2})$ is equal to pq_1q_2, then there exists an integer $m|pq_1q_2$ such that,
$\sqrt{m\varepsilon_{pq_1q_2}} \in \mathbb{Q}(\sqrt{pq_1q_2})$, since $\varepsilon_{pq_1q_2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$ then $m \notin \{1, pq_1q_2\}$, therefore:

$$\sqrt{m\varepsilon_{pq_1q_2}} \in \mathbb{Q}(\sqrt{pq_1q_2}) \ (4)$$

With $m \in \{p, q_1, q_2, pq_1, pq_2, q_1q_2\}$ similarly we have,

$$\sqrt{m\varepsilon_{pq_1q_3}} \in \mathbb{Q}(\sqrt{pq_1q_3}) \ (5)$$

With $m \in \{p, q_1, q_3, pq_1, pq_3, q_1q_3\}$ consequently using (1), (4) and (5) we obtain that exactly one of the units

$$\sqrt{\varepsilon_{q_1q_3}\varepsilon_{pq_1q_2}},\ \sqrt{\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}},\ \sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}}$$

is contained in K, so $Q_K = 2$. □

Lemma 3.3. *Let $p,\ q_1,\ q_2$ and q_3 be distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1(mod4)$, and $(\frac{p}{q_1}) = (\frac{p}{q_2}) = -(\frac{p}{q_3}) = 1$. Then the biquadratic number field, $L = \mathbb{Q}(\sqrt{q_1q_3}, \sqrt{pq_1q_2})$ contains exactly one of the following units:*

$$\sqrt{\varepsilon_{q_1q_3}\varepsilon_{pq_1q_2}},\ \sqrt{\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}},\ \sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}}$$

Consequently $Q_L = 2$.

Proof. We have:

$$\sqrt{q_1\varepsilon_{q_1q_3}} \in \mathbb{Q}(\sqrt{q_1q_3}) \qquad (6)$$

Using (3), (4) and (6) we obtain that exactly one of the unit,
$\sqrt{\varepsilon_{q_1q_3}\varepsilon_{pq_1q_2}}$, $\sqrt{\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}}$, $\sqrt{\varepsilon_{q_1q_2}\varepsilon_{pq_1q_2}\varepsilon_{pq_2q_3}}$
is contained in L so $Q_L = 2$. □

As a consequence we have the list of real biquadratic number fields $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ such that $C_{2,K}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Theorem 3.4. *Let $p,\ q_1,\ q_2$ and q_3 be distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1\ (mod\ 4)$ and let $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ be a biquadratic number field. The 2-ideal class group of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if the following condition is satisfied:*

$$(\frac{p}{q_1}) = (\frac{p}{q_2}) = -(\frac{p}{q_3}) = 1$$

Proof. In [7] the class number for K is given by:

$$h(K) = \frac{Q_K h(pq_1q_3)h(pq_2q_3)}{4}$$

assume that $C_{2,K} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then rank($C_{2,K}$)= 2. By lemma 2.6 we have

$$(\frac{p}{q_1}) = (\frac{p}{q_2}) = 1.$$

1) If $(\frac{p}{q_3}) = 1$, by [3] we have $4|h(pq_1q_3)$ and $4|h(pq_2q_3)$. On other hand by lemma 3.2, $Q_K = 2$, therefore $8|h(K)$. Consequently $C_{2,K}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2) If $(\frac{p}{q_3}) = -1$, by [3] we have, $h(pq_1q_3) \equiv h(pq_2q_3) \equiv 2\ (mod\ 4)$ and by lemma 3.1 we have $Q_K = 4$, then $h_2(K) = 4$. The 2-ideal class group of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Suppose now that $(\frac{p}{q_1}) = (\frac{p}{q_2}) = -(\frac{p}{q_3}) = 1$, by lemma 2.6 we have rank($C_{2,K}$)= 2, by [3] we have,
$h(pq_1q_3) \equiv h(pq_2q_3) \equiv 2\ (mod\ 4)$ and by lemma 3.1 we have $Q_K = 4$, then $h_2(K) = 4$. Consequently $C_{2,K}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The theorem follows. □

4. Proof of theorem 1.1

Throughout this section we suppose that:
$C_{2,K} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4.1. Necessary and sufficient conditions such that $K_2^{(1)} \neq K_2^{(2)}$

The genus field of biquadratic field $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ is $K^{(*)} = \mathbb{Q}(\sqrt{p}, \sqrt{q_1q_2}, \sqrt{q_1q_3})$. We introduce the biquadratic number field $L = \mathbb{Q}(\sqrt{q_1q_3}, \sqrt{pq_1q_2})$, then $K^{(*)}/L$ is unramified. The 2-ideal class group of L is cyclic [see lemma 2.7], then the fields $K^{(*)}$ and L have the same Hilbert 2- class field $K_2^{(2)}$. Therefore

$$h(K_2^{(1)}) = \frac{1}{2}h(K^{(*)}) = \frac{1}{4}h(L)$$

Consequently

$$K_2^{(1)} \neq K_2^{(2)} \Leftrightarrow 2|h(K_2^{(1)}) \Leftrightarrow 4|h(K^{(*)}) \Leftrightarrow 8|h(L)$$

Lemma 4.1. *Let p, q_1 and q_2 be a distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv 1 \ (mod 4)$, and $(\frac{p}{q_1}) = (\frac{p}{q_2}) = (\frac{q_2}{q_1}) = 1$. There exist X, Y, k, l such that $pq_1 = k^2X^2 + 2lXY + 2mY^2, -q_2 = l^2 - 2k^2m$, denote $\alpha = (\frac{q_1q_2}{p})_4$ and $\beta = \frac{2(k^2X+Y)}{p}$, we have:*

$$8|h(pq_1q_2) \ if \ and \ only \ if \ \alpha = \beta = 1.$$

Proof. See [3] □

Let α and β the integres defined in theorem 5 we have a following theorem.

Theorem 4.2. *Let p, q_1, q_2 and q_3 be a distinct prime numbers such that, $p \equiv -q_1 \equiv -q_2 \equiv -q_3 \equiv 1 (mod 4)$. If the biquadratic number field, $K = \mathbb{Q}(\sqrt{q_1q_2}, \sqrt{pq_1q_3})$ has 2-ideal class group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then: $K_2^{(1)} \neq K_2^{(2)}$ if and only if $(\frac{q_2}{q_1}) = 1$ and $\alpha = \beta = 1$.*

Proof. We have $K_2^{(1)} \neq K_2^{(2)} \Leftrightarrow 8|h(L)$. Suppose now that the condition of theorem 3.4 are satisfied. A class number of L is given by:

$$h(L) = \frac{Q_L h(pq_1q_3)h(pq_1q_2)}{4}$$

By lemma 3.3, we have $Q_L = 2$. On other hand by [3] we have,
$h(pq_1q_3) \equiv 2 \ (mod \ 4)$, then $h(L) = h(pq_1q_2)$. The lemma 4.1 gives then necessary and sufficient conditions such that $K_2^{(1)} \neq K_2^{(2)}$. □

4.2. Generators of 2-ideal class group of K

Since $(\frac{q_1q_2}{p}) = 1$ the ideal p splits completely in $\mathbb{Q}(\sqrt{q_1q_2})$, we have $po_{\mathbb{Q}(\sqrt{q_1q_2})} = \mathcal{P}_1\mathcal{P}_2$ where $\mathcal{P}_i, i \in \{1, 2\}$ are two distinct prime ideals in $\mathbb{Q}(\sqrt{q_1q_2})$. Moreover, since p is ramified in K then $\mathcal{P}_io_K = \mathcal{Y}_i^2$, where $\mathcal{Y}_i, i \in \{1, 2\}$ are two distinct prime ideals in K wich remain inert in $K^{(*)} = \mathbb{Q}(\sqrt{p}, \sqrt{q_1q_2}, \sqrt{q_1q_3})$.

Theorem 4.3. *Assume that the 2-ideal class group of K is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then the two ideal class $[\mathcal{Y}_1^l]$ and $[\mathcal{Y}_2^l]$ generates the 2-ideal class group of K. With l is the class number of $\mathbb{Q}(\sqrt{q_1q_2})$.*

Proof. 1) Show that $\mathcal{Y}_1^l$ and $\mathcal{Y}_2^l$ are not principal ideals. Since l is the class number of $\mathbb{Q}(\sqrt{q_1q_2})$, the prime ideal $\mathcal{P}_1^l$ and $\mathcal{P}_2^l$ are principal. Therfore $[\mathcal{Y}_1^l]$ and $[\mathcal{Y}_2^l]$ are in $C_{2,K}$. Applying the Artin reciprocity laws in the extension $K^{(*)}/K$ we find that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not principal ideals. It follows that $\mathcal{Y}_1^l$ and $\mathcal{Y}_2^l$ are not principal ideals.

2) show that $\mathcal{Y}_1^l\mathcal{Y}_2^l$ is not principal ideal.
We have $N_{K/\mathbb{Q}(\sqrt{q_1q_2})}(\mathcal{Y}_1^l\mathcal{Y}_2^l) = \mathcal{P}_1^l\mathcal{P}_2^l = p^l o_{\mathbb{Q}(\sqrt{q_1q_2})}$, supposing that $\mathcal{Y}_1^l\mathcal{Y}_2^l$ is principal then $\mathcal{Y}_1^l\mathcal{Y}_2^l = (a)$ with $a \in K$. So $N_{K/\mathbb{Q}(\sqrt{q_1q_2})} = p^l o_{\mathbb{Q}(\sqrt{q_1q_2})}$. It follows that there exists a unit u of $\mathbb{Q}(\sqrt{q_1q_2})$ such that p^lu is a norm in $K/\mathbb{Q}(\sqrt{q_1q_2})$. Then we must have

$$(\frac{p^lu, pq_1q_3}{\mathcal{P}_1}) = 1 \ (7).$$

Using the properties of Hilbert's 2-th power norm residue symbol mod $\mathcal{P}_1$, we have $(\frac{-1,pq_1q_3}{\mathcal{P}_1}) = 1$ and $(\frac{\varepsilon_{q_1q_2},pq_1q_3}{\mathcal{P}_1}) = (\frac{q_1,pq_1q_3}{\mathcal{P}_1}) = 1$. So

$$(\frac{u, pq_1q_3}{\mathcal{P}_1}) = 1 \ for \ any \ unit \ u \ of \ \mathbb{Q}(\sqrt{q_1q_2}).$$

Moreover we have $(\frac{p^l,pq_1q_3}{\mathcal{P}_1}) = (-\frac{p}{q_3})^l = (-1)^l = -1$, consequently $(\frac{p^lu,pq_1q_3}{\mathcal{P}_1}) = -1$ which is in contradiction with (7). Finally $\mathcal{Y}_1^l\mathcal{Y}_2^l$ is not principal ideal. □

4.3. Determination of the 2-ideal class group of K which capitulates in $K^{(*)}$

We have $[\mathcal{Y}_1^l]$ and $[\mathcal{Y}_2^l]$ generates the 2-ideal class group of K. We denote by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the two prime ideals in $K^{(*)}$ such that $\mathcal{Y}_1o_{K^{(*)}} = \mathcal{Q}_1$ and $\mathcal{Y}_2o_{K^{(*)}} = \mathcal{Q}_2$.

Theorem 4.4. *All 2-ideal classes group of K capitulate in $K^{(*)}$.*

Proof. 1) show that $\mathcal{Y}_1^l\mathcal{Y}_2^l$ capitulates in $K^{(*)}$.
Since $(\frac{q_1q_3}{p}) = -1$ the number of prime ideals of $\mathbb{Q}(\sqrt{q_1q_2})$ which ramify in $L' = \mathbb{Q}(\sqrt{q_1q_3}, \sqrt{p})$ is equal to 1, by lemma 2.1 we have $rang(C_{2,L'}) = 0$. Moreover, since p is ramified in $L' = \mathbb{Q}(\sqrt{q_1q_3}, \sqrt{p})$ we have $po_L = \mathcal{P'}^2$. The class number of L' is odd, so $\mathcal{P}'$ is principal ideal. On other hand $\mathcal{P}'o_{K^{(*)}} = \mathcal{Q}_1\mathcal{Q}_2$ consequently $\mathcal{Q}_1\mathcal{Q}_2$ is principal ideal. And we have $\mathcal{Q}_1^l\mathcal{Q}_2^l$ is principal ideal, it follows that $\mathcal{Y}_1^l\mathcal{Y}_2^l$ capitulates in $K^{(*)}$.

2) show that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ capitulate in $K^{(*)}$.
Let $L = \mathbb{Q}(\sqrt{q_1q_3}, \sqrt{pq_1q_2})$ since $(\frac{q_1q_3}{p}) = -1$ and p is ramified in L we have $po_L = \mathcal{S}^2$ with $\mathcal{S}$ is a prime ideal in L, therfore $\mathcal{S}o_{K^{(*)}} = \mathcal{Q}_1\mathcal{Q}_2$. We have $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are principal if and only if $\mathcal{S}$ is principal, indeed: We now that if $\mathcal{Q}_1$ is a principal ideal, then $N_{K^{(*)}/L}(\mathcal{Q}_1) = \mathcal{S}$ is a principal ideal. Conversely if

$\mathcal{S}$ is a principal ideal, by Artin reciprocity law applied in the extension $K_2^{(2)}/L$, $\mathcal{S}$ split completely in $K_2^{(2)}$. Therefore $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are splits completely in $K_2^{(2)}$, by Artin reciprocity law applied in the extension $K_2^{(2)}/K^{(*)}$ we have $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are principal ideals.
-show that $\mathcal{S}$ is principal ideal.
We know that $\sqrt{\varepsilon_{q_1q_3}} = u_1\sqrt{q_1} + u_2\sqrt{q_3}$ with $u_1, u_2 \in \mathbb{Q}$ [see proof of lemma 2.5] and $\sqrt{\varepsilon_{pq_2q_3}} = v_1\sqrt{q_2} + v_2\sqrt{pq_3}$ with $v_1, v_2 \in \mathbb{Q}$ [see proof of lemma 3.1], therefore $\sqrt{p}\sqrt{\varepsilon_{q_1q_3}\varepsilon_{pq_2q_3}}$ is a integr of L. Since $po_L = (\sqrt{p}\sqrt{\varepsilon_{q_1q_3}\varepsilon_{pq_2q_3}})^2(\varepsilon_{q_1q_3}\varepsilon_{pq_2q_3})^{-1}o_L = \mathcal{S}^2$, $\mathcal{S}$ is a principal ideal. We conclude that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are principal, then $\mathcal{Y}_1$ and $\mathcal{Y}_2$ capitulate in $K^{(*)}$. Hence the theorem 1.1 follows.

□

Acknowledgments

The First author would like to thank the editor and the anonymous reviewer for their invaluable comments and constructive suggestions used to improve the quality of the manuscript.

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