The construction the new way for the solving of the Volterra integral equations with the symmetric boundaries

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***Abstract*—** One of the priorities directions in modern computational mathematics is to define certain criteria for comparison of existing methods, which are applied to the solving of the different certain problems of natural science. In this connection here we will try to give some criteria for the comparison, of the onestep and multistep methods, which have applied to solving of the Volterra integral equation with the symmetric boundaries. By these comparison we are constructed more accurate methods and prove the advantages of the proposed methods have constructed at the junction of the forward jumping and hybrid methods. For the illustration the results obtained here, we used the model equation to solving of which here, have constructed the specific effective methods having certain accuracy.

***Keywords*—** Volterra integral equation, symmetric boundaries, model equation.

I. Introduction

One of the main questions in the study of numerical methods consist in finding reliable information about of the solution of the considering problem. As is known, all the numerical methods are used to find approximately solutions of the considering problems. Note that the estimates for these methods, are valid for sufficiently small values of h, which is commonly referred to as the step size of integration. Therefore, some scientists have proposed to construct methods, after application which to solve a practical problems, can be obtain a discrete solution having some properties of the solution of considering problems as increases and decreases, as well as some of the other behavior of the exact solution of that problems. Such approaches are relevant with the solution of the Volterra integral equations with the symmetric boundaries, which are connected with the fact that in solving such equations we must to fined the value $y(x)$ and $y(-x)$ of the solution of original problem.

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Note that to obtain reliable results in solving some problems, one can use the two sided methods or by using the predictor-corrector method that can help to find the interval for the changing of step size h. However, here we want to construct the methods for solving Volterra integral equations with the symmetric boundaries by using the information about the solution of the considering problem in the previous and the next mesh points.

Consider to solving of the following Volterra integral equation with the symmetric boundaries:

$$y(x) = f(x) + \int_{-x}^{x} K(x, s, y(s))ds, \; x_0 \le s \le x \le X. \quad (1)$$

Assume that the equation (1) has a unique continuous solution defined on the segment $[-X, X]$. To find the approximately values of the solution of equation (1) on some mesh points, let us divide the segment $[x_0, X]$ into N equal parts by the mesh points $x_i = x_0 + ih (i = 0,1,..., N)$. Here $0 < h$ - is a step size. Let us also denote by the y_i and y_{-i} approximately values, but by the $y(x_i)$ and $y(-x_i)$ through, the exact value of the solution of equation (1) at the mesh points $x_{\pm i}$ $(i = 0,1,..., N)$, respectively.

By the solving equations similarly of the equation (1) face in the study of earthquakes and natural periodic seismic processes, studying of the variation of the tension on the thickness rod the investigation to transmit of the signals and etc. (see [1]-[7]). It should be noted that the relationship between Volterra integral equations with symmetric boundary and the symmetric methods studied in the work [8], in which is given the way for determining the effective methods for solving of the Volterra integral equation with the symmetric boundaries. Here, in contrast to the title of the work is suggested a method for the construction of some algorithms for possessing any properties of the solutions of the equation (1). Such methods are applied to solving of the initial value problem for ordinary differential equations which are studied by different authors (see eg. [9]-[12]). One can be fined the information about of the two-sided methods in the work [13].

As is known, depending from the accuracy of the considering method, which have applied to solving of the equation (1), the various conditions imposed on the kernel $K(x,s,z)$. Here, we assume that the function $K(x,s,z)$, is continuous to the totality of variables and is defined in the set $G=\{x_0 \le s \le x \le X;\ |z| \le a\}$, and also it has the continuous partial derivatives up to some order p, inclusively. Sometimes it is necessary to investigation of the equation (1) in a ε-extension area G, which is defined as: $\overline{G}=\{x_0 \le s \le x+\varepsilon \le X+\varepsilon;\ |z| \le a\}$ (see eg. [14]). However, the use of such an extension domain of the function $K(x,s,z)$ is not essential in the study of the numerical solution of the equation (1). Therefore, we further believe that $\varepsilon = 0$.

II. On a way to construction an algorithm to solving of the equation (1)

Obviously, equation (1) can be written as follows:

$$y(x) = f(x) + \int_0^x \varphi(x,s,y(s))ds,\ \ x_0 \le s \le x \le X, \quad (2)$$

where the integral kernel $\varphi(x,s,z)$ is defined as:

$$\varphi(x,s,y(s)) = K(x,s,y(s)) + K(x,-s,y(-s)) \quad (3)$$

Thus, formally to solving of the equation (1) can be applied one of the known methods, which have used in solving of the Volterra integral equations (2) (see eg.[15]-[22]). In this case, meet with the needed to calculation of the approximately values of the quantity $y(-x_i)$ $(i>0)$. Therefore, using the methods application to the solving of the Volterra integral equations with the fixed boundary do not taking into account the properties of the integral kernel, is not always possible for obtain acceptable results. Therefore, here consider the methods, which are constructed by using some of the properties of the integral kernel. To this end, let us consider to the following method with the constant coefficients:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i}\ \ (m \ge 1,\ n=0,1,...,N-k). \quad (4)$$

Note that the notion of the degree and stability for the method (4) can be determined in the following form:

Definition 1. The integer value p is called a degree of the method (4), if the following is holds:

$$\sum_{i=0}^{k-m}\alpha_i y(x+ih) - h\sum_{i=0}^{k}\beta_i y'(x+ih) = O(h^{p+1}),\ h \to 0.$$

Definition 2. The method (4) is stable if the roots of its characteristic polynomial:

$$\rho(\lambda) = \alpha_{k-m}\lambda^{k-m} + \alpha_{k-m+1}\lambda^{k-m+1} + ... + \alpha_1\lambda + \alpha_0,$$

lies inside of the unit circle, on the boundary which there is no multiple roots.

The stability of the methods has as theoretical and practical interest. By using the definition 2 we define the characteristic polynomial as the following:

$$\rho(\lambda) = \alpha_{k-m}\lambda^{1+j} + \alpha_{k-m-1-j}\ (j=0,1).$$

Here $\alpha_{k-m} \times \alpha_{k-m-1-j} < 0$.

By using the above mentioned the method (4) can be written as follows:

$$y_{n+k-m} = y_{n+k-m-1-j} + h\sum_{i=0}^{k}\beta_i y'_{n+i}\ . \quad (5)$$

Note that from the method of (4) by the selection of coefficients $\alpha_i\ (i=0,1,...,k-m)$ can be obtained various stable methods, the maximum degree for which coincides with the maximum accuracy of the method (5). It is clear, that the condition $k-m>0$ is holds.

If we take into account that the value of k and m is integer, then receive, that $k \ge 2$. In the case $k=2$, from the method of (5) we have:

$$y_{n+1} = y_n + h(5y'_n + 8y'_{n+1} - y'_{n+2})/12. \quad (6)$$

If the method (6) by using the relation (3), is applied to solving of equation (2), then, we have:

$$\begin{aligned} y_{i+1} = y_i + f_{i+1} - f_i + h(3k(x_{i+1},x_{i+1},\overline{y}_{i+1}) +\\ +3k(x_i,x_i,y_i) + 5k(x_{i+2},x_{i+1},\overline{y}_{i+1})\\ +2k(x_{i+1},x_i,y_i) - k(x_{i+2},x_{i+2},\hat{y}_{i+2}))/12 +\\ h(3k(x_{i+1},-x_{i+1},\overline{y}_{-i-1}) + 5k(x_{i+2},-x_{i+1},\overline{y}_{-i-1}) +\\ +3k(x_i,-x_i,y_{-i}) +\\ +2k(x_{i+1},-x_i,y_{-i}) - k(x_{i+2},-x_{i+2},\hat{y}_{-i-2}))/12. \end{aligned} \quad (7)$$

$$\begin{aligned} y_{-i-1} = y_{-i} + f_{-i-1} - f_{-i} - h(3k(-x_{i+1},x_{i+1},\overline{y}_{i+1})\\ +3k(-x_i,x_i,y_i) + 5k(-x_{i+2},x_{i+1},\overline{y}_{i+1}) +\\ +2k(-x_{i+1},x_i,y_i) - k(-x_{i+2},x_{i+2},\hat{y}_{i+2}))/12 -\\ h(3k(-x_{i+1},-x_{i+1},\overline{y}_{-i-1}) + 3k(-x_i,-x_i,y_{-i}) +\\ +5k(-x_{i+2},-x_{i+1},\overline{y}_{-i-1}) +\\ 2k(-x_{i+1},-x_i,y_{-i}) - k(-x_{i+2},-x_{i+2},\hat{y}_{-i-2}))/12. \end{aligned} \quad (8)$$

Note that to calculation of the values $\overline{y}_{n+1}$ and $\overline{y}_{-n-1}$ one can be use the predictor-corrector methods in which as the predictor and corrector methods may be proposed the Euler's method and the trapezoidal rule, respectively. But to calculate the values $\hat{y}_{n+2}$ and $\hat{y}_{-n-2}$ one can be use the following midpoint method:

$$\begin{aligned} \hat{y}_{n+2} = y_n + f_{n+2} - f_n + h(k(x_{i+1},x_{i+1},y_{i+1}) +\\ k(x_{i+2},x_{i+1},y_{i+1})) +\\ h(k(x_{i+1},-x_{i+1},y_{-i-1}) + k(x_{i+2},-x_{i+1},y_{-i-1}). \end{aligned}$$

Method (6) is symmetrical (see.[8]), because for application that to the determination the values of the solution of equation (2) at the mesh point x_{n+1}, must be known the values of the solution of the original problem in the previous mesh point x_n and the next mesh point x_{n+2}. Therefore, its application to solving of equation (1) gives the best result. Indeed, to finding the numerical solution of the equation (1), we used Simpson's method and the method (6). The results obtained by the method (6) are more accurate. To illustration the effect by using the information about the solution of the considering problem in the next mesh point, consider the following forward-jumping method:

$$y_{n+2} = (8y_{n+1} + 11y_n)/19 + \\ + h(10y'_n + 57y'_{n+1} + 24y'_{n+2} - y'_{n+3})/57 . \quad (9)$$

The both methods (6) and (9) have the type of the forward-jumping methods. If formally put $m = 0$, then the method (4) is transformed into a multistep method. As is known stable methods received from the multistep method in the case $k = 2$ and $k = 3$ has the maximum degree $p = 4$. As follows from here the degree of the forward-jumping methods in the first case is less, and in the second case more than the degree of the corresponding multistep methods. But the best results are obtained when these methods are using to solving of practical problems.

Remark that in using forward-jumping methods we have meet some difficulties in the selection of the methods which are applied to calculating the approximately values of the solution of the considering problem in the next mesh points.

Note, that some authors are considered the hybrid method as successful (see.eg.[23]-[27]). However, in their application appeases difficulty for computing the values of the problem in the hybrid points, which can be solved by the predictor-corrector method (see. eg.[27]). In the work [23] is proved, that in the class of methods (4) there are stable methods with the degree $p \le k + m + 1$. It seems that, by the increases the value of m, the values of p also is increases. But it is not right. Remark, that there exist some relation between quantity p and m, which can be written as: $p \le 2k - m$.

From here, receive relation between quantity m and k in the form: $m \le [(k-1)/2] + 1([a]$) is whole parte of a. But the increases values of quantity m are complicate to application of the forward-jumping methods. Therefore we are basically investigated method of (4) for the value $m = 1$ and $m = 2$. Note that there are works in which proved the advantage of the forward-jumping methods and hybrid methods, which are applied to solving of the integral equations of Volterra type (see. eg. [8], [21], [24], [12]) And so, here for solving of integral equations of type (1) and (2) proposed to use the methods from the following classes:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h\sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i} . \quad (10)$$
$$(|\nu_i| < 1, i = 0,1,...,k)$$

Here $\alpha_i, \beta_i, \gamma_i (i = 0,1,...,k)$ - some real numbers.

III. On the Illustration of the Advantage of the Forward-Jumping Methods

We assume that the coefficients of the method (10) satisfies the following conditions:

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \ne 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i; \quad \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i} .$$

have no common multipliers different from the constant.

C: $\delta(1) + \gamma(1) \ne 0$ and $p \ge 1$.

And now to consider construction the stable methods of type (10) having a high degree. To this end, put $m = 1$ and $k = 2$. Then, to determine the values of variables $\alpha_i, \beta_i, \gamma_i, \nu_i (i = 0,1,...,k)$, we obtain the following system nonlinear algebraic equations:

$$\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 1,$$
$$2^j \beta_2 + \beta_1 + l_2^j \gamma_2 + l_1^j \gamma_1 + l_0^j \gamma_0 = 1/(j+1) \ (j = 1,...,7). \quad (11)$$

By using the solution of the system (11) one can be constructed stable methods with the degree $p \le 8$. If in the system (11) using the values $l_2 = 1 + \alpha$, $l_1 = 1$, $l_0 = 1 - \alpha \ (|\alpha| < 1)$ then receive the following solution:

$$\beta_2 = -1/180; \beta_1 = 6/90; \beta_0 = 29/180; l_0 = 1/2,$$
$$\gamma_2 = 1/45, \quad \gamma_1 = 6/90; \quad \gamma_0 = 31/45; \quad l_2 = 3/2.$$

In this case the stable method can be written in the following form:

$$y_{n+1} = y_n + h(29y'_n + 24y'_{n+1} - y'_{n+2})/180 + \\ + h(62y'_{n+1/2} + 2y'_{n+3/2})/90. \quad (12)$$

This method is stable and has the degree $p = 5$.

In [12, p. 277] is proved that if the method (10) for $m = 1$ is stable and has a maximum degree then the coefficients β_k and β_{k-1} satisfied the following conditions $\beta_{k-1} > 0$, $\beta_{k-1}, \beta_k < 0$ and $|\beta_{k-1}| > |\beta_k|$. As is known, if the method (10) for sufficiently smooth solution of the original problem has the degree of p, then it is local error can be written in the following form

$$C_1 h^{p+1} y^{p+1} + C_2 h^{p+2} y^{n+2} + ... + O(h^{p+2}), \ h \to 0.$$

Usually when increases the value of p, the value of the coefficient C_1 is decreases. These parameters for the method of (12), are defined as $p = 5$ and $C_1 = -1/720$. For the obtained coefficients $\alpha_i, \beta_i, \gamma_i \quad (i = 0,1,2,...,k)$ and variables $\nu_i \ (i = 0,1,...,k)$ one can be use the same standard program. If p is sufficiently large (in the redistribution of p = 10), the values of the constant C_1 may be at a short distance from the machine zero. In this case the degree of the method (10) can be extended. Therefore, in such cases it is desirable compliance with caution.

Now consider the application of the method (12) to solving of the equation (2). Then we receive have:

$$y_{n+1} = y_n + f_{n+1} - f_n + h(10\varphi(x_n, x_n, y_n) + 10\varphi(x_{n+1}, x_n, y_n) + 9\varphi(x_{n+2}, x_n, y_n) + 6\varphi(x_{n+1}, x_{n+1}, y_{n+1}) + 6\varphi(x_{n+2}, x_{n+1}, y_{n+1}) - \varphi(x_{n+2}, x_{n+2}, y_{n+2})/180 + h(31\varphi(x_{n+1}, x_{n+1/2}, y_{n+1/2}) + 31\varphi(x_{n+1/2}, x_{n+1/2}, y_{n+1/2}) + 6\varphi(x_{n+1}, x_{n+1}, y_{n+1}) + \varphi(x_{n+2}, x_{n+3/2}, y_{n+3/2}) + \varphi(x_{n+3/2}, x_{n+3/2}, y_{n+3/2}))/90.$$

In the work [8] have investigated the comparison of the method (6) with the following hybrid method by using the solutions of equation (1):

$$y_{n+1} = y_n + h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha})\,(\alpha = \sqrt{3}/6). \quad (13)$$

Here, we study the comparison of the method (6) with the method (9) and show that these methods are use information about the solution of the considering problem, as in the previous and so in the next mesh points, and therefore they give the best results. To this end, we apply the method of (9) to solving of the equation (1). Then we have:

$$y_{i+2} = (8y_{i+1} + 11y_i)/19 + (19f_{i+2} - 8f_{i+1} - 11f_i)/19 - h(k(x_{i+3,}x_{i+3,}y_{i+3}) - 12k(x_{i+2}, x_{i+2}, y_{i+2}) - 12k(x_{i+3}, x_{i+2}, y_{i+2}) - 37k(x_{i+1}, x_{i+1}, y_{i+1}) - 20k(x_{i+2}, x_{i+1}, y_{i+1}) - 10k(x_{i+1}, x_i, y_i))/57 - h(k(x_{i+3}, -x_{i+3}, y_{-i-3}) - 12k(x_{i+2}, -x_{i+2}, y_{-i-2}) - 12k(x_{i+3}, -x_{i+2}, y_{-i-2}) - 37k(x_{i+1}, -x_{i+1}, y_{-i-1}) - 20k(x_{i+2}, -x_{i+1}, y_{-i-1}) - 10k(x_{i+1}, -x_i, y_{-i}))/57, \quad (14)$$

$$y_{-i-2} = (8y_{-i-1} + 11y_{-i})/19 + (19f_{-i-2} - 8f_{-i-1} - 11f_{-i})/19 + h(k(-x_{i+3,}x_{i+3,}y_{i+3}) - 12k(-x_{i+2}, x_{i+2}, y_{i+2}) - 12k(-x_{i+3}, x_{i+2}, y_{i+2}) - 37k(-x_{i+1}, x_{i+1}, y_{i+1}) - 20k(-x_{i+2}, x_{i+1}, y_{i+1}) - 10k(-x_{i+1}, x_i, y_i))/57 + h(k(-x_{i+3}, -x_{i+3}, y_{-i-3}) - 24k(-x_{i+2}, -x_{i+2}, y_{-i-2}) - 57k(-x_{i+1}, -x_{i+1}, y_{-i-1}) - 10k(-x_i, -x_i, y_{-i}))/57. \quad (15)$$

For the selection of effective methods, we have considered application of some multistep methods to solving of the Volterra integral equations. To illustrate the way to selection of effective methods, here have used compares of the methods (6), (9) Simpson, trapezoidal and midpoints rules. In the construction of algorithms for the applications of the implicit methods to solving of the equations (1) and (2) (these equations are considered to be equivalent only in the case when the integral kernel has the form (3)) was used predictor and corrector method, in which in the capacity of the predictor methods proposed the same methods in all the algorithms. Therefore, the order of accuracy of these algorithms can be considers the same.

IV. The Illustration of the Received Results

Note that in all the cases the results received by the forward jumping methods and the hybrid methods were the best. Following are proposed some of these fragments. To this end, investigated the following examples:

Example 1. $y(x) = 1 + \int_0^x y(s)ds,\ 0 \le x \le 1$ (exact solution $y(x) = \exp(x)$),

Example 2. $y(x) = 1 + x^2/2 + \int_0^x y(s)ds,\ 0 \le x \le 1$ (exact solution $y(x) = \exp(x) - x - 1$),

Example 3. $y(x) = \exp(-mx) + m\int_{-x}^{x} y(s)ds, 0 \le x \le 1$ (exact solution $y(x) = \exp(mx)$).

Here we have applied the method (6) and the hybrid method (13) to solving of the equation (1). The results can be considered identical. But to solving of the example 2 have applied the method of (13) and the Simpson's rules. The results are placed in the following table 1:

Table 1.

Variable x	Hybrid	Simpson
0.1	2.6E-11	1.6E-06
0.2	5.4E-11	3.6E-06
0.3	8.3E-11	6.0E-06
0.5	1.4E-11	1.2E-05
1.00	3.4E-11	4.0E-05

For the comparison methods of (6) and (13), here considered to solving of the example 3 by the methods (6) and (13). And the next to solving example 3 are applied the methods (9) and the method (12). Results for them are placed in the table 2 and 3.

Table 2.

Step size	Variable x	method (6)	method (13)
$h = 0{,}05$	0.05	7.8E-07	1.6E-10
	0.1	1.6E-06	2.2E-08
	1.0	4.05E-05	6.3E-06

Table 3.

Step size	Variable x	method (9)	method (12)
$h = 0{,}05$	0.1	1.4E-11	7.1E-9
	0.4	1.5E-8	4.8E-8
	0.7	5.9E-8	9.4E-8
	1.0	1.4E-7	1.4E-7
$h = 0{,}01$	0.1	1.5E-12	1.9E-11
	0.4	3.2E-11	8.6E-11
	0.7	1.09E-10	1.6E-10
	1.0	2.4E-10	2.5E-10

V. CONCLUSION

As is known, each method has its advantages and disadvantages. Often are available to construct methods, that when applied to the solving of some problems gives irregular results, than the known methods (see. eg. [28, p. 410-411]). And it is also known, that by modifying some specific problem one can receive the define class of problems for which purpose specially constructed methods gives the best results, that the known. However, it does not follow from here that the proposed method is the best. For example, one can be shows that the result obtained by the following method:

$$y_{n+1} = y_n + 50hf_n / 49$$

applied to solving of the problem $y' = y,\ y(0) = 1$ is not worse than the result obtained by the explicit Euler's method. As is known the methods which are constructed at the junction of some methods usually are the best than the methods which are used for its construction. Consequently by the above mentioned, we are comparing the quality of some of the known numerical methods, such as stability and accuracy, the region of stability, etc. constructed, here the methods which have the best qualities than the using methods. Remark that the forward-jumping and hybrid methods are constructed on the above said method. Therefore, we have constructed here a new method at the junction of those methods, believing that the method of (10) will be one of the most promising directions in the theory of numerical methods.

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