A Weight Restriction Approach for Evaluating Two-Stage Decision Making Units

A. Payan, A.R Hajihosseini

***Abstract*—**In this paper, the relative efficiency of two-stage decision making units (DMUs) is estimated by modifying the product model proposed by Kao and Hwang (2008) and the additive model by Chen *et al.* (2009). Articles that have so far addressed the evaluation of two-stage units have been mainly unable to calculate relative efficiency. When no calculation of relative efficiency is available, it is not possible to form efficiency frontier, determine benchmark units, estimate returns to scale and so on. Based on the nature of two-stage models, we propose to consider them as data envelopment analysis models with the assurance region type II. In this direction, the relative efficiency of two-stage DMUs is estimated. The validity of the method is also proved. An example is presented to explain the method and draw a comparison between this method and other available methods of two-stage units.

***Keywords*—**Data envelopment analysis; assurance region type ii; relative efficiency; two-stage DMUs.

I. Introduction

DATA envelopment analysis (DEA), which was proposed by Charnes *et al.* [1], is a mathematical programming technique used to measure the relative efficiency of a group of decision making units (DMUs). It ascribes a weight to every input and output. In the DEA method, the efficiency of DMUs is expressed as the ratio of weighted sum of outputs to the weighted sum of inputs. Weights are considered as decision variables in DEA models.

DMUs do not always follow the simple input-output pattern. Multi-stage and multi-component systems are common examples of decision making units with more complex structures. In some decision making units, using a series of inputs leads to production of outputs that are not considered the final outputs of the system. These outputs, which are known as the middle data, form the input of the second stage and yield final outputs. Such decision making units are called two-stage decision making units. The overall structure of two-stage units is shown in Fig. 1.

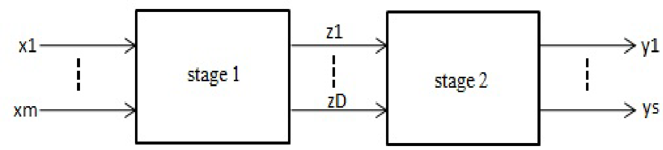

Fig. 1 Two-stage decision making unit

Two-stage decision making units are widely used for real world purposes. For instance, they are employed in banks, social insurance systems, etc. [2]. Numerous studies have been initiated since 1999 to study two-stage decision making units. Kao and Hwang [3] introduced a model for calculation the first and second stage efficiencies as well as overall system efficiency using the multiplier form of CCR model. They defined overall system efficiency as a product of the first and second stage efficiencies. They introduced a model, known as the product model, for calculation the efficiency of two-stage units. Chen *et al.* [4] considered overall system efficiency to be a convex combination of the first and second stage efficiencies and proposed a model known as the additive model. In addition to the aforementioned methods, which were based on the multiplier forms of DEA models, researchers also have focused on the assessment of performance of two-stage units based on the envelopment forms of DEA. Chen and Zhu [5] introduced an envelopment form for the assessment of performance of two-stage units with variable return to scale assumption. Chen *et al.* [6] also proved equivalence of the model developed by Chen and Zhu [5] in constant return to scale situation and dual model by Kao and Hwang [3]. Chen *et al.* [7] reported that the existing envelopment models are not capable of determining efficient frontier and frontier projection (benchmark unit) and this is a basic problem in evaluating two-stage units. For more information on two-stage DEA models see the article by Cook *et al.* [8].

One of the issues with data envelopment analysis is the study of the effect of assurance regions on DEA models. In DEA, assurance regions are divided into two groups: 1) assurance regions type I, which only impose constraints on inputs or outputs; 2) assurance regions type II, which impose constraints on both inputs and outputs. Application of the assurance regions type II on DEA models often leads to impossibility of calculation of relative efficiency [9]. The most important problems in such cases are lack of formation of an efficient frontier and the inability to produce a frontier projection. Recently, Khalili *et al.* [10] have introduced a

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method for calculation of relative efficiency units with linear assurance regions type II. Their model is a non-linear programming problem.

The multiplier models developed for calculation the efficiency of two-stage DMUs include two types of constraints, which have the nature of the assurance regions type II. Hence, it is possible for these models to fail to calculate relative efficiency. As a result, the aforementioned problems emerge. Some examples are provided in the second section of this paper to clarify the point. By considering two-stage multiplier models as DEA models with assurance region type II, these models (productive and additive models) are then modified such that they become capable of calculating relative efficiency.

Hence, the present paper includes the following sections. The second section discusses two-stage DEA models and their disadvantages. The third section proposes a method for calculation of relative efficiency of two-stage decision making units. An example is also provided for a better understanding of the method. The final section of the paper presents a conclusion.

II. Two-stage DEA

Consider a two-stage process and assume that there are n decision making units. Every DMU_j $(j=1,\dots,n)$ uses m index x_{ij} $(i=1,\dots,m)$ as its inputs to produce D index z_{dj} $(d=1,\dots,D)$ as outputs for the first stage. Next, the D outputs are used as the input for the second stage and produce the outputs of this stage, which are shown by y_{rj} $(r=1,\dots,s)$. This process is depicted in Fig. 1.

A. Productive two-stage DEA

Kao and Hwang [3] developed a model for calculation the efficiency in two-stage DEA. Their method is able to calculate overall efficiency as well as first and second stage efficiencies by an objective function as:

$$\left(\sum_{r=1}^{s}u_r y_{ro} \Big/ \sum_{d=1}^{D}w_d z_{do}\right)*\left(\sum_{d=1}^{D}w_d z_{do} \Big/ \sum_{i=1}^{m}v_i x_{io}\right)=\sum_{r=1}^{s}u_r y_{ro} \Big/ \sum_{i=1}^{m}v_i x_{io}$$

This model is known as product model and is expressed as follows:

$$\begin{aligned}
&max\ \sum_{r=1}^{s}u_r y_{ro} \Big/ \sum_{i=1}^{m}v_i x_{io},\\
&s.t\left(\sum_{r=1}^{s}u_r y_{rj} \Big/ \sum_{d=1}^{D}w_d z_{dj}\right)\le 1,\ j=1,\dots,n, &(1\text{-}1)\\
&\left(\sum_{d=1}^{D}w_d z_{dj} \Big/ \sum_{i=1}^{m}v_i x_{ij}\right)\le 1,\ j=1,\dots,n, &(1\text{-}2)\\
&v_i\ge 0,\ i=1,\dots,m,\\
&u_r\ge 0,\ r=1,\dots,s,\\
&w_d\ge 0,\ d=1,\dots,D, &(1)
\end{aligned}$$

Results of the product model (1) applied to measure the efficiency of 5 two-stage units presented in Table I are shown in the second column of Table II. Accordingly, none of the DMUs where shown to be efficient. That is to say, the product model (1) is not capable for calculating the relative efficiency of two-stage decision making units.

Table I Five Two-stage DMUs

DMU	X1	X2	Z1	Z2	Y1	Y2
1	1178744	673512	7451757	856735	984143	681687
2	1381822	1352755	10020274	1812894	1228502	834754
3	1177494	592790	4776548	560244	293613	658428
4	601320	594259	3174851	371863	248709	177331
5	6699063	3531614	37392862	1753794	7851229	3925272

B. Additive two-stage DEA

Chen *et al.* [4] studied the overall efficiency of two-stage decision making units as the convex combination of the first and second stage efficiencies. Hence, the overall efficiency of DMU_o is expressed as follows:

$$w_1\left(\sum_{d=1}^{D}w_d z_{do} \Big/ \sum_{i=1}^{m}v_i x_{io}\right)+w_2\left(\sum_{r=1}^{s}u_r y_{ro} \Big/ \sum_{d=1}^{D}w_d z_{do}\right)\quad(2)$$

where, w_1 and w_2 are weights satisfying the relationship $w_1+w_2=1$.

Chen *et al.* [4] developed the following model based on Relation (2). This model is known as the additive model and is designed to assess the performance of two-stage decision making units.

$$\begin{aligned}
&\max\ w_1\left(\sum_{d=1}^{D}w_d z_{do} \Big/ \sum_{i=1}^{m}v_i x_{io}\right)+w_2\left(\sum_{r=1}^{s}u_r y_{ro} \Big/ \sum_{d=1}^{D}w_d z_{do}\right),\\
&s.t\left(\sum_{r=1}^{s}u_r y_{rj} \Big/ \sum_{d=1}^{D}w_d z_{dj}\right)\le 1,\ j=1,\dots,n,\\
&\left(\sum_{d=1}^{D}w_d z_{dj} \Big/ \sum_{i=1}^{m}v_i x_{ij}\right)\le 1,\ j=1,\dots,n,\\
&v_i\ge 0,\ i=1,\dots,m,\\
&u_r\ge 0,\ r=1,\dots,s,\\
&w_d\ge 0,\ d=1,\dots,D, &(3)
\end{aligned}$$

They defined w_1 and w_2 as follows in order to solve the above model [4]:

$$w_1=\sum_{i=1}^{m}v_i x_{io} \Big/ \left(\sum_{i=1}^{m}v_i x_{io}+\sum_{d=1}^{D}w_d z_{do}\right)$$

$$w_2=\sum_{d=1}^{D}w_d z_{do} \Big/ \left(\sum_{i=1}^{m}v_i x_{io}+\sum_{d=1}^{D}w_d z_{do}\right)$$

Hence, model (3) is transformed into the following model as:

$$\begin{aligned}
&\max\left(\sum_{d=1}^{D}w_d z_{do}+\sum_{r=1}^{s}u_r y_{ro}\right)\Big/\left(\sum_{i=1}^{m}v_i x_{io}+\sum_{d=1}^{D}w_d z_{do}\right),\\
&s.t\left(\sum_{r=1}^{s}u_r y_{rj} \Big/ \sum_{d=1}^{D}w_d z_{dj}\right)\le 1,\ j=1,\dots,n, &(4\text{-}1)\\
&\left(\sum_{d=1}^{D}w_d z_{dj} \Big/ \sum_{i=1}^{m}v_i x_{ij}\right)\le 1,\ j=1,\dots,n, &(4\text{-}2)\\
&v_i\ge 0,\ i=1,\dots,m,\\
&u_r\ge 0,\ r=1,\dots,s,\\
&w_d\ge 0,\ d=1,\dots,D, &(4)
\end{aligned}$$

As seen in the fourth column of Table II, none of the DMUs in Table I are shown to be efficient by solving model (4). That is to say, the additive model (4) is not capable of calculating the relative efficiency based on the data presented in Table I.

Table II Measuring overall efficiency of DMUs in Table I

DMU	Kao and Hwang [3]	Model (7)	Chen *et al.* [4]	Model (10)
1	0.79	0.98	0.89	1
2	0.72	1	0.86	1
3	0.73	0.99	0.84	1
4	0.35	0.48	0.62	0.84
5	0.95	1	0.97	1

Therefore, there is a basic problem in two-stage models which they are not able to calculate relative efficiency, while the first aim of DEA is comparing DMUs together.

III. Relative Efficiency

As mentioned, constraints (1-1) and (1-2) in model (1) and constraints (4-1) and (4-2) in model (4) have the nature of the assurance region type II. Presence of assurance region type II in these models is the main cause of their inability to calculate relative efficiency. Hence, these models are considered as DEA models with assurance region type II and modifications were performed to calculate the relative efficiency of two-stage units.

A. Relative efficiency in product two-stage DEA

By applying the method developed by Thompson and Thrall [11] to model (1), the following fractional model is obtained for calculation of relative efficiency of two-stage decision making units:

$$f\,(v,u,w)=max\;\frac{\left(\sum_{r=1}^{s}u_r y_{ro}\Big/\sum_{i=1}^{m}v_i x_{io}\right)}{\max\limits_{j=1,\dots,n}\left\{\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right\}},$$

$$s.t\left(\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{d=1}^{D}w_d z_{dj}\right)\le 1,\; j=1,\dots,n, \quad (5\text{-}1)$$

$$\left(\sum_{d=1}^{D}w_d z_{dj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right)\le 1,\; j=1,\dots,n, \quad (5\text{-}2)$$

$$v_i\ge 0,\; i=1,\dots,m,$$
$$u_r\ge 0,\; r=1,\dots,s,$$
$$w_d\ge 0,\; d=1,\dots,D, \quad (5)$$

We define $c=\max\limits_{j=1,\dots,n}\left\{\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right\}$. According to constraints (5-1) and (5-2), $c\le 1$. The values of all input and output factors as well as middle data are larger than zero and thus $c\ge 0$. Hence, based on Khalili *et al.* [10], the following model is developed as:

$$g\,(v,u,w,c)=max\;\frac{1}{c}\left(\sum_{r=1}^{s}u_r y_{ro}\Big/\sum_{i=1}^{m}v_i x_{io}\right),$$

$$s.t\left(\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right)\le c,\; j=1,\dots,n,$$

$$\left(\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{d=1}^{D}w_d z_{dj}\right)\le 1,\; j=1,\dots,n,$$

$$\left(\sum_{d=1}^{D}w_d z_{dj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right)\le 1,\; j=1,\dots,n,$$

$$v_i\ge 0,\; i=1,\dots,m,$$
$$u_r\ge 0,\; r=1,\dots,s,$$
$$w_d\ge 0,\; d=1,\dots,D, \quad (6)$$

The above model is rewritten based on the variable transformation introduced by Charnes and Cooper [12] as follows:

$$k\,(v,u,w,c)=max\;\sum_{r=1}^{s}u_r y_{ro},$$

$$s.t\;\; c\sum_{i=1}^{m}v_i x_{io}=1,$$

$$\sum_{r=1}^{s}u_r y_{rj}-c\left(\sum_{i=1}^{m}v_i x_{ij}\right)\le 0,\quad j=1,\dots,n,$$

$$\sum_{d=1}^{D}w_d z_{dj}-\sum_{i=1}^{m}v_i x_{ij}\le 0,\quad j=1,\dots,n,$$

$$\sum_{r=1}^{s}u_r y_{rj}-\sum_{d=1}^{D}w_d z_{dj}\le 0,\quad j=1,\dots,n,$$

$$v_i\ge 0,\; i=1,\dots,m,$$
$$u_r\ge 0,\; r=1,\dots,s,$$
$$w_d\ge 0,\; d=1,\dots,D, \quad (7)$$

Definition 1: DMU_o is an efficient two-stage decision making unit if the optimal value of model (7) is 1 and all variables have positive values, for at least one optimal solution.

Theorem 2: Models (5) and (6) are equivalent as they have equal optimal objective values.

Proof: Assume (v,u,w,c) is an optimal solution to model (6). Therefore, (v,u,w) is a feasible solution to model (5). Moreover, since $c\ge\max\limits_{j=1,\dots,n}\left\{\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right\}$ and $0\le c\le 1$ then:

$$1\Big/\max_{j=1,\dots,n}\left\{\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right\}\ge\frac{1}{c}\Rightarrow$$

$$\frac{\left(\sum_{r=1}^{s}u_r y_{ro}\Big/\sum_{i=1}^{m}v_i x_{io}\right)}{\max\limits_{j=1,\dots,n}\left\{\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right\}}\ge\frac{\left(\sum_{r=1}^{s}u_r y_{ro}\Big/\sum_{i=1}^{m}v_i x_{io}\right)}{c}$$

$$\Rightarrow g\,(v,u,w,c)\le f\,(v,u,w)$$

Hence, the optimal value of model (6) is less than or equal to the optimal value of model (5).

On the other hand, if $(\bar{v},\bar{u},\bar{w})$ is an optimal solution to model (5) and $\bar{c}=\max\limits_{j=1,\dots,n}\left\{\sum_{r=1}^{s}\bar{u}_r y_{rj}\Big/\sum_{i=1}^{m}\bar{v}_i x_{ij}\right\}$, then $(\bar{v},\bar{u},\bar{w},\bar{c})$ is a feasible solution to model (6). In addition,

$$1\Big/\max_{j=1,\dots,n}\left\{\sum_{r=1}^{s}\bar{u}_r y_{rj}\Big/\sum_{i=1}^{m}\bar{v}_i x_{ij}\right\}=\frac{1}{\bar{c}}\Rightarrow$$

$$\left(\sum_{r=1}^{s}\bar{u}_r y_{ro}\Big/\sum_{i=1}^{m}\bar{v}_i x_{io}\right)\Big/\max_{j=1,\dots,n}\left\{\sum_{r=1}^{s}\bar{u}_r y_{rj}\Big/\sum_{i=1}^{m}\bar{v}_i x_{ij}\right\}$$

$$=\left(\sum_{r=1}^{s}\bar{u}_r y_{ro}\Big/\sum_{i=1}^{m}\bar{v}_i x_{io}\right)\Big/\bar{c}\Rightarrow g\,(\bar{v},\bar{u},\bar{w},\bar{c})=f\,(\bar{v},\bar{u},\bar{w})$$

Therefore, if (v,u,w,c) is an optimal solution to model (6), then $g(v,u,w,c)\geq g(\bar{v},\bar{u},\bar{w},\bar{c})$ and $g(v,u,w,c)\geq f(\bar{v},\bar{u},\bar{w})$. Hence, the optimal value of model (5) is less than or equal to the optimal value of model (6). It is concluded that the optimal values of (5) and (6) are equal.

Accordingly, the efficiency values obtained by models (5) and (6) are equal. Therefore, instead of solving model (5), which gives the relative efficiency of two-stage decision making units, it is possible to use model (6) to calculate the relative efficiency of two-stage units.

Theorem 3: Models (6) is equivalent to model (7).

Proof: Assume (v,u,w,c) is an optimal solution to model (6). If $t=1\Big/c\left(\sum_{i=1}^{m}v_i x_{io}\right)$, then $(V,U,W,c)=(tv,tu,tw,c)$ is a feasible solution to model (7). On the other hand,

$$k\,(V,U,W,c)=\sum_{r=1}^{s}U_r y_{ro}=t\sum_{r=1}^{s}u_r y_{ro}$$

$$=\sum_{r=1}^{s}u_r y_{ro}\Big/c\left(\sum_{i=1}^{m}v_i x_{io}\right)=g\,(v,u,w,c)$$

Hence, the optimal value of model (6) is less than or equal to the optimal value of model (7).

If $(\bar{V},\bar{U},\bar{W},\bar{C})$ is the optimal solution to model (7), it is a feasible solution to model (6) too. Moreover,

$$g\,(\bar{V},\bar{U},\bar{W},\bar{C})=\sum_{r=1}^{s}\bar{U}_r y_{ro}\Big/\bar{C}\left(\sum_{i=1}^{m}\bar{V}_i x_{io}\right)$$

$$=\sum_{r=1}^{s}\bar{U}_r y_{ro}\Big/1=k\,(\bar{V},\bar{U},\bar{W},\bar{C})$$

Therefore, the optimal value of model (7) is less than or equal to the optimal value of model (6). In this case, the optimal values of models (6) and (7) are equal and these models are equivalent accordingly.

Result 4: Models (5) and (7) are equivalent in terms of their optimal values.

Result 5: Model (6) calculates the relative efficiency of two-stage units.

Result 6: Model (7) calculates the relative efficiency of two-stage units.

Based on the above mentioned theorems and results, we can define the concept of reference point (DMU) for non-efficient units.

Definition 7: Consider (v,u,w,c) as an optimal solution for evaluating DMU_o by model (10). DMU_p is a reference for DMU_o, if

$$\sum_{r=1}^{s}u_r y_{rp}-c\left(\sum_{i=1}^{m}v_i x_{ip}\right)=0$$

B. Relative efficiency in additive two-stage DEA

In order to calculate relative efficiency by the additive two-stage DEA model the above process is iterated and model (4) is transformed into the following fractional model as:

$$F\,(v,u,w)=max\ \left(\frac{\sum_{d=1}^{D}w_d z_{do}+\sum_{r=1}^{s}u_r y_{ro}}{\sum_{d=1}^{D}w_d z_{do}+\sum_{i=1}^{m}v_i x_{io}}\right)\Bigg/\max_{j=1,\dots,n}\left\{\frac{\sum_{d=1}^{D}w_d z_{dj}+\sum_{r=1}^{s}u_r y_{rj}}{\sum_{d=1}^{D}w_d z_{dj}+\sum_{i=1}^{m}v_i x_{ij}}\right\}$$

$$s.t\left(\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{d=1}^{D}w_d z_{dj}\right)\leq 1,\ j=1,\dots,n,$$

$$\left(\sum_{d=1}^{D}w_d z_{dj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right)\leq 1,\ j=1,\dots,n,$$

$$v_i\geq 0,\ i=1,\dots,m,$$

$$u_r\geq 0,\ r=1,\dots,s,$$

$$w_d\geq 0,\ d=1,\dots,D, \qquad (8)$$

If $c=\max_{j=1,\dots,n}\left\{\left(\sum_{d=1}^{D}w_d z_{dj}+\sum_{r=1}^{s}u_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D}w_d z_{dj}+\sum_{i=1}^{m}v_i x_{ij}\right)\right\}$, the following fractional model is derived from model (8):

$$G\,(v,u,w,c)=max\ \frac{1}{c}\left(\left(\sum_{d=1}^{D}w_d z_{do}+\sum_{r=1}^{s}u_r y_{ro}\right)\Big/\left(\sum_{d=1}^{D}w_d z_{do}+\sum_{i=1}^{m}v_i x_{io}\right)\right),$$

$$s.t\left(\sum_{d=1}^{D}w_d z_{dj}+\sum_{r=1}^{s}u_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D}w_d z_{dj}+\sum_{i=1}^{m}v_i x_{ij}\right)\leq c,$$

$$j=1,\dots,n,$$

$$\left(\sum_{r=1}^{s}u_r y_{rj}\Big/\sum_{d=1}^{D}w_d z_{dj}\right)\leq 1,\ j=1,\dots,n,$$

$$\left(\sum_{d=1}^{D}w_d z_{dj}\Big/\sum_{i=1}^{m}v_i x_{ij}\right)\leq 1,\ j=1,\dots,n,$$

$$v_i\geq 0,\ i=1,\dots,m,$$

$$u_r\geq 0,\ r=1,\dots,s,$$

$$w_d\geq 0,\ d=1,\dots,D, \qquad (9)$$

The above model can be rewritten as follows according to the method introduced by Charnes and Cooper [12]:

$$K\,(v,u,w,c)=max\ \sum_{d=1}^{D}w_d z_{do}+\sum_{r=1}^{s}u_r y_{ro},$$

$$s.t\ c\left(\sum_{d=1}^{D}w_d z_{do}+\sum_{i=1}^{m}v_i x_{io}\right)=1,$$

$$\sum_{d=1}^{D}w_d z_{dj}+\sum_{r=1}^{s}u_r y_{rj}-c\left(\sum_{d=1}^{D}w_d z_{dj}+\sum_{i=1}^{m}v_i x_{ij}\right)\leq 0,$$

$$j=1,\dots,n,$$

$$\sum_{d=1}^{D}w_d z_{dj}-\sum_{i=1}^{m}v_i x_{ij}\leq 0,\quad j=1,\dots,n,$$

$$\sum_{r=1}^{s}u_r y_{rj}-\sum_{d=1}^{D}w_d z_{dj}\leq 0,\quad j=1,\dots,n,$$

$$v_i \geq 0,\ i = 1,\ldots,m,$$
$$u_r \geq 0,\ r = 1,\ldots,s,$$
$$w_d \geq 0,\ d = 1,\ldots,D, \tag{10}$$

Definition 8: DMU_o is an efficient two-stage unit if the optimal value of model (10) is equal to 1 and all variables have positive values, for at least one optimal solution.

Theorem 9: Models (8) and (9) are equivalent.

Proof: If (v,u,w,c) is the optimal solution to model (9), then (v,u,w) is a feasible solution to model (8). In addition, if $(\bar{v},\bar{u},\bar{w})$ is an optimal solution to model (8), then $F(v,u,w) \leq F(\bar{v},\bar{u},\bar{w})$. In model (9) $c \geq \max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{r=1}^{s} u_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{i=1}^{m} v_i x_{ij}\right)\right\}$ and $0 \leq c \leq 1$, hence:

$$1\Big/\max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{r=1}^{s} u_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{i=1}^{m} v_i x_{ij}\right)\right\} \geq \frac{1}{c} \Rightarrow$$

$$\frac{\left(\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{r=1}^{s} u_r y_{ro}\right)\Big/\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{i=1}^{m} v_i x_{io}\right)\right)}{\max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{r=1}^{s} u_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} w_d z_{dj} + \sum_{i=1}^{m} v_i x_{ij}\right)\right\}} \geq \frac{\left(\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{r=1}^{s} u_r y_{ro}\right)\Big/\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{i=1}^{m} v_i x_{io}\right)\right)}{c}$$

Therefore, $G(v,u,w,c) \leq F(v,u,w) \leq F(\bar{v},\bar{u},\bar{w})$ and the optimal value of model (9) is less than or equal to the optimal value of model (8). On the contrary, if $(\bar{v},\bar{u},\bar{w})$ is the optimal value of model (8), then $(\bar{v},\bar{u},\bar{w},\bar{c})$ is a feasible solution to model (9) with $\bar{c} = \max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{r=1}^{s} \bar{u}_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{i=1}^{m} \bar{v}_i x_{ij}\right)\right\}$.

Moreover, if (v,u,w,c) is the optimal solution to model (9), then $G(v,u,w,c) \geq G(\bar{v},\bar{u},\bar{w},\bar{c})$. Since

$$1\Big/\max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{r=1}^{s} \bar{u}_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{i=1}^{m} \bar{v}_i x_{ij}\right)\right\} = \frac{1}{\bar{c}} \Rightarrow$$

$$\frac{\left(\left(\sum_{d=1}^{D} \bar{w}_d z_{do} + \sum_{r=1}^{s} \bar{u}_r y_{ro}\right)\Big/\left(\sum_{d=1}^{D} \bar{w}_d z_{do} + \sum_{i=1}^{m} \bar{v}_i x_{io}\right)\right)}{\max_{j=1,\ldots,n}\left\{\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{r=1}^{s} \bar{u}_r y_{rj}\right)\Big/\left(\sum_{d=1}^{D} \bar{w}_d z_{dj} + \sum_{i=1}^{m} \bar{v}_i x_{ij}\right)\right\}} = \frac{\left(\left(\sum_{d=1}^{D} \bar{w}_d z_{do} + \sum_{r=1}^{s} \bar{u}_r y_{ro}\right)\Big/\left(\sum_{d=1}^{D} \bar{w}_d z_{do} + \sum_{i=1}^{m} \bar{v}_i x_{io}\right)\right)}{\bar{c}}$$

, the following relation is true: $G(v,u,w,c) \geq G(\bar{v},\bar{u},\bar{w},\bar{c}) = F(\bar{v},\bar{u},\bar{w},\bar{c})$.

Hence, the optimal value of model (8) is less than or equal to the optimal value of model (9) and thus the optimal values of models (8) and (9) are equivalent.

The efficiency values obtained for models (8) and (9) are equal. Therefore, it is possible to use model (9) to calculate the relative efficiency of two-stage units instead of model (8).

Theorem 10: Models (9) and (10) are equivalent.

Proof: Assume (v,u,w,c) is the optimal solution to model (9). If $t = 1\Big/c\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{i=1}^{m} v_i x_{io}\right)$, then $(tv,tu,tw,c) = (V,U,W,c)$ is a feasible solution to model (10). Moreover, if $(\bar{V},\bar{U},\bar{W},\bar{C})$ is the optimal solution to model (10), then $K(V,U,W,c) \leq K(\bar{V},\bar{U},\bar{W},\bar{C})$. On the other hand,

$$K(V,U,W,c) = \sum_{d=1}^{D} W_d z_{do} + \sum_{r=1}^{s} U_r y_{ro}$$
$$= t\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{r=1}^{s} u_r y_{ro}\right) = \frac{\sum_{d=1}^{D} w_d z_{do} + \sum_{r=1}^{s} u_r y_{ro}}{c\left(\sum_{d=1}^{D} w_d z_{do} + \sum_{i=1}^{m} v_i x_{io}\right)}$$
$$= G(v,u,w,c)$$

Hence, $G(v,u,w,c) \leq K(\bar{V},\bar{U},\bar{W},\bar{C})$ and the optimal value of model (9) is less than or equal to the optimal value of model (10).

On the contrary, if $(\bar{V},\bar{U},\bar{W},\bar{C})$ is the optimal solution to model (10), then it is considered as a feasible solution to model (9). In addition, if (V,U,W,C) is an optimal solution to model (9), then $G(\bar{V},\bar{U},\bar{W},\bar{C}) \leq G(V,U,W,C)$.

On the other hand, $G(\bar{V},\bar{U},\bar{W},\bar{C})$

$$= \left(\sum_{d=1}^{D} \bar{W}_d z_{do} + \sum_{r=1}^{s} \bar{U}_r y_{ro}\right)\Big/\bar{C}\left(\sum_{d=1}^{D} \bar{W}_d z_{do} + \sum_{i=1}^{m} \bar{V}_i x_{io}\right).$$
$$= \sum_{d=1}^{D} \bar{W}_d z_{do} + \sum_{r=1}^{s} \bar{U}_r y_{ro}\Big/1 = K(\bar{V},\bar{U},\bar{W},\bar{C})$$

Hence, $K(\bar{V},\bar{U},\bar{W},\bar{C}) \leq G(V,U,W,C)$ and the optimal value of model (10) is less than or equal to that of model (9). It is therefore concluded that the optimal values of models (9) and (10) are equal and these models are equivalent accordingly.

Result 11: Models (8) and (10) are equivalent for having equal optimal values.

Result 12: Model (9) calculates the relative efficiency of two-stage units.

Result 13: Model (10) calculates the relative efficiency of two-stage units.

Definition 14: Consider (v,u,w,c) as an optimal solution for evaluating DMU_o by model (10). DMU_p is a reference for DMU_o, if

$$\sum_{d=1}^{D} w_d z_{dp} + \sum_{r=1}^{s} u_r y_{rp} - c\left(\sum_{d=1}^{D} w_d z_{dp} + \sum_{i=1}^{m} v_i x_{ip}\right) = 0$$

IV. EXAMPLE

Here, we evaluate the performance of 5 two-stage units in Table I by using models (7) and (10). According to model (7), 5 two-stage decision making units in Table I are assessed which are shown in the third column of Table II. By this direction to measure performance, units 2 and 5 are evaluated efficient. Also, as seen in the last column of Table II, the relative efficiency of two-stage units is expressed based on model (10). According to the results, units 1, 2, 3 and 5 are efficient. A comparison between classic models of two-stage DEA and new proposed models are provided in Figs. 2 and 3.

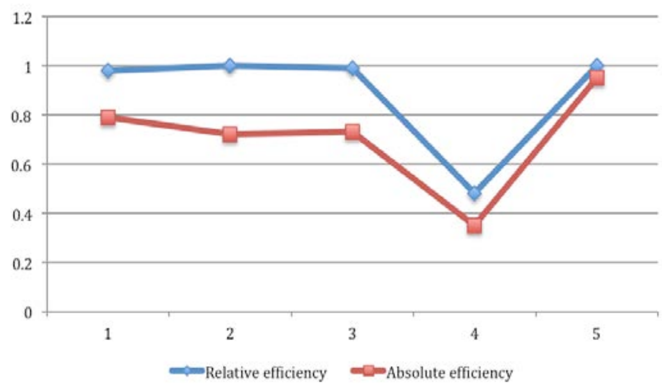

Fig. 2 Comparison between absolute efficiency of model (1) and relative efficiency of model (7).

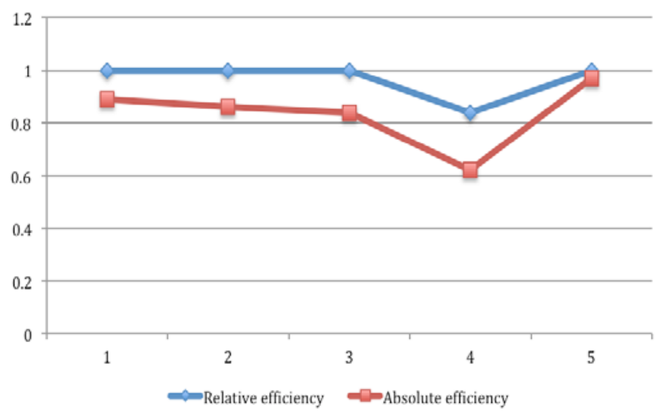

Fig. 3 Comparison between absolute efficiency of model (4) and relative efficiency of model (10).

Data on 24 Taiwanese companies, which was prepared by Kao and Hwang [3], are shown in Table III. Insurance costs and research costs form the inputs of the first stage, while direct premiums and self-insurance premiums are the middle data. In addition, commitment profits and investment incomes are considered as the final outputs of the system.

Table IV shows the efficiency scores of units are obtained using models (7) and (10) as well as the models proposed by Kao and Hwang [3] and Chen *et al.* [4]. According to the models proposed in this research, some units are efficient but according to previous models, none of the units are efficient. By solving model (7), units 2, 5, 12 and 22 are efficient and by model (10), units 1, 2, 5, 9, 12, 15, 19, 20, 22 and 24 are efficient.

Table V reports reference two-stage DMUs for non-efficient two-stage units. For example, units 5, 12 and 22 are references of unit 8, based on model (7).

Table III Data of 24 Taiwanese companies

DMU	X1	X2	Z1	Z2	Y1	Y2
1	1178744	673512	7451757	856735	984143	681687
2	1381822	1352755	10020274	1812894	1228502	834754
3	1177494	592790	4776548	560244	293613	658428
4	601320	594259	3174851	371863	248709	177331
5	6699063	3531614	37392862	1753794	7851229	3925272
6	2627707	668363	9747908	952326	1713598	415058
7	1942833	1443100	10685457	643412	2239593	439039
8	3789001	1873530	17267266	1134600	3899530	622868
9	1567746	950432	11473162	546337	1043778	264098
10	1303249	1298470	8210389	504528	1697941	554806
11	1962448	672414	7222378	643178	1486014	18259
12	2592790	650952	9434406	1118489	1574191	909295
13	2609941	1368802	13921464	811343	3609236	223047
14	1396002	988888	7396396	465509	1401200	332283
15	2184944	651063	10422297	749893	3355197	555482
16	1211716	415071	5606013	402881	854054	197947
17	1453797	1085019	7695461	342489	3144484	371984
18	757515	547997	3631484	995620	692731	163927
19	159422	182338	1141950	483291	519121	46857
20	145442	53518	316829	131920	355624	26537
21	84171	26224	225888	40542	51950	6491
22	15993	10502	52063	14574	82141	4181
23	54693	28408	245910	49864	0.1	18980
24	163297	235094	476419	644816	142370	16976

Table IV Results of different methods

DMU	Kao and Hwang [3]	Model (7)	Chen *et al.* [4]	Model (10)
1	0.699	0.984	0.849	1
2	0.625	1	0.812	1
3	0.690	0.988	0.817	0.9805
4	0.304	0.488	0.596	0.8404
5	0.767	1	0.873	1
6	0.390	0.594	0.689	0.982
7	0.277	0.470	0.580	0.874
8	0.275	0.415	0.579	0.855
9	0.223	0.327	0.612	1
10	0.466	0.781	0.713	0.931
11	0.164	0.283	0.509	0.851
12	0.760	1	0.880	1
13	0.208	0.353	0.557	0.937
14	0.289	0.470	0.577	0.859
15	0.614	0.979	0.807	1
16	0.320	0.472	0.639	0.951
17	0.360	0.635	0.613	0.923
18	0.259	0.427	0.587	0.884
19	0.411	0.822	0.706	1
20	0.547	0.935	0.765	1
21	0.201	0.333	0.541	0.8578
22	0.590	1	0.742	1
23	0.420	0.599	0.685	0.934
24	0.135	0.257	0.544	1

Table V Reference points of inefficient two-stage units

DMU	Reference DMUs by Model (7)	Reference DMUs by Model (10)
1	2	1
2	2	2
3	5,12	12
4	2	2,9,19
5	5	5
6	12	12,15
7	5,22	1,9,19,22
8	5,12,22	1,9,15,19,22
9	5,22	9
10	5,22	2,9,19
11	22	9,15
12	12	12
13	12,22	9,15,22
14	5,22	1,9,19,22
15	12,22	15
16	5,12,22	9,15,19
17	5,22	1,9,19,22
18	5,22	15,19
19	5,22	19
20	12,22	20
21	12,22	12,19,20
22	22	22
23	5,12	1,12,15,19
24	5,22	24

V. CONCLUSION

In this research, the models developed by Kao and Hwang [3] and Chen *et al.* [4] were modified to be able to calculate the relative efficiency of two-stage decision making units. Some theorems were used to indicate that the proposed models are always capable for calculating the relative efficiency in two-stage DEA. By extending the dual models of multiplier forms extended in this paper, envelopment models can be obtained for production frontier projection of two-component inefficient units. Moreover, it is possible to test ranking, productivity, benchmarking based on the proposed models. Similar direction can be done to measure the relative efficiency in two-stage DEA with variable returns to scale assumption.

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