On the existence of weak solutions for a nonlinear system of Atmosphere dynamics counting with humidity and heat transfer

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***Abstract*—**A nonlinear model of the mathematical fluid dynamics of the Atmosphere is considered. The model is a generalization of the nonlinear Navier-Stokes system with the addition of the equations for changeable density, humidity, moisture content in the clouds and heat transfer. An explicit algorithm for a weak solution is constructed by Galerkin method, the "a priori" estimates for the weak solution are obtained and the proof of the existence of the weak solution is given.

***Keywords*—**A Priori Estimates, Galerkin Method, Navier-Stokes Equations, Nonlinear Partial Differential Equations, Weak Solutions.

I. INTRODUCTION

LET us consider a bounded domain $\Omega \in R^3$ with a smooth boundary, and the following nonlinear system of fluid dynamics

$$\begin{cases} \frac{\partial v_1}{\partial t} - \omega v_2 - \nu_1 \Delta v_1 + v' \cdot \nabla v_1 + \frac{\partial p}{\partial x_1} = f_1 \\ \frac{\partial v_2}{\partial t} + \omega v_1 - \nu_1 \Delta v_2 + v' \cdot \nabla v_2 + \frac{\partial p}{\partial x_2} = f_2 \\ \frac{\partial v_3}{\partial t} - \nu_1 \Delta v_3 + v' \cdot \nabla v_3 - \mu v_4 + g v_5 + \frac{\partial p}{\partial x_3} = f_3 \\ \frac{\partial v_4}{\partial t} - \nu_2 \Delta v_4 + v' \cdot \nabla v_4 + s v_3 = f_4 \\ \frac{\partial v_5}{\partial t} - \nu_2 \Delta v_5 + v' \cdot \nabla v_5 = f_5 \\ \frac{\partial v_6}{\partial t} - \nu_2 \Delta v_6 + v' \cdot \nabla v_6 = f_6 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \, . \end{cases} \tag{1}$$

Here $x = (x_1, x_2, x_3)$ is the space variable, $v(x,t) = (v', v_4, v_5, v_6)$, $v'(x,t) = (v_1, v_2, v_3)$ is the velocity field, v_4 is the temperature, v_5 is the humidity, v_6 is moisture content in the clouds, $p(x,t)$ is the scalar field of the pressure, $f(x,t) = (f_1, f_2, f_3, f_4, f_5, f_6)$ is a known function from $L_2(Q_T)$, $Q_T = \Omega \times [0,T)$, $\nu_1 > 0$ is the kinematic viscosity parameter, $\nu_2 > 0$ is the heat conductivity coefficient, and ω, μ, g, s are positive continuous functions of x. The system (1) describes the nonlinear motions of three-dimensional incompressible viscous fluid which is rotating over the vertical axis with the angular velocity $\vec{\omega} = [0,0,\omega]$, also with consideration of moisture, humidity and heat transfer. The deduction of the equations (1) for linear non-viscous case is given, for example, in [1]. For non-linear case without the stratification, the equations (1) appear, for example, in [2], where the considered model was used for numerical calculations. For the simplified case of linearized compressible fluid without rotation, the system (1) was studied in [3], where the structure and localization of the essential spectrum of normal vibrations was established. Due to the presence of the fourth, fifth and sixth equations for the unknown functions of temperature, humidity and moisture,and also due to the presence of the rotation parameter, the equations (1) represent a novelty with respect to classical Navier-Stokes equations. In [4], the system (1) was considered for four equations without rotation, heat transfer, humidity and moisture, and there were established some properties of the corresponding weak solution.

Our aim is to obtain the "a priori" estimates for a weak solution of the system (1), to establish the existence and uniqueness of the weak solution, as well as to construct an explicit algorithm for that solution.

II. PROBLEM FORMULATION

If we introduce the following notations; $\tilde{v} = (v', v_4, v_5, v_6)$,

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$$Bv = \begin{bmatrix} -\omega v_2 \\ \omega v_1 \\ -\mu v_4 + g v_5 \\ s v_3 \\ 0 \\ 0 \end{bmatrix}, \qquad \nu \Delta v = \begin{bmatrix} \nu_1 \Delta v_1 \\ \nu_1 \Delta v_2 \\ \nu_1 \Delta v_3 \\ \nu_2 \Delta v_4 \\ \nu_2 \Delta v_5 \\ \nu_2 \Delta v_6 \end{bmatrix},$$

then we can write the system (1) as follows.

$$\begin{cases} \frac{\partial v}{\partial t} + \left(v' \cdot \nabla \right) v - \nu \Delta v + Bv + \nabla p = f \\ \operatorname{div} v' = 0, \quad x \in \Omega, \ t \geq 0. \end{cases} \tag{2}$$

We associate the system (2) with the conditions

$$\begin{cases} v\big|_{t=0} = 0 \\ v\big|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

in the bounded domain $Q_T = \Omega \times [0, T]$.

Let us multiply the system (2) by $2v$ in $L_2(\Omega)$:

$$\left(2v, v_t\right) + 2\left(v, \left(v' \cdot \nabla\right) v\right) - 2\left(v, \nu \Delta v\right) + \\ + \left(2v, Bv\right) + \left(2v, \nabla p\right) = \left(2v, f\right).$$

We integrate the last relation by parts and also with respect to $\tau \in [0, T]$:

$$\|v\|^2_{L_2(\Omega)} + 2\nu_1 \|\nabla v'\|^2_{L_2(Q_t)} + 2\nu_2 \|\nabla v''\|^2_{L_2(Q_t)} + \\ + \int_{Q_t} \left(2v, Bv\right) dx d\tau = \int_{Q_t} \left(2v, f\right) dx d\tau.$$

We denote $\sup_{\bar{Q}_t} \left(|\omega|, |\mu|, |g|, |s|\right) = b < \infty$, $\nu = \min\left(\nu_1, \nu_2\right)$ and use the inequalities of Cauchy, Poincare-Friedrichs and the obvious estimate $ac \leq \frac{1}{2}\left(a^2 + c^2\right)$. Thus, we obtain

$$\|v\|^2_{L_2(\Omega)} + 2\nu_1 \|\nabla v'\|^2_{L_2(Q_t)} + 2\nu_2 \|\nabla v''\|^2_{L_2(Q_t)} \geq \\ \geq \|v\|^2_{L_2(\Omega)} + \nu \|\nabla v\|^2_{L_2(Q_t)} + \nu K(\Omega) \|v\|^2_{L_2(Q_t)} - 2b \|v\|^2_{L_2(Q_t)}.$$

After applying Cauchy inequality, we get

$$\|v\|^2_{L_2(\Omega)} + \nu \|v_x\|^2_{L_2(Q_t)} \leq -\nu K(\Omega) \|v\|^2_{L_2(Q_t)} + \|v'\|^2_{L_2(Q_t)} + \\ + 2b \|v\|^2_{L_2(Q_t)} + \frac{1}{2} \|v\|^2_{L_2(Q_t)} + \frac{1}{2} \|f\|^2_{L_2(Q_t)}.$$

In this way, we have

$$\|v\|^2_{L_2(\Omega)} + \nu \|v_x\|^2_{L_2(Q_t)} \leq C_1 \|v\|^2_{L_2(Q_t)} + C_2 \|f\|^2_{L_2(Q_t)},$$

which, of course, implies the estimate

$$\|v\|^2_{L_2(\Omega)} \leq C_1 \|v\|^2_{L_2(Q_t)} + C_2 \|f\|^2_{L_2(Q_t)}.$$

Let us introduce the notations

$$F(t) = \|v\|^2_{L_2(\Omega)}, \ a(t) = C_2 \|f\|^2_{L_2(Q_1)}, \ g(t) = \int_0^t F(\tau) d\tau.$$

Then, the obtained inequality will take the following form:

$$g'(t) \leq a(t) + C_1 g(t).$$

Evidently, $\left(g(t) e^{-C_1 t}\right)' \leq a(t) e^{-C_1 t}$. After integrating the last estimate with respect to t, we have

$$g(t) e^{-C_1 t} \leq \int_0^t a(\tau) e^{-C_1 \tau} d\tau,$$

which implies the relations

$$g(t) \leq a(t) e^{C_1 t} \int_0^t e^{-C_1 \tau} d\tau = \frac{a(t)}{C_1}\left(e^{C_1 t} - 1\right),$$

and

$$C_1 g(t) \leq a(t) e^{C_1 t}.$$

In this way, we get the estimate

$$C_1 \|v\|^2_{L_2(Q_t)} \leq C_2 e^{C_1 t} \|f\|^2_{L_2(Q_t)}.$$

We observe that it follows from the last relation that the "a priori" estimate is valid:

$$\|v\|^2_{L_2(\Omega)} + \nu \|v_x\|^2_{L_2(Q_t)} \leq C \|f\|^2_{L_2(Q_t)},$$

where the positive constant C depends only on b, T, Ω.

Let us choose an orthonormal complete set of functions $\{u_k\}$ in the Hilbert functional space

$$H = \left\{ \varphi(x) = \left(\varphi_1, ..., \varphi_6\right) \ : \ \varphi \in \overset{0}{W}{}^1_2(\Omega), \operatorname{div} \varphi' = 0 \right\}.$$

Now, let $\Phi(x, t) = \left(\Phi_1, ..., \Phi_6\right)$ be test functions from $L_2(Q_T)$, which for every $0 \leq t \leq T$ belong to the Sobolev space $\overset{0}{W}{}^1_2(\Omega)$, and which also satisfy the conditions:

$$\operatorname{div} \Phi' = 0, \quad \Phi\big|_{t=T} = 0, \quad \Phi\big|_{\partial\Omega} = 0.$$

For the weak solution v we require the same conditions as for the functions Φ. We will call $v(x, t)$ a *weak solution* of the problem (2), (3), if v satisfies the integral identity

$$\int_{Q_T} \Big[-\left(v, \Phi_t\right) + \nu_1 \sum_{i=1}^{3} \left(\nabla v_i, \nabla \Phi_i\right) + \nu_2 \sum_{i=4}^{6} \left(\nabla v_i, \nabla \Phi_i\right) + \\ + \left(v, \left(v' \cdot \nabla\right) \Phi\right) + \left(Bv, \Phi\right) \Big] dx dt = \int_{Q_T} \left(f, \Phi\right) dx dt, \tag{4}$$

for all the functions Φ. Our aim is to prove the result of the existence of the weak solution.

III. Problem Solution

To construct the weak solution, we will use the Galerkin method. We find the approximate solutions of the problem (2), (3) in the following form

$$v^N(x, t) = \sum_{k=1}^{N} C_k^N(t) u_k(x). \tag{5}$$

In the system (2) we put $v = v^N$, multiply by u_k in sense of $L_2(\Omega)$ and integrate by parts in Ω. In this way, we obtain a Cauchy problem for the system of ordinary differential equations of the type

$$\frac{d}{dt}C_k^N(t)+\sum_{j=1}^{N}C_j^N\{u_j,u_k\}-\int_\Omega\left(v^N,\left(v'^N\cdot\nabla\right)u_k\right)dx+ \\ +\int_\Omega\left(Bv^N,u_k\right)dx=F_k(t)\ ,\ k=1,...,N, \tag{6}$$

where

$$\{\vec{u},\vec{v}\}=\int_\Omega\left[\nu_1\sum_{i=1}^{3}\left(\nabla u_i,\nabla v_i\right)+\nu_2\sum_{4}^{6}\left(\nabla u_i,\nabla v_i\right)\right]dx,$$

$$F_k(t)=\int_\Omega(f,u_k)dx\ \ ,\ k\geq 1.$$

To prove that (6) is solvable uniquely, we have to verify the "a priori" boundedness of the functions $C_k^N(t)$, $t\in[0,T]$, in the norm $L_2(\Omega)$. Evidently, the required property follows from the inequalities:

$$\left\|v^N\right\|_{L_2(\Omega)}^2+\nu\left\|v_x{}^N\right\|_{L_2(Q_T)}^2\leq C\|f\|_{L_2(Q_T)}^2 \\ \left\|v^N\right\|_{L_2(Q_T)}^2\leq C\|f\|_{L_2(Q_T)}^2. \tag{7}$$

The relations (7) are obtained by the same reasoning as in the previous section. It follows from (7) that the Galerkin approximations (5) are "a priori" bounded. From the sequence $\left\{v^N\right\}_{N=1}^{\infty}$, keeping in mind the estimates (7), we can choose the subsequence $\left\{v^{N_k}\right\}$ which is weakly convergent to some function $v(x,t)$ in $L_2(Q_T)$, together with its first derivatives with respect to x_k, $k=1,2,3$. The last fact follows from the weak compactness of bounded sets in the Hilbert space $L_2(Q_T)$. It is easy to see that the subsequence $\left\{v^{N_k}\right\}$ also tends strongly to v in sense of $L_2(Q_T)$, which follows from the generalized Friedrichs lemma ([5]):

$$\left\|v^{N^k}-v^{N^m}\right\|_{L_2(Q_T)}^2\leq\sum_{l=1}^{N^\varepsilon}\int_0^T\left(v^{N^k}-v^{N^m},u_l\right)^2+ \\ +\varepsilon\left\|v_x{}^{N^k}-v_x{}^{N^m}\right\|_{L_2(Q_T)}^2.$$

Therefore, the sequence $\left\{v^{N_k}\right\}$ tends strongly to v.

Now, we have to prove that $v(t,x)$ is a weak solution.

From the definition of the Galerkin approximations it follows that the following relation is valid

$$\int_0^T\int_\Omega\left(\frac{\partial v^N}{\partial t},\psi_k u_k\right)dxdt+\int_0^T\left\{v^N,\psi_k u_k\right\}dt+ \\ +\int_0^T\int_\Omega\left(v^N,\left(v'^N,\nabla\right)\psi_k u_k\right)dxdt+\int_0^T\int_\Omega\left(Bv^N,\psi_k u_k\right)dxdt= \\ =\int_0^T\int_\Omega(f,\psi_k u_k)dxdt,$$

for each function $\psi_k(t)\in C_0^\infty(0,T)$ and for every $k=1,...,N$.

Since the approximations $v^{N_k^k}(t,x)$ converge strongly $v^{N_k^k}(t,x)\Rightarrow v(t,x)$ in $L_2(Q_T)$; then, for every fixed $k\geq 1$ and for every function $\psi_k(t)\in C_0^\infty(0,T)$, the equality is valid:

$$-\int_{Q_T}\left(v,\psi_k'(t)u_k(x)\right)dxdt+\int_{Q_T}\left(\nu v,\psi_k\Delta u_k\right)dxdt- \\ -\int_{Q_T}\left(v,\left(v',\nabla\right)\psi_k u_k\right)dxdt+\int_{Q_T}\left(Bv,\psi_k u_k\right)dxdt= \\ =\int_{Q_T}\left(f,\psi_k u_k\right)dxdt.$$

Therefore, we have that $v(x,t)$ is a weak solution of the problem (2)-(3), in other words, the function $v(x,t)$ satisfies the integral identity (4) with the test functions

$$\Phi^N(x,t)=\sum_{k=1}^{N}\psi_k(t)u_k(x).$$

Evidently, the relations

$$\mathrm{div}v'^N=0,\ \ v'^N\big|_{\partial\Omega}=0$$

are also fulfilled for the limit function $v(x,t)$.

It remains to verify that $\|v\|_{L_2(\Omega)}$ tends strongly to zero as $t\to 0$.

Indeed, since the "a priori" estimates from the previous section are valid as well for $\left\{v^N\right\}$, we have, in particular, that

$$\left\|v^N\right\|_{L_2(\Omega)}^2\leq C_1\int_0^t\|f\|_{L_2(\Omega)}^2d\tau.$$

As $v^N(x,t)$ tends weakly to v in $L_2(\Omega)$, from the last estimate we obtain

$$\|v\|_{L_2(\Omega)}^2\leq C\int_0^t\|f\|_{L_2(\Omega)}^2d\tau,$$

which implies the property

$$\lim_{t\to 0}\|v\|_{L_2(\Omega)}=0.$$

In this way, we have proved the following theorem.

Theorem. There exists at least one weak solution for the problem (2)-(3), which can be found as the limit of the approximations (5).

IV. Conclusion

The results of this paper, particularly the explicit algorithm for construction of the solution, may be applied in the theoretical and computational study of the Atmosphere and the Ocean, for the models which consider the rotation of the earth, the heat transfer, the humidity and the moisture content.

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