Mathematical description of the resonance effect in the problem of oscillations of rotating stratified fluid

A. Giniatoulline

***Abstract*—**For various models of three-dimensional fluid which describe the flows in the Atmosphere and the Ocean, we find a relation between the essential spectrum of normal vibrations of internal waves and non-uniqueness of the limit amplitude of vibrations induced by external mass forces. We consider the both cases of incompressible and compressible and fluid and find a new mathematical description of the resonance of the internal waves. Since all the considered models correspond to of the stratified density in a homogeneous gravitational field, the obtained results may find their application in the study of the Atmosphere and the Ocean.

***Keywords*—**Turbulence and multiphase flows, computational fluid dynamics, compressible flows, internal waves, spectral theory, uniqueness of mathematical solutions, stratified flows.

I. Introduction

LET us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^1 and the following system of fluid dynamics

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \operatorname{div} \vec{u} = 0 \qquad x \in \Omega, \quad t \geq 0. \end{cases} \tag{1}$$

Here $\vec{u} = (u_1, u_2, u_3)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure and $\rho(x,t)$ is the dynamic density. In this model, the stationary distribution of density is described by the function e^{-Nx_3}, so N is a positive constant.

We also suppose that ω is a positive constant so that the system (1) describes linear motions of incompressible stratified inviscid fluid which is rotating over the vertical axis with a constant angular velocity $\vec{\omega} = (0,0,\omega)$.

The equations (1) are deduced in [1]-[3].

The fundamental solution of internal waves in stratified flows was first constructed in [4]. Particularly, it is easy to see that the system (1) with the parameter $\omega = 0$ is equivalent to the scalar equation

$$\frac{\partial^2}{\partial t^2}\Delta u + N^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0,$$

and it was proved in [4] that the singular (fundamental) solution of the last equation has the form

$$E(x,t) = \frac{1}{4\pi|x_3|}\int_0^{\eta} J_0(\alpha)\,d\alpha,$$

where J_0 is Bessel function of order zero and $\eta = \dfrac{|x_3| Nt}{|x|}$.

For the model of the stratified fluid, the stationary distribution of density takes the form of an exponentially decreasing function of the altitude (which, in fact is the Boltzmann-type distribution), thanks to the homogeneous gravity force. If we introduce some disturbance at the initial moment, then, in order to re-establish the initial stationary distribution of density, the action of the gravity force will result in appearance of the internal waves in the fluid which are described mathematically by various expressions which involve the above singular solution $E(x,t)$.

The solution for a Cauchy problem for the system (1), was constructed in [5], where it was proved that

$$\vec{v}(x,t) = \vec{v}^0(x)\Phi(t) + \iiint_{R^3} \vec{v}^0(y)\Gamma(x-y,t)\,dy, \text{ where}$$

$$\Gamma(x,t) = G(x,t) - \int_0^t J_1(t-\tau)G(x,\tau)\,d\tau, \text{ and}$$

$$G(x,t) = \frac{1}{4\pi}\left[\frac{t^2\left(x_1^2 + x_2^2\right)}{r^5} J_0\left(\frac{\rho t}{r}\right) + \frac{t}{r^3}\left(\frac{\rho}{r} + \frac{r}{\rho}\right) J_0'\left(\frac{\rho t}{r}\right)\right],$$

$$\rho = |x_3|, \quad r = |x|.$$

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It can easily be seen that the solution of the Cauchy problem for stratified fluid is closely related to the function

$$V = \frac{1}{r} J_0\left(\frac{\rho}{r} t\right) = \frac{1}{r} J_0\left(t\cos\theta\right).$$

Let us discuss the conduct of the function V as a function of t. We consider a sphere of a constant radius. On the sphere, for every t, the function V depends only on the polar angle θ. The argument of the Bessel function on the sphere changes from 0 to t. With t growing, we will have more and more waves generated by maxima and minima of the Bessel function, all of them situated between the pole and the equator of the sphere. The waves will appear on the pole and then will move towards the equator, accumulating but not disappearing. Thus large waves will generate more and more short ones. For rotating non-stratified fluid (with the parameter $N = 0$), the singular solution was constructed in [6]:

$$E(x,t) = \frac{1}{4\pi\sqrt{x_1^2 + x_2^2}} \int_0^{\frac{\omega t\sqrt{x_1^2+x_2^2}}{\|x\|}} J_0(\alpha) d\alpha.$$

Comparing the singular solutions for the corresponding models of stratified and rotating fluids, we can express certain conjecture of similarity of behavior of inner waves in both cases. From the mathematical point of view, we would like to compare the interaction of the external induced vibrations and the proper inner oscillations of the waves in the model where the stratification and the rotation are considered simultaneously. Particularly, we would like to describe the conditions and the properties of the resonance effect caused by the exterior induced vibrations.

II. Statement of the problem and preliminaries

We will study first the spectrum of the operators generated by the system (1). Let us consider the system (1) with the boundary condition

$$\vec{u}\cdot\vec{n}\big|_{\partial\Omega} = 0. \qquad (2)$$

We consider the following problem of normal oscillations

$$\begin{aligned} \vec{u}(x,t) &= \vec{v}(x)e^{-\lambda t} \\ \rho(x,t) &= Nv_4(x)e^{-\lambda t} \\ p(x,t) &= v_5(x)e^{-\lambda t} \quad , \quad \lambda \in \mathrm{C}. \end{aligned} \qquad (3)$$

Without loss of generality, we put $g = 1$, denote $\tilde{v} = (\vec{v}, v_4, v_5)$ and write (1) as

$$L\tilde{v} = 0, \qquad (4)$$

where $L = M - \lambda I_4$ and

$$M = \begin{pmatrix} 0 & -\omega & 0 & 0 & \frac{\partial}{\partial x_1} \\ \omega & 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}. \qquad (5)$$

We define the domain of the differential operator M as follows.

$$D(M) = \left\{ \begin{array}{l} \vec{v} \in (L_2(\Omega))^3 \mid \exists f \in L_2(\Omega): \\ (\vec{v}, \nabla\varphi) = (f, \varphi) \ \forall \varphi \in W_2^1(\Omega) \end{array} \right\} \times W_2^1(\Omega) \times W_2^1(\Omega).$$

We observe first that the operator M is a closed operator, and its domain is dense in $(L_2(\Omega))^5$.

Let us denote by $\sigma_{ess}(M)$ the essential spectrum of the linear operator M. Let us recall that the essential spectrum

$$\sigma_{ess}(M) = \{\lambda \in C : (M - \lambda I) \text{ is not of Fredholm type}\},$$

is composed of the points which belong to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity [7], [8].

Therefore, every spectral point outside of the essential spectrum, is an eigenvalue of finite multiplicity.

To find the essential spectrum of the operator M, we use the following property which is attributed to Weyl [8], [9]: a necessary and sufficient condition that a real finite value λ be a point of the essential spectrum of a self-adjoint operator M is that there exist a sequence of elements $v_n \in D(M)$ such that

$$\|v_n\| = 1, \quad v_n \to 0 \text{ weakly, and } \|(M - \lambda I)v_n\| \to 0.$$

Evidently, the operator M is skew-selfadjoint and its spectrum belongs to the imaginary axis.

Theorem 1.

Let $a = \min\{\omega, N\}$, $A = \max\{\omega, N\}$. Then, the essential spectrum of M is the following symmetrical set of the imaginary axis:

$$\{0\} \cup [-iA, -ia] \cup [ia, iA]. \qquad (6)$$

Moreover, the points $\{0\}, \pm\{ia\}, \pm\{iA\}$ are eigenvalues of infinite multiplicity.

Proof.

For the operator L defined in (4), we observe that the main symbol $\tilde{L}(\xi)$ takes the following form:

$$\tilde{L}(\xi)=\begin{pmatrix} -\lambda & -\omega & 0 & 0 & \xi_1 \\ -\omega & -\lambda & 0 & 0 & \xi_2 \\ 0 & 0 & -\lambda & N & \xi_3 \\ 0 & 0 & -N & -\lambda & 0 \\ \xi_1 & \xi_2 & \xi_3 & 0 & 0 \end{pmatrix}$$

and thus

$$\det \tilde{L}(\xi)=\lambda\left[\left(\lambda^2+N^2\right)\left(\xi_1^2+\xi_2^2\right)+\left(\lambda^2+\omega^2\right)\xi_3^2\right]. \quad (7)$$

We can see from (7) that if

$$\lambda \notin \left[\{0\}\cup(-iA,-ia)\cup(ia,iA)\right],$$

then the operator L is elliptic in sense of Douglis-Nirenberg.

Using (7), we consider $\lambda_0 \in \pm(ia,iA)\setminus\{0\}$ and choose a vector $\xi \neq 0$ such that

$$\left(\lambda_0{}^2+N^2\right)\left(\xi_1^2+\xi_2^2\right)+\left(\lambda_0{}^2+\omega^2\right)\xi_3^2=0.$$

Therefore, there exists the vector η such that

$$\tilde{L}(\xi)\eta=0. \quad (8)$$

After solving (8) with respect to η, we obtain one of possible solutions:

$$\begin{cases} \eta_1=\dfrac{\lambda_0\xi_1-\omega\xi_2}{\lambda_0^2+\omega^2}\ ,\ \ \eta_2=\dfrac{\lambda_0\xi_2+\omega\xi_1}{\lambda_0^2+\omega^2}, \\ \eta_3=\dfrac{\lambda_0\xi_3}{\lambda_0^2+N^2}\ ,\ \ \eta_4=\dfrac{-N\xi_3}{\lambda_0^2+N^2}\ ,\ \ \eta_5=1\,. \end{cases}$$

We observe that $\eta_i \neq 0$, $i=1,2,3,4,5$. Now, we choose a function $\psi_0 \in C_0^\infty(\Omega)$, $\int\limits_{\|x\|\le 1}\psi_0^2(x)dx=1$. Let us fix $x_0\in\Omega$ and put

$$\psi_k(x)=k^{\frac{3}{2}}\psi_0\left(k\left(x-x_0\right)\right),\ k=1,2,...$$

We define the Weyl sequence $\tilde{v}^k$ as follows:

$$\begin{cases} \tilde{v}_j^k(x)=\eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k+\dfrac{i}{k^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right),\ j=1,2,3 \\ \tilde{v}_4^k(x)=\eta_4\psi_k e^{ik^3\langle x,\xi\rangle} \\ \tilde{v}_5^k(x)=-\dfrac{i}{k^3}\psi_k e^{ik^3\langle x,\xi\rangle} \\ \langle x,\xi\rangle=x_1\xi_1+x_2\xi_2+x_3\xi_3\ , \qquad k=1,2,... \end{cases} \quad (9)$$

It is easy to verify that the sequence (9) satisfies the Weyl criterion, the details are analogous to the corresponding result in [10].

Let us investigate now the structure of the boundary points of the essential spectrum. For $\lambda=0$, the system (4) takes the form

$$\begin{cases} -\omega v_2+\dfrac{\partial v_5}{\partial x_1}=0 \\ \omega v_1+\dfrac{\partial v_5}{\partial x_2}=0 \\ Nv_4+\dfrac{\partial v_5}{\partial x_3}=0 \\ -Nv_3=0 \\ \operatorname{div}\vec{v}=0 \end{cases}.$$

As it can be easily verified, every vector-function of the type

$$v=\left(\frac{-1}{\omega}\frac{\partial\varphi}{\partial x_2},\frac{1}{\omega}\frac{\partial\varphi}{\partial x_1},0,\frac{-1}{N}\frac{\partial\varphi}{\partial x_3},\varphi\right),\ \ \varphi\in C_0^\infty(\Omega),$$

satisfies the last system, and therefore $\lambda=0$ is an eigenvalue of infinite multiplicity.

Since the essential spectrum of a linear operator is always a closed set, the points $\pm\{ia\},\pm\{iA\}$, belong to it. These boundary points are also eigenvalues of infinite multiplicity. Let us consider, for example, the case $\lambda=iN$, $N>\omega$.

Then, the system (4) transforms into

$$\begin{cases} -iNv_1-\omega v_2+\dfrac{\partial v_5}{\partial x_1}=0 \\ \omega v_1-iNv_2+\dfrac{\partial v_5}{\partial x_2}=0 \\ -iNv_3+Nv_4+\dfrac{\partial v_5}{\partial x_3}=0 \\ -Nv_3-iNv_4=0 \\ \operatorname{div}\vec{v}=0 \end{cases}.$$

As it can be seen, any vector function of the form

$$\left(0,0,\varphi(x_1,x_2),i\varphi(x_1,x_2),0\right),\ \ \varphi\in C_0^\infty,$$

satisfies the last system and thus the theorem is proved.

Now, let us consider the following non-homogeneous system corresponding to system (1):

$$\begin{aligned} &\frac{\partial\vec{v}}{\partial t}+B\vec{v}+\nabla p=\vec{F}(x,t) \\ &\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0 \end{aligned}. \quad (10)$$

Here $\vec{F}(x,t)$ represents mass forces acting on the fluid, and B is a matrix operator which defines the model:

$$B(\vec{v},v_4)=\begin{pmatrix} -\omega v_2 \\ \omega v_1 \\ v_4 \\ -N^2v_3 \end{pmatrix}.$$

We consider the Cauchy problem for the system (10) for the case when the right-hand side $\vec{F}(x,t)$ depends on time harmonically, i.e.

$$\vec{F}(x,t)=\vec{f}(x)e^{-i\lambda_0 t}\ ,\ \lambda_0\geq 0 .$$

From the physical point of view, for this case, the solution of the Cauchy problem should stabilize and describe the mode of the induced vibrations with frequency λ_0.

Theorem 2.

Let the exterior mass force be a periodic function with frequency $\lambda_0 \geq 0$:

$$\vec{F}(x,t)=\vec{f}(x)e^{-i\lambda_0 t} .$$

Then, the solution $v(x,t)$ of the Cauchy problem for (10) is also periodic and the following stabilization property is valid:

$$\lim_{t\to\infty} e^{i\lambda_0 t}\vec{v}(x,t)=\vec{U}(x),$$

where $\vec{U}(x)$ is the solution of the stationary system with the external force $\vec{f}(x)$.

The proof of the above statement of stabilization for the case of rotating fluid may be found in [11]. In [12], the analogous result was proved for stratified fluid. The proof of Theorem 2 for the general case of rotating stratified fluid is similar to the corresponding proof in [12]. Following the traditionally accepted terminology, we will call the function $\vec{U}(x)$ the *limit amplitude* of the stabilized induced vibrations.

Our next objective is to compare the uniqueness of the limit amplitude $\vec{U}(x)$ and the variation of the values of the frequency of external vibrations λ_0 with respect to the essential spectrum of normal vibrations of the fluid.

III. On the uniqueness of Laplace and wave equations in $L_p(R^3)$

Theorem 3.

Let $u\in L_p(R^3)$, $1\leq p<\infty$. If $u(x)$ is a solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=0 \qquad (11)$$

in R^3, then $u(x)=0$ almost everywhere.

Proof.

From Schwartz theorem [13], we obtain that for $u\in L_p(R^3)$ every linear functional

$$\int_{R^3} u(x)\phi(x)dx\ ,\ \ \phi\in S ,$$

is an element of S'.

After applying the Fourier transform to (11), we have

$$\left(\xi_1^2+\xi_2^2+\xi_3^2\right)\hat{u}=0 ,$$

where $\hat{u}$ is the Fourier image of u.

In this way, the support of $\hat{u}(\xi)$ consists of one only point $\{0\}$. Then, from the theory of generalized functions [14] we have

$$\hat{u}(\xi)=\sum_{|\alpha|\leq N} C_\alpha D^\alpha \delta(\xi) ,$$

from which it follows that

$$u(x)=\sum_{|\alpha|\leq N} C'_\alpha x^\alpha .$$

In other words, u is a polynomial of x.

Since $u\in L_p(R^3)$, we finally have $u(x)=0$ almost everywhere in R^3 and thus the theorem is proved.

Theorem 4.

Let $1<p<\frac{3}{2}$, $f(x)\in L_p(R^3)$. Then, the solution of the Poisson equation in R^3

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=f(x), \qquad (12)$$

belongs to the class of uniqueness $L_q(R^3)$, $3<q<\infty$.

Proof.

The singular (fundamental) solution of the Laplace equation in R^3 is the function $E(x)=-\frac{1}{4\pi|x|}$.

We will use the following property [16]: in $x\in R^n$, the solution of the equation $Lu=f$ is unique in the class of functions for which there exists the convolution $E*f$.

Therefore, from theorem 3 we obtain that the solution of (12) $u(x)$ is performed by

$$u(x)=\frac{-1}{4\pi}\int_{R^3}\frac{f(x)}{|x-y|}dy .$$

From Hardy-Littlewood inequalities [15] and Sobolev inequalities for integrals of potential type [6], it follows that for $1<p<3/2$, $3<q<\infty$, the property holds:

$$\|u\|_{L_q(R^3)}\leq C\|f\|_{L_p(R^3)} ,$$

where p and q satisfy the relation $\frac{1}{q}=\frac{1}{p}-\frac{2}{3}$.

In this way, the theorem is proved.

Now, let us consider the following norm in the non-isotropic space $L_{\vec{p}}(R^3)$, $\vec{p}=(p,p,p_3)$

$$\| f \|_{L_{\vec{p}}(R^3)} = \left(\int_{R^2} \left(\int_{R^1} | f(x) |^{p_3} \, dx_3 \right)^{p/p_3} dx' \right)^{1/p} \quad , \quad x' = (x_1, x_2) .$$

The proof of the following theorem is similar to [17], [12], though the original model is different.

Theorem 5.

Let $u(x)$ be a solution of the wave equation in R^3

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0 \qquad (13)$$

such that $u(x) \in L_{\vec{p}}(R^3)$, $\vec{p} = (p, p, 2)$, $4 < p < \infty$.

Then, there exist nonzero solutions of (13) in the considered class of functions.

Proof.

We consider the Fourier transform in x_3 : $F_{x_3 \to \xi}$.

It can be easily seen that for arbitrary $A(\xi)$, the function $v(x', \xi) = A(\xi) J_0(\xi |x'|)$ for almost every $\xi \in R^1$, satisfies the Helmholtz equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \xi^2 v = 0 ,$$

where $|x'| = \sqrt{x_1^2 + x_2^2}$ and $J_0(\xi r)$ is the Bessel function of order zero which is the solution of the Bessel equation

$$r^2 v'' + r v' + r^2 \xi^2 v = 0 .$$

We choose an arbitrary function $A(\xi) \in C_0^\infty(R^1)$ such that $A(\xi) \equiv 0$ for $|\xi| \le 1$, and $A(\xi) \ne 0$ in a set of non-zero measure in R^1.

In this way, the function

$$u(x) = \frac{1}{2\pi} \int_{R^1} v(x', \xi) e^{-i x_3 \xi} d\xi$$

represents a non-trivial solution of (13).

By Hausdorff-Young inequality for Fourier transforms [18], We have that the estimate holds:

$$\|u\|_{L_2(R^1_{x_3})} \le C \|v\|_{L_2(R^1_\xi)} .$$

In this way, we obtain the inequality

$$\Big\| \|u\|_{L_2(R^1_{x_3})} \Big\|_{L_p(R^2)} \le C \Big\| \|v\|_{L_2(R^1_\xi)} \Big\|_{L_p(R^2)} .$$

Since $\int_1^\infty |J_0(\xi r)|^p \, r dr < \infty$ for $p > 4$, then the norm in the right side of the last relation is finite.

Finally, we have the estimate $\|u\|_{L_{\vec{p}}(R^3)} < \infty$, $\vec{p} = (p, p, 2)$ and thus the theorem is proved.

Remark 1.

For the results concerning the structure and the localization of the essential spectrum, instead of the system (1), we can also consider the corresponding model for compressible fluid:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \dfrac{\partial p}{\partial t} + \operatorname{div} \vec{u} = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases}$$

In this case, the operator L in (4) will take the form

$$L = M - \lambda I_5$$

where

$$I_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

From the definition of the ellipticity in sense of Douglis-Nirenberg [19], we have that, for the operator $L = M - \lambda I_5$ it is possible to consider the main symbol of the operator L as the main symbol of the operator $M - \lambda I_4$,

Indeed, from the definitions in [19], [20] for calculating the main symbol, in this case we can assume $(M - \lambda I_5)_{55} = 0$. Therefore, the main symbol of L will coincide with the main symbol of the operator $M - \lambda I_4$ and thus the considered spectral results obtained in for incompressible fluid, can be easily extended to the case of the compressible fluid.

Remark 2.

For the model which counts with salinity and heat transfer, the operator B in (10) acts on 5-dimensional vector:

$$B(\vec{v}, v_4, v_5) = \begin{pmatrix} 0 \\ 0 \\ -\alpha v_4 - \beta v_5 \\ \alpha v_3 \\ \beta v_3 \end{pmatrix} ,$$

where v_4 is the temperature of the fluid, and v_5 is the salinity. The equations of this model are deduced in [21], and the spectral properties for viscous fluid were studied in [22]. From the explicit form of the operator B we can conclude, without loss of generality, that the spectral results for inviscid fluid considered in section II of this paper, are also valid for the flows modeling salinity and heat transfer.

IV. UNIQUENESS AND NON-UNIQUENESS OF THE LIMIT AMPLITUDE AND THE RESONANCE EFFECT

We will use here the notations from theorem 2.

For the model of rotating fluid, it was proved in [11] that, if the frequency of the external vibrations λ_0 was close to ω, then the limit amplitude $U(x)$ assumed unbounded growth. In this paper, we would like to present a different description of the resonance effect. Particularly, for rotating stratified fluid, as well as for the models of salinity and heat transfer, we will show that there exists an explicit relation between the belonging of λ_0 to the essential spectrum of normal inner vibrations, and the non-uniqueness of the limit amplitude.

By consecutive differentiation and corresponding substitution, it can be easily verified that non-homogeneous system (1) is equivalent to the scalar equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_3^2}\right)+N^2\left(\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}\right)+\omega^2\frac{\partial^2 v}{\partial x_3^2}=F(x,t).$$

We put $F(x,t)=f(x)e^{-i\lambda_0 t}$ and use the periodicity of the stabilized solution from theorem 2: $v(x,t)=u(x)e^{-i\lambda_0 t}$.
In this way, we obtain the following equation:

$$\left(N^2-\lambda_0^2\right)\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+\left(\omega^2-\lambda_0^2\right)\frac{\partial^2 u}{\partial x_3^2}=f(x),\ x\in R^3. \quad (14)$$

Evidently, the function $U(x)$ from theorem 2 which is the limit amplitude of the induced oscillations of the external forces with frequency λ_0, is a solution of the equation of the stabilized vibrations (14).

We observe that the equation (14) can be also obtained from the expression of the mail symbol (7).

Let us express the homogeneous equation (14) as follows.

$$\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+\frac{\left(\omega^2-\lambda_0^2\right)}{\left(N^2-\lambda_0^2\right)}\frac{\partial^2 u}{\partial x_3^2}=0. \quad (15)$$

We consider first the case when the external frequency does not belong to the essential spectrum of normal vibrations

$$\lambda_0\notin\{[-A,-a]\cup[a,A]\},\ a=\min\{\omega,N\},\ A=\max\{\omega,N\}.$$

The last term in (15) satisfies the estimate $\frac{\left(\omega^2-\lambda_0^2\right)}{\left(N^2-\lambda_0^2\right)}>0$ and thus, by a contraction of the axis x_3, the equation (15) can be transformed to the Laplace equation (11).

Let us consider $f(x)\in W_1^1\left(R^3\right)$. We note that from Sobolev inclusion theorems [6] for $1<p<\frac{3}{2}$ the property holds:

$$W_1^1\left(R^3\right)\subset L_p\left(R^3\right).$$

From theorems 3 and 4 we obtain that the solution of the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}=f(x)$$

is unique, and for $f(x)\in W_1^1\left(R^3\right)$ belongs to the class of uniqueness $L_q\left(R^3\right)$, $3<q<\infty$.

Now we consider the case when λ_0 belongs to the essential spectrum of normal vibrations:

$$\lambda_0\in\{\{-A,-a\}\cup\{a,A\}\}.$$

For this case, we have the property

$$\frac{\left(\omega^2-\lambda_0^2\right)}{\left(N^2-\lambda_0^2\right)}<0,$$

and therefore, by a transform of linear contraction of the axis x_3, the equation (15) can be transformed into the wave equation (13).

The singular solution of (13) is

$$E(x)=-\frac{1}{4\pi\sqrt{x_1^2+x_2^2-x_3^2}}.$$

In this way, the solution of the non-homogeneous equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 u}{\partial x_3^2}=f(x),$$

is performed by the function

$$U(x)=\frac{-1}{4\pi}\int_{R^3}\frac{f(y)dy}{\sqrt{|x'-y'|^2-(x_3-y_3)^2}},\ x'=(x_1,x_2).$$

Again, let $f(x)\in W_1^1\left(R^3\right)$. Using Minkovsky inequality, Sobolev inclusion theorems and the properties of the integrals of potential type [6], we obtain the estimate

$$\|U\|_{L_{\vec{p}}\left(R^3\right)}\le C\|f\|_{W_1^1\left(R^3\right)},\ \vec{p}=(p,p,2),\ 4<p<\infty.$$

Therefore, the assumptions for the theorem 5 are fulfilled, and we have that the limit amplitude $U(x)$ is not unique. Moreover, it belongs to

$$u(x)\in L_{\vec{p}}\left(R^3\right),\ \vec{p}=(p,p,2),\ 4<p<\infty,$$

which is the class of non-uniqueness of the equation (13).

We note that the established non-uniqueness of the limit amplitude of induced vibrations for the case when the external frequency belongs to the essential spectrum of proper normal vibrations, can be interpreted as a new mathematical description of the resonance effect of inner waves.

Now, we can sum up the results of this section as the following statement.

Theorem 6.

Let $f(x) \in W_1^1(R^3)$. Then, for the models of rotating stratified fluid, as well as for the models for salinity, humidity and heat transfer, for both incompressible and compressible fluid, the limit amplitude of oscillations which are induced by the external mass forces of the type

$$F(x,t) = f(x)e^{-i\lambda_0 t}$$

is unique when the external frequency λ_0 is outside of the essential spectrum of the proper normal vibrations. This limit amplitude belongs to the class of uniqueness of the equation (12)

$$L_q(R^3), \ 3 < q < \infty .$$

On the other hand, if the external frequency λ_0 belongs to the essential spectrum of normal vibrations, then the limit amplitude is not unique and belongs to the class of non-uniqueness of the equation (13)

$$L_{\vec{p}}(R^3), \ \vec{p} = (p, p, 2), \ 4 < p < \infty.$$

Remark 3.

The Weyl sequence (9) from theorem 1, evidently, is not unique as a result of an arbitrary selection of the function ψ_0. Due to the property $\|(M - \lambda I)v_n\| \to 0$ it can also be interpreted as an explicit example of non-uniqueness of the solution.

V. Conclusion

Traditionally, the resonance effect has been considered mathematically as an unbounded growth of the amplitude as a result of the superposition of the external vibrations and the proper vibrations of the system. However, for the considered models of rotating stratified fluid as well as for the models counting with humidity, salinity and heat transfer, rather than the growth of the solution, there may be valid one more mathematical tool to describe the resonance: uniqueness and non-uniqueness of the limit amplitude.

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