A Study of the Problem-Solving Process Using Fuzzy Relation Equations

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***Abstract*—**Fuzzy relation equations are associated with the composition of binary fuzzy relations. In the present paper fuzzy relation equations are used as a tool for studying student problem-solving skills. A classroom application and other suitable examples connected to student problem-solving are also presented illustrating our results and useful conclusions are obtained.

***Keywords*—**Fuzzy Sets, Binary Fuzzy Relations, Fuzzy Relation Equations (FRE), Problem-Solving (PS)

I. Introduction

As the human society moved from an industrial to a knowledge society, it can be argued that the nature of many problems has been changed and new problems have arisen which may require a different approach to overcome them. Educational institutions and governments have recognized long ago the importance of **Problem–Solving (PS)** and volumes of research have been written about it. Universities and other higher learning institutions are entrusted with the task of producing graduates that have higher order PS skills among other skills.

Mathematics by its nature is a subject whereby PS forms its essence. In Voskoglou [1, 2] we have examined the role of the problem in learning mathematics and we have attempted a review of the evolution of research on PS in mathematics education from the time that Polya presented his first ideas on the subject until today. Here is a rough chronology of that progress:

1950's – 1960's: Introduction of the **heuristic strategies** by Polya ([3] – [5], etc.) for teaching the PS process.

1970's: Emergency of mathematics education as a self – sufficient science. The research on PS was still based on Polya's ideas, while the research methods were almost exclusively statistical.

1980's: A framework appeared describing the PS process, and reasons for success or failure in PS, e.g. Schoenfeld's [6] **Expert Performance Model**, etc.

1990's: Models of teaching mathematics were developed using PS, e.g. constructivist view of learning (von Claserfeld [7], etc.), Mathematical Modelling and applications (e.g. see [8]), etc.

2000's - today: In contrast to the earlier work on PS, which was focused mainly on analyzing the PS process and on describing the proper heuristic strategies to be used in each of its stages, the research has turned mainly on solvers' behavior and required attributes during the PS process; e. g. **Multidimensional PS Framework (MPSF)** of Carlson and Bloom [9], Schoenfeld's theory of **Goal-Directed Behavior** [10], etc.

Carlson and Bloom [9] drawing from the large amount of literature related to PS developed a broad taxonomy to characterize major PS attributes that have been identified as relevant to PS success. This taxonomy gave genesis to their MPSF model, which includes the following steps: **Orientation, Planning, Executing** and **Checking**. It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases a solver obtained the solution of a problem by making this cycle only once. Thus embedded in the framework are two cycles, one cycling forward and one cycling backward, each of which includes the three out of the four steps, i.e. planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning step of the PS process, the solvers were at times engaged in a **conjecture-imagine-evaluate (accept/reject)** sub-cycle. Therefore, apart of the two main cycles, embedded in the MPSF is also the above sub-cycle, which is connected to the step of planning ([9], Figure 1).

Note that the main phases of the MPSF are actually the same to the steps of Schoenfeld's [6] Expert Performance Model for PS; only their names are stated differently. In fact, a careful inspection of the two PS models shows that Orientation corresponds to Schoenfeld's **Analysis** of the problem, Planning corresponds to the **Design** of the solution, the conjecture-imagine-evaluate sub-cycle corresponds to Schoenfeld's step of **Exploration**, Executing corresponds to the **Implementation** of the solution and finally Checking corresponds to Schoenfeld's **Verification** of the solution [1]. However, there exists a basic qualitative difference between the two models: While in the MPSF the emphasis is turned to the solver's behavior and required attributes, the EPM is oriented towards the PS process itself describing the proper heuristic strategies that may be used at each step of the PS process.

In earlier works the present author based on the above two models for PS (EPM and MPSF) has used the theory of finite

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Markov chains and principles of fuzzy logic to develop several models for assessing the student progress during the PS process (e.g. see [11, 12] and the relevant references contained in these books).

Here a new approach will be developed involving the use of **Fuzzy Relation Equations (FRE)** for studying the student PS skills. The rest of the paper is formulated as follows: Section II contains the background from fuzzy binary relations and FRE which is necessary for the understanding of the paper. In Section III the model using FRE for studying the process of PS is developed, while in Section IV a classroom application and other suitably chosen examples are presented illustrating the new model in practice. Finally, Section V is devoted to our conclusion and to some hints for future research on the subject.

II. FUZZY RELATION EQUATIONS

For general facts on fuzzy sets we refer to the book [13].

Definition 1: Let X, Y be two crisp sets. Then a **fuzzy binary relation** R(X, Y) is a fuzzy set on the Cartesian product X x Y of the form:

$$R(X, Y) = \{(r, m_R(r)) : r = (x, y) \in X \times Y\},$$

where $m_R : X \times Y \rightarrow [0, 1]$ is the corresponding membership function.-

When X = $\{x_1, \ldots\ldots\ldots, x_n\}$ and Y = $\{y_1, \ldots\ldots, y_m\}$, then a fuzzy binary relation R(X, Y) can be represented by a n X m matrix of the form:

$$R = \begin{array}{c} \\ x_1 \\ . \\ . \\ . \\ x_n \end{array} \begin{array}{c} y_1 \cdots\cdots\cdots\cdots \quad y_m \\ \begin{pmatrix} r_{11} & . & . & . & r_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ r_{m1} & . & . & . & r_{mn} \end{pmatrix} \end{array} = [r_{ij}],$$

where $r_{ij} = m_R(x_i, y_j)$, with i = 1,…, n and j =1,…, m. The matrix R is called the **membership matrix** of the fuzzy relation R(X, Y).

The basic ideas of fuzzy relations, which were introduced by Zadeh [14] and were further investigated by other researchers, are extensively covered in the book [15].

Definition 2: Consider two fuzzy binary relations P(X, Y) and Q(Y, Z) with a common set Y. Then, the standard composition of these relations, which is denoted by P(X, Y) $\circ$ Q(Y, Z) produces a binary relation R(X, Z) with membership function m_R defined by:

$$m_R(x_i, z_j) = \underset{y \in Y}{Max} \ \min\,[m_P(x_i, y)\,,\, m_Q(y, z_j)] \qquad (1),$$

for all i=1,…,n and all j=1,…,m. This composition is often referred as **max-min composition.**

Compositions of binary fuzzy relations are conveniently performed in terms of membership matrices of the relations. In fact, if P = $[p_{ik}]$ and Q=$[q_{kj}]$ are the membership matrices of the relations P(X, Y) and Q(Y, Z) respectively, then by relation (1) we get that the membership matrix of R(X, Y) = P(X, Y) $\circ$ Q(Y, Z) is the matrix R = $[r_{ij}]$, with

$$r_{ij} = \underset{k}{Max} \ \min(p_{ik}, q_{kj}) \qquad (2)$$

Example 1:

If $$P = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0.8 \\ 0.1 & 0.5 & 1 \\ 0.4 & 0.7 & 0.3 \end{pmatrix} \end{array}$$ and

$$Q = \begin{array}{c} \\ y_1 \\ y_2 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.2 & 0.7 & 0 & 0.4 \\ 0.8 & 0.1 & 0.5 & 0.6 \\ 1 & 0.3 & 0.2 & 0.9 \end{pmatrix} \end{array}$$

are the membership matrices of P(X, Y) and Q(Y, Z) respectively, then by relation (2) the membership matrix of R(X, Z) is the matrix

$$R = P \circ Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.3 & 0.4 & 0.8 \\ 1 & 0.3 & 0.5 & 0.9 \\ 0.7 & 0.4 & 0.5 & 0.6 \end{pmatrix} \end{array}.$$

Observe that the same elements of P and Q are used in the calculation of m_R as would be used in the regular multiplication of matrices, but the product and sum operations are here replaced with the min and max operations respectively.

Definition 3: Consider the fuzzy binary relations P(X, Y), Q(Y, Z) and R(X, Z), defined on the sets, X = $\{x_i : i \in N_n\}$, Y = $\{y_j : j \in N_m\}$, Z= $\{z_k : k \in Ns\}$, where $N_t = \{1,2,\ldots,t\}$, for t = n, m, k, and let P=$[p_{ij}]$, Q=$[q_{jk}]$ and R=$[r_{ik}]$ be the membership matrices of P(X, Y), Q(Y, Z) and R(X, Z) respectively. Assume that the above three relations constrain each other in such a way that

$$P \circ Q = R \qquad (3),$$

where $\circ$ denotes the max-min composition. This means that

$$\underset{j \in J}{Max} \ \min\,(p_{ij}\,,\, q_{jk}) = r_{ik} \qquad (4),$$

for each i in N_n and each k in N_s. Therefore the matrix equation (3) encompasses nXs simultaneous equations of the form (4). When two of the components in each of the equations (4) are given and one is unknown, these equations are referred as fuzzy relation equations *(FRE)*.

The notion of FRE was first proposed by Sanchez [16] and later was further investigated by other researchers [17] – [19].

III. A STUDY OF PROBLEM- SOLVING SKILLS USING FRE

Let us consider the crisp sets X = {M}, Y = {A, B, C, D, F} and Z = {O, P, E, Ch}, where M denotes the **"average student"** of a class, A = Excellent, B = Very Good, C = Good, D = Fair and F = Failed are **linguistic labels (grades)** used for the assessment of the student performance and O = Orientation, P = Planning, E = Execution and Ch = Checking represent the states of the PS process.

Further, let *n* be the total number of students of a certain class and let n_i be the numbers of students who obtained the grade i assessing their performance, $i \in Y$. Then one can represent the average student of the class as a **fuzzy set on Y** of the form

$$M = \{(i, \frac{n_i}{n}): i \in Y\}.$$

The fuzzy set M induces a fuzzy binary relation P(X, Y) with membership matrix

$$P = [\frac{n_A}{n} \ \frac{n_B}{n} \ \frac{n_C}{n} \ \frac{n_D}{n} \ \frac{n_F}{n}]$$

In an analogous way the average student of a class can be represented as a **fuzzy set on Z** of the form

$$M = \{(j, m(j)): j \in Z\},$$

where $m: Z \rightarrow [0, 1]$ is the corresponding membership function. In this case the fuzzy set M induces a fuzzy binary relation R(X, Z) with membership matrix

$$R = [m(O)\ m(P)\ \ m(E)\ \ m(Ch)].$$

We consider also the fuzzy binary relation Q(Y, Z) with membership matrix the 5X4 matrix $Q = [q_{ij}]$, where $q_{ij} = m_Q(i, j)$ with $i \in Y$ and $j \in Z$ and the FRE encompassed by the matrix equation (3), i.e. by $P \circ Q = R$.

When the matrix Q is fixed and the row-matrix P is known, then the equation (3) has always a unique solution with respect to R, which enables the representation of the average student of a class as a fuzzy set on the set of the steps of the learning process. This is useful for the instructor for designing his/her future teaching plans. On the contrary, when the matrices Q and R are known, then the equation (3) could have no solution or could have more than one solution with respect to P, which makes the corresponding situation more complicated.

All the above will be illustrated in the next section with a classroom application and other suitable examples.

IV. A CLASSROOM APPLICATION AND OTHER EXAMPLES

The Classroom Application

The following experiment took place at the Graduate Technological Educational Institute of Western Greece, in the city of Patras, when I was teaching to a group of 60 students of the School of Technological Applications (future engineers) the use of the derivative for the maximization and minimization of a function. A written test was performed after the end of the teaching process involving the two problems listed in the Appendix at the end of the paper. The results of the test are depicted in Table 1.

Table 1: Student Performance

Grade	No. of Students
A	20
B	15
C	7
D	10
F	8
Total	60

Then the average student M of the class can be represented as a fuzzy set on Y = {A, B, C, D, F} by

$$M = \{(A, \frac{20}{60}), (B, \frac{15}{60}), (C, \frac{7}{60}), (D, \frac{10}{60}), (F, \frac{8}{60})\}$$

$$\approx \{(A, 0.33), (B, 0.25), (C, 0.12), (D, 0.17), (F, 0.13)\}$$

Therefore M induces a binary fuzzy relation P(X, Y), where X = {M}, with membership matrix

$$P = [0.33\ \ 0.25\ \ 0.12\ 0.17\ 0.13].$$

Also, using statistical data of the last five academic years on learning mathematics from the students of the School of Technological Applications of the Graduate Technological Educational Institute of Western Greece, we fixed the membership matrix Q of the binary fuzzy relation Q(Y, Z), where Z = {O, P, E, Ch}, in the form:

$$Q = \begin{array}{c} \\ A \\ B \\ C \\ D \\ F \end{array} \begin{array}{c} \begin{array}{cccc} O & P & E & Ch \end{array} \\ \begin{pmatrix} 0.7 & 0.5 & 0.3 & 0 \\ 0.4 & 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.7 & 0.5 \\ 0 & 0.1 & 0.5 & 0.8 \end{pmatrix} \end{array}$$

The statistical data used to form the matrix Q were collected by the instructor who was inspecting the student reactions during the solution of several problems in the classroom.
Next, using the max-min composition of fuzzy binary relations one finds that the membership matrix of
R(X, Z) = P(X, Y) o Q (Y, Z) is equal to

$$R = P \text{ o } Q = [0.33\ \ 0.33\ \ 0.3\ 0.17].$$

Therefore the average student of the class can be expressed as a fuzzy set on Z by

$$M = \{(O, 0.33), (P, 0.33), (E, 0.3), (Ch, 0.17)\}.$$

The conclusions obtained from the above expression of M are the following:

- Only the $\frac{1}{3}$ of the students of the class were ready to use contents of their memory (background knowledge, etc.) in order to facilitate the solution of the given problems.
- All the above students were able to plan and almost all of them were able to execute the solutions of the given problems.
- On the contrary, half of the above students did not succeed in checking the correctness of the solutions found.

The first conclusion was not a surprising one, since the majority of the students have the wrong habit to start studying their courses the last month before the final exams. On the other hand, the second conclusion shows that the instructor's teaching procedure was successful enabling the diligent students to plan and execute successfully the solutions of the given problems. Finally, the last conclusion is due to the fact that students, when solving mathematical modeling problems, they usually omit to check if their solutions are compatible to the restrictions imposed by the real world (for example see the two problems in the Appendix). Therefore, it is very useful for the instructor to emphasize that the last step of the PS process (checking) is not simply a formality, but it has its own importance for preventing several mistakes.

Other Examples

Let us now consider the case where the membership matrices Q and R are known and we want to determine the matrix P representing the average student of the class as a fuzzy set on Y. This is a complicated case because we may have more than one solution or no solution at all. The following two examples illustrate this situation:

Example 2: Consider the membership matrices Q and R of the previous application and set

$$P = [p_1\ p_2\ p_3\ p_4\ p_5].$$

Then the matrix equation P o Q = R encompasses the following equations:

$$\max\{\min(p_1, 0.7), \min(p_2, 0.4), \min(p_3, 0.2), \min(p_4, 0.1), (p_5, 0)\} = 0.33$$

$$\max\{\min(p_1, 0.5), \min(p_2, 0.6), \min(p_3, 0.7), \min(p_4, 0.5), (p_5, 0.1)\} = 0.33$$

$$\max\{\min(p_1, 0.3), \min(p_2, 0.3), \min(p_3, 0.6), \min(p_4, 0.7), (p_5, 0.5)\} = 0.3$$

$$\max\{\min(p_1, 0), \min(p_2, 0.1), \min(p_3, 0.2), \min(p_4, 0.5), (p_5, 0.8)\} = 0.17$$

The first of the above equations is true if, and only if, $p_1 = 0.33$ or $p_2 = 0.33$, values that satisfy the second and third equations as well. Also, the fourth equation is true if, and only if, $p_3 = 0.17$ or $p_4 = 0.17$ or $p_5 = 0.17$. Therefore, any combination of values of p_1, p_2, p_3, p_4, p_5 in [0, 1] such that $p_1 = 0.33$ or $p_2 = 0.33$ and $p_3 = 0.17$ or $p_4 = 0.17$ or $p_5 = 0.17$ is a solution of P o Q = R.

Let S(Q, R) = {P: P o Q = R } be the set of all solutions of P o Q = R. Then one can define a partial ordering on S(Q, R) by

$$P \leq P' \Leftrightarrow p_i \leq p'_i,\ \forall \iota = 1, 2, 3, 4, 5.$$

It is well established that whenever S(Q, R) is a non empty set, it always contains a unique maximum solution and it may contain several minimal solutions [16]. It is further known that S(Q, R) is fully characterized by the maximum and minimal solutions in the sense that all its other elements are between the maximal and each of the minimal solutions [16]. A method of determining the maximal and minimal solutions of P o Q = R with respect to P is developed in [19].

Example 3: Let Q = $[q_{ij}]$, i = 1, 2, 3, 4, 5 and j = 1, 2, 3, 4 be as in Example 2 and let R = [1 0.33 0.3 0.17]. Then the first equation encompassed by the matrix equation P o Q = R is

$$\max\{\min(p_1, 0.7), \min(p_2, 0.4), \min(p_3, 0.2), \min(p_4, 0.1), (p_5, 0)\} = 1.$$

In this case it is easy to observe that the above equation has no solution with respect to p_1, p_2, p_3, p_4, p_5, therefore P o Q = R has no solution with respect to P.
In general, writing R = $\{r_1\ \ r_2\ \ r_3\ \ r_4\}$, it becomes evident that we have no solution if $\max\limits_j q_{ij} < r_j$.

V. CONCLUSION

In the present article we used FRE for assessing student PS skills. In this way we have managed to express the "average student" of a class as a fuzzy set on the set of the steps of the PS process (orientation, planning, execution and checking), which gives valuable information to the instructor for designing his future teaching plans. On the contrary, we have realized that the problem of representing the "average student" of a class as a fuzzy set on the set of the linguistic grades characterizing his performance using FRE is complicated, since it may have more than one solutions or no solution at all.

In general, the use of FRE looks as a powerful tool for the assessment of human skills and therefore our future research plans include the effort of using them in other human activities apart from the PS process, like learning, modeling, decision-making, etc.

APPENDIX

The Problems of the Classroom Application

Problem 1: We want to construct a channel to run water by folding the two edges of a rectangle metallic leaf having sides of length 20 cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of water is constant, how we can run the maximum possible quantity of the water through the channel?

Solution: Folding the two edges of the metallic leaf by length x across its longer side the vertical cut of the constructed channel forms an orthogonal with sides x and 32-2x (Figure 1).

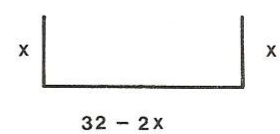

Fig.1. The vertical cut of the channel

The area of the rectangle, which is equal to $E(x) = x(32-2x) = 32x-2x^2$, has to be maximized. Taking the derivative $E'(x)$ the equation $E'(x) = 32-4x = 0$ gives that x = 8 cm. But $E''(x) = -4 < 0$, therefore $E(8) = 128\ cm^2$ is the maximum possible quantity of water to run through the channel.

Remark: A number of students folded the edges of the other side of the leaf and they found that $E(x) = x(20-2x) = 20x-2x^2$. In this case the equation $E'(x) = 0$ gives that x = 5 cm, while $E(5) = 50\ cm^2$. Their solution was of course mathematically correct, but many of them failed to realize that it is not acceptable in practice (real world).

Problem 2: Among all the cylindrical towers having a total surface of 180π m^2, which one has the maximal volume?

Solution: Let R be the radius of the basement of the tower and let h be its height. Then its total surface is equal to $2\pi Rh+2\pi R^2=180\pi \Rightarrow h = \frac{90-R^2}{R}$. Therefore the volume of the tower as a function of R is equal to V(R) = $\pi R^2 \frac{90-R^2}{R}=90\pi R-\pi R^3$. But $V'(R) = 90\pi-3\pi R^2 = 0$ gives that $R = \sqrt{30}$ m, while $V''(R) = -6\pi R < 0$. Thus, the maximal volume of the tower is equal to $V(\sqrt{30}) = 90\pi\sqrt{30} - \pi(\sqrt{30})^3 = 60\sqrt{30}\pi \approx 1032\, m^3$

Remark: A number of students considered the total surface of the tower as being equal to $2\pi Rh$, not including to it the areas of its basement and its roof. In this case they found that $h=\frac{90}{R}$, $V(R) =90\pi R$ *and* $V'(R) = 90\pi > 0$, which means that under these conditions there is no tower having a maximal volume. However, some of these students failed to correct their model in order to find the existing solution of the real problem (unsuccessful transition from the model to the real world).

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