Event-Triggered Global Robust Adaptive Stabilization for a Class of Nonlinear Systems

Wei Liu and Jie Huang

***Abstract*—In this paper, we study the event-triggered global robust adaptive stabilization problem for a class of nonlinear systems with unity relative degree, which contain not only disturbances, but also static parameter uncertainties and dynamic uncertainties. By combining the robust control technique and the adaptive control technique, we design a digital control law, a digital adaptive law, and an event-triggered mechanism to stabilize the system in the sense that the state of the closed-loop system is globally ultimately bounded. What's more, we show that the Zeno behavior does not happen. Finally, we illustrate our approach by applying it to the controlled Lorenz system.**

***Index Terms*—Event-triggered control, adaptive control, robust control, nonlinear systems.**

I. INTRODUCTION

Nowadays, in order to take advantage of microchips and computers in fast computation and signal processing, many controllers are implemented in digital platforms. In such an implementation, we first need to sample the states or the outputs of the plant, and then we need to compute and implement the actuator signals. Conventionally, the data sampling and the control actuator are performed periodically, since this allows us to design the controller and analyze the stability of the closed-loop system based on the well-developed theory on sampled-data systems, see, e.g., [3], [7]. Although periodic sampling is convenient for us to design the controller and analyze the stability, it has some drawbacks in utilizing the limited system resources. Namely, data samplings and control actuation take place when the system has achieved the control goal with sufficient accuracy, which is clearly a waste of the system resources. To overcome these drawbacks, a new digital control approach called event-triggered control (ETC) has been proposed, see, e.g., [2], [4], [8], [15]. In ETC, the data samplings and the control actuation are aperiodic and are triggered by some specific conditions depending on the states or the outputs of the plant. As a result, it is more efficient in balancing the resource utilization and the control performance. It is worth noting that one main challenge of the ETC is to guarantee the existence of the minimum inter-execution time. This is very important, because when the minimum inter-execution time does not exist, the number of the execution times may become infinite in finite time, i.e., the so-called Zeno behavior happens, which leads to that the ETC scheme is infeasible for implementation in digital platforms.

Up to now, significant contributions have been made on the ETC problems for both linear systems and nonlinear systems. For example, in [8], [15], the event-triggered stabilization problem was studied for linear systems and nonlinear systems, respectively. In [1], [11], the robust event-triggered stabilization problem was further studied for nonlinear systems based on the hybrid control technique and the small gain theorem technique, respectively. It is noted that one common assumption in [1], [8], [11], [15] is that there exists a controller such that the closed-loop system is input-to-state stable (ISS) with respect to the measurement error. In [12], the event-triggered global robust output regulation problem was studied based on the internal model approach. In particular, reference [17] studied the event-triggered adaptive control problem for a class of nonlinear systems subject to linearly parameterized uncertainties by proposing a switching threshold event-triggered mechanism. Reference [10] studied the event-triggered adaptive control problem for a class of parametric strict-feedback nonlinear systems satisfying the global Lipchitz condition by developing an impulsive adaptive control law. Some other relevant results can be found in [6], [13], [16], [19], etc.

In this paper, we will further study event-triggered global robust adaptive stabilization problem for a class of nonlinear systems. Compared with the existing results on the ETC problems for nonlinear systems, the main challenges of our problem consist of the following three aspects. First, the systems considered in this paper contain not only disturbances, but also static parameter uncertainties and dynamic uncertainties. Second, we remove the assumption that there exists a controller such that the closed-loop system is input-to-state stable (ISS) with respect to the measurement error, and do not impose the global Lipchitz condition on the nonlinear functions. Third, our control law and adaptive law are both digital and thus can be directly implemented in digital platforms. To overcome these challenges, by combining the robust control technique and the adaptive control technique, we design an event-triggered adaptive digital control law and an event-triggered mechanism to guarantee that the state of the closed-loop system is globally ultimately bounded and the Zeno behavior does not happen.

Notation. Denote $\mathrm{col}(c_1, ..., c_s) = [c_1^T, ..., c_s^T]^T$, where c_i, $i = 1, ..., s$, are any column vectors. $\|\cdot\|$ denotes the Euclidean norm of a vector or the induced norm of a matrix. The set of all nonnegative integers is denoted by $\mathbb{Z}^+$. The base of the natural logarithm is denoted by $\mathbf{e}$. The maximum eigenvalue

This work has been supported in part by the Research Grants Council of the Hong Kong Special Administration Region under grant No. 14200515, and in part by National Natural Science Foundation of China under Project 61633007.

Wei Liu and Jie Huang are with the Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. E-mail: wliu@mae.cuhk.edu.hk, jhuang@mae.cuhk.edu.hk

Corresponding author: Jie Huang.

and the minimum eigenvalue of a symmetric real matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. In this paper, we use the notation x to denote $x(t)$ for simplicity when no ambiguity occurs.

II. Problem formulation and preliminaries

In this paper, we consider the following class of nonlinear systems:

$$\begin{aligned}\dot{z} &= f(z,y,d)\\ \dot{y} &= g(z,y,d)+b\varphi^T(y)\theta+bu\end{aligned} \tag{1}$$

where $z\in\mathbb{R}^{n_z}$ and $y\in\mathbb{R}$ are the states, $u\in\mathbb{R}$ is the input, $d:[0,\infty)\to\mathbb{D}\subset\mathbb{R}^{l_1}$ with $\mathbb{D}$ a compact set represents the disturbance, $\theta\in\mathbb{R}^{l_2}$ is the unknown constant parameter, the parameter b is an unknown positive real number, the functions $f(\cdot)$, $g(\cdot)$ and $\varphi(\cdot)$ are sufficiently smooth and satisfy $f(0,0,d)=0$ and $g(0,0,d)=0$ for all $d\in\mathbb{D}$. As noted in [5], the inverse dynamics governing z in (1) can be viewed as the dynamic uncertainty.

Then we consider an adaptive control law of the following form

$$\begin{aligned}u(t) &= \hat{f}(\hat{\theta}(t_k),y(t_k),t_k)\\ \dot{\hat{\theta}}(t) &= \hat{g}(\hat{\theta}(t),y(t_k),t_k),\ \forall t\in[t_k,t_{k+1}),\ k\in\mathbb{Z}^+\end{aligned} \tag{2}$$

where $\hat{f}(\cdot)$ and $\hat{g}(\cdot)$ are some nonlinear functions, $\hat{\theta}\in\mathbb{R}^{l_2}$ is the estimation of the unknown parameter θ, and the triggering time instants t_k with $t_0=0$ are determined by an event-triggered mechanism of the following form

$$t_{k+1}=\inf\{t>t_k \mid h(\tilde{\theta}(t),\tilde{y}(t),\hat{\theta}(t),y(t))\geq\delta\} \tag{3}$$

where $h(\cdot)$ is some nonlinear function, $\delta>0$ is some constant, and

$$\begin{aligned}\tilde{y}(t) &= y(t_k)-y(t)\\ \tilde{\theta}(t) &= \hat{\theta}(t_k)-\hat{\theta}(t),\ \forall t\in[t_k,t_{k+1}),\ k\in\mathbb{Z}^+.\end{aligned} \tag{4}$$

The structure of the event-triggered closed-loop system is shown in Figure 1.

Remark 2.1: The control law (2) is called as an adaptive event-triggered control law. As will be seen from Remark 3.1 that we can further discretize the adaptive law in (2) and get an equivalent digital adaptive control law as follows:

$$\begin{aligned}u(t) &= \hat{f}(\hat{\theta}(t_k),y(t_k),t_k)\\ \hat{\theta}(t_{k+1}) &= \bar{g}(\hat{\theta}(t_k),y(t_k),t_k)\end{aligned} \tag{5}$$

for any $t\in[t_k,t_{k+1})$ with $k\in\mathbb{Z}^+$. It is noted that the control law in [17] is piecewise constant but the adaptive law in [17] is still continuous and depends on the continuous-time states of the plant, whereas here both the control law and the adaptive law in (5) have the discrete-time form and only depend on the sampled output $y(t_k)$ and state $\hat{\theta}(t_k)$, and thus can be directly implemented in a digital platform.

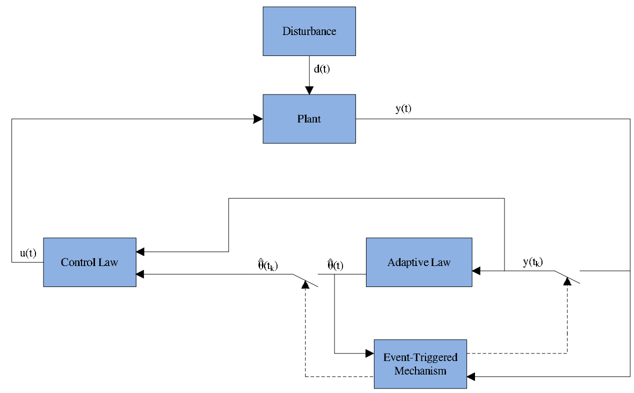

Fig. 1: Structure of the event-triggered closed-loop system.

Let

$$\begin{aligned}\bar{\theta}(t) &= \hat{\theta}(t)-\theta\\ x_c(t) &= \mathrm{col}(z(t),y(t),\hat{\theta}(t))\\ \bar{x}_c(t) &= \mathrm{col}(z(t),y(t),\bar{\theta}(t)).\end{aligned} \tag{6}$$

Then we formulate our problem as follows.

Problem 2.1: Given the plant (1), and some compact subset $\mathbb{D}\subset\mathbb{R}^{l_1}$, design a control law of the form (2) and an event-triggered mechanism of the form (3) such that, for any $d\in\mathbb{D}$, and any initial states $z(0)$, $y(0)$, $\hat{\theta}(0)$,

1) the solution $x_c(t)$ of the closed-loop system composed of (1) and (2) exists and is bounded for all $t\geq 0$;
2) $\lim_{t\to\infty}\sup\|\bar{x}_c(t)\|\leq\epsilon$ for some real number $\epsilon>0$.

Remark 2.2: Problem 2.1 is called as the event-triggered global robust adaptive stabilization problem. It is worth mentioning that the global robust adaptive stabilization problem for the system (1) has been studied in Section 5.3.1 of [5] by a continuous-time adaptive control law. Compared with the result in Section 5.3.1 of [5], our problem is more challenging in the following three ways. First, we need to design a digital adaptive control law instead of the analog adaptive control law in [5], and we also need to design an extra event-triggered mechanism to determine the triggering time sequence $\{t_k\}$. Second, we need to make an extra effort to prevent the Zeno behavior from happening. Third, under the event-triggered control, the closed-loop system is hybrid, which results in that the stability analysis of the closed-loop system is more complex.

To solve our problem, we introduce two standard assumptions as follows.

Assumption 2.1: There exist some known positive real numbers b_m and b_M such that $b_m\leq b\leq b_M$.

Assumption 2.2: For any compact subset $\mathbb{D}\subset\mathbb{R}^{l_1}$, there exists a $\mathcal{C}^1$ function $V(z)$ such that, for any $d\in\mathbb{D}$, and any z and y,

$$\underline{\alpha}(\|z\|)\leq V(z)\leq\bar{\alpha}(\|z\|) \tag{7}$$

$$\frac{\partial V(z)}{\partial z}f(z,y,d)\leq-\alpha(\|z\|)+\gamma(y) \tag{8}$$

where $\underline{\alpha}(\cdot)$, $\bar{\alpha}(\cdot)$ and $\alpha(\cdot)$ are some known class $\mathcal{K}_\infty$ functions with $\alpha(\cdot)$ satisfying $\lim_{s\to 0^+}\sup(s^2/\alpha(s))<\infty$, and $\gamma(\cdot)$ is a known smooth positive definite function.

Remark 2.3: Assumptions 2.1 and 2.2 are the same as Assumptions 5.1 and 5.2 of [5]. Under Assumption 2.2, the subsystem $\dot{z} = f(z, y, d)$ is input-to-state stable (ISS) with y as the input [14].

III. Main Result

In this section, we will present our main result.

First, we introduce some notation. Let

$$\begin{aligned} v(t) &= -\varphi^T(y(t))\hat{\theta}(t) - \rho(y(t))y(t) \\ \psi(t) &= y(t)\varphi(y(t)) \end{aligned} \tag{9}$$

where $\rho(\cdot)$ is a positive function which will be specified later. Then we consider the following adaptive control law

$$\begin{aligned} u(t) &= v(t_k) \\ \dot{\hat{\theta}}(t) &= \Lambda\psi(t_k) - a\Lambda\hat{\theta}(t), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}^+ \end{aligned} \tag{10}$$

where a is a positive real number and Λ is a positive definite matrix.

Remark 3.1: Note that the control law (10) can be further discretized into the following form

$$\begin{aligned} u(t) &= v(t_k) \\ \hat{\theta}(t_{k+1}) &= \mathbf{e}^{-a\Lambda(t_{k+1}-t_k)}\hat{\theta}(t_k) \\ &\quad + \int_{t_k}^{t_{k+1}} \mathbf{e}^{-a\Lambda(t_{k+1}-\tau)} d\tau \Lambda\psi(t_k). \end{aligned} \tag{11}$$

Clearly, the control law (11) has the same form as (5) and can be directly implemented in a digital platform. Since the control law (11) is equivalent to the control law (10), for convenience, we will still use the control law (10) to analyze the closed-loop stability.

Next, we further define

$$\begin{aligned} \tilde{v}(t) &= v(t_k) - v(t) \\ \tilde{\psi}(t) &= \psi(t_k) - \psi(t), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}^+. \end{aligned} \tag{12}$$

Then we consider the following event-triggered mechanism

$$t_{k+1} = \inf\{t > t_k \mid |\tilde{v}|^2 + \frac{b_M}{a}\|\tilde{\psi}\|^2 - \sigma\rho(y)y^2 \geq \delta\} \tag{13}$$

where $\sigma > 0$ and $\delta > 0$ are some real numbers. Note that, under the event-triggered mechanism (13), we have

$$|\tilde{v}|^2 + \frac{b_M}{a}\|\tilde{\psi}\|^2 \leq \sigma\rho(y)y^2 + \delta \tag{14}$$

for any $t \in [t_k, t_{k+1})$ with $k \in \mathbb{Z}^+$.

The closed-loop system composed of (1) and (10) can be put into the following form:

$$\begin{aligned} \dot{z} &= f(z, y, d) \\ \dot{y} &= g(z, y, d) + b\varphi^T(y)\theta + b(\tilde{v} + v) \\ \dot{\hat{\theta}} &= \Lambda(\tilde{\psi} + \psi) - a\Lambda\hat{\theta}, \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{Z}^+. \end{aligned} \tag{15}$$

According to (6), we have

$$\bar{x}_c(t) = x_c(t) - \begin{bmatrix} \mathbf{0}_{n_z \times 1} \\ 0 \\ \theta \end{bmatrix}. \tag{16}$$

Clearly, the existence of the solution $\bar{x}_c(t)$ immediately implies the existence of the solution $x_c(t)$, so next we will focus on the $\bar{x}_c(t)$ system. For this purpose, we further let

$$f_c(\bar{x}_c, \tilde{v}, \tilde{\psi}, d) = \begin{bmatrix} f(z, y, d) \\ g(z, y, d) + b\varphi^T(y)\theta + b(\tilde{v} + v) \\ \Lambda(\tilde{\psi} + \psi) - a\Lambda(\bar{\theta} + \theta) \end{bmatrix}$$

for any $t \in [t_k, t_{k+1})$ with $k \in \mathbb{Z}^+$. Then we obtain the following compact form:

$$\dot{\bar{x}}_c = f_c(\bar{x}_c, \tilde{v}, \tilde{\psi}, d). \tag{17}$$

Suppose that the solution $\bar{x}_c(t)$ of (17) is right maximally defined for all $t \in [0, T_M)$ under the event-triggered mechanism (13), where $0 < T_M \leq \infty$. Then we give the following lemma.

Lemma 3.1: Under Assumptions 2.1 and 2.2, there exists a $\mathcal{C}^1$ function $U(\bar{x}_c)$, such that, for any $d \in \mathbb{D}$, and any $\bar{x}_c$,

$$\underline{\gamma}(\|\bar{x}_c\|) \leq U(\bar{x}_c) \leq \bar{\gamma}(\|\bar{x}_c\|) \tag{18}$$

$$\begin{aligned} &\frac{\partial U(\bar{x}_c)}{\partial \bar{x}_c} f_c(\bar{x}_c, \tilde{v}, \tilde{\psi}, d) \\ &\leq - a_m\|\bar{x}_c\|^2, \ \forall \|\bar{x}_c\| \geq \sqrt{\frac{\delta + ab_M\|\theta\|^2}{a_m}} \end{aligned} \tag{19}$$

where a_m is a positive real number to be specified later, $\underline{\gamma}(\cdot)$ and $\bar{\gamma}(\cdot)$ are two class $\mathcal{K}_\infty$ functions.

Proof: First, under Assumption 2.2, by applying the changing supply pair technique in [14], for any smooth function $\Delta(z) \geq 0$, there exists a $\mathcal{C}^1$ function $\bar{V}(z)$, such that, for any $d \in \mathbb{D}$, and any z, y,

$$\underline{\beta}(\|z\|) \leq \bar{V}(z) \leq \bar{\beta}(\|z\|) \tag{20}$$

$$\frac{\partial \bar{V}(z)}{\partial z} f(z, y, d) \leq -\Delta(z)\|z\|^2 + \beta(y)y^2 \tag{21}$$

where $\underline{\beta}(\cdot)$ and $\bar{\beta}(\cdot)$ are some known class $\mathcal{K}_\infty$ functions, and $\beta(\cdot)$ is some known smooth positive function.

Let

$$U(\bar{x}_c) = \bar{V}(z) + \frac{1}{2}y^2 + \frac{1}{2}b\bar{\theta}^T\Lambda^{-1}\bar{\theta}. \tag{22}$$

Clearly, $U(\bar{x}_c)$ is positive definite and radially unbounded. Thus there exist two class $\mathcal{K}_\infty$ functions $\underline{\gamma}(\cdot)$ and $\bar{\gamma}(\cdot)$ such that (18) is satisfied.

Since $g(0, 0, d) = 0$ for all $d \in \mathbb{R}^{l_1}$, by Part (ii) of Corollary 11.1 of [5], there exist two sufficiently smooth functions $\pi(z) \geq 0$ and $\varpi(y) \geq 0$ such that, for any $d \in \mathbb{D}$, and any z, y,

$$|g(z, y, d)| \leq \pi(z)\|z\| + \varpi(y)|y|. \tag{23}$$

Then, according to (9), (12), (15) and (23), we have

$$
\begin{aligned}
y\dot{y} =& y(g(z,y,d) + b\varphi^T(y)\theta + b(\tilde{v}+v)) \\
\leq& |y|\pi(z)\|z\| + \varpi(y)y^2 + by(\varphi^T(y)\theta \\
&+ v + \tilde{v}) \\
\leq& \frac{1}{4}y^2 + \pi^2(z)\|z\|^2 + \varpi(y)y^2 + by(\varphi^T(y)\theta \\
&- \varphi^T(y)\hat{\theta} - \rho(y)y + \tilde{v}) \\
\leq& \frac{1}{4}y^2 + \pi^2(z)\|z\|^2 + \varpi(y)y^2 - by\varphi^T(y)\bar{\theta} \\
&- b_m\rho(y)y^2 + \frac{b_M^2}{4}y^2 + |\tilde{v}|^2 \\
=& \pi^2(z)\|z\|^2 - (b_m\rho(y) - \frac{1+b_M^2}{4} - \varpi(y))y^2 \\
&- by\varphi^T(y)\bar{\theta} + |\tilde{v}|^2.
\end{aligned} \tag{24}
$$

Also, according to (6), (9) and (15), we have

$$
\begin{aligned}
b\bar{\theta}^T\Lambda^{-1}\dot{\bar{\theta}} =& b\bar{\theta}^T\Lambda^{-1}\dot{\hat{\theta}} \\
=& b\bar{\theta}^T\Lambda^{-1}(\Lambda(\tilde{\psi}+\psi) - a\Lambda\hat{\theta}) \\
=& b\bar{\theta}^T(\tilde{\psi}+\psi - a\hat{\theta}) \\
=& b\bar{\theta}^T(y\varphi(y) + \tilde{\psi} - a\bar{\theta} - a\theta) \\
\leq& by\varphi^T(y)\bar{\theta} + b(\frac{a}{4}\|\bar{\theta}\|^2 + \frac{1}{a}\|\tilde{\psi}\|^2) \\
&- ab\|\bar{\theta}\|^2 + ab(\frac{1}{4}\|\bar{\theta}\|^2 + \|\theta\|^2) \\
\leq& by\varphi^T(y)\bar{\theta} - \frac{1}{2}ab_m\|\bar{\theta}\|^2 \\
&+ \frac{b_M}{a}\|\tilde{\psi}\|^2 + ab_M\|\theta\|^2.
\end{aligned} \tag{25}
$$

Combining (14), (21), (24) and (25), we have

$$
\begin{aligned}
&\frac{\partial U(\bar{x}_c)}{\partial \bar{x}_c} f_c(\bar{x}_c, \tilde{v}, \tilde{\psi}, d) \\
\leq& -\Delta(z)\|z\|^2 + \beta(y)y^2 + \pi^2(z)\|z\|^2 \\
&- (b_m\rho(y) - \frac{1+b_M^2}{4} - \varpi(y))y^2 \\
&- by\varphi^T(y,t)\bar{\theta} + |\tilde{v}(t)|^2 + by\varphi^T(y,t)\bar{\theta} \\
&- \frac{1}{2}ab_m\|\bar{\theta}\|^2 + \frac{b_M}{a}\|\tilde{\psi}(t)\|^2 + ab_M\|\theta\|^2 \\
=& -(\Delta(z) - \pi^2(z))\|z\|^2 - \Bigg(b_m\rho(y) - \frac{1+b_M^2}{4} \\
&- \varpi(y) - \beta(y)\Bigg)y^2 - \frac{1}{2}ab_m\|\bar{\theta}\|^2 \\
&+ |\tilde{v}(t)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t)\|^2 + ab_M\|\theta\|^2 \\
\leq& -(\Delta(z) - \pi^2(z))\|z\|^2 - \Bigg((b_m - \sigma)\rho(y) \\
&- \frac{1+b_M^2}{4} - \varpi(y) - \beta(y)\Bigg)y^2 - \frac{1}{2}ab_m\|\bar{\theta}\|^2 \\
&+ \delta + ab_M\|\theta\|^2.
\end{aligned} \tag{26}
$$

Choose

$$
\begin{aligned}
&0 < \sigma < b_m \\
&\Delta(z) \geq \pi^2(z) + a_1 \\
&\rho(y) \geq \frac{1}{b_m - \sigma}(\frac{1+b_M^2}{4} + \varpi(y) + \beta(y) + a_2)
\end{aligned} \tag{27}
$$

where a_1, a_2 are some positive real numbers, and further let

$$
\begin{aligned}
a_3 &= \frac{1}{2}ab_m, \\
a_m &= \frac{1}{2}\min\{a_1, a_2, a_3\}.
\end{aligned} \tag{28}
$$

Then we have

$$
\begin{aligned}
&\frac{\partial U(\bar{x}_c)}{\partial \bar{x}_c} f_c(\bar{x}_c, \tilde{v}, \tilde{\psi}, d) \\
\leq& -(a_1\|z\|^2 + a_2y^2 + a_3\|\bar{\theta}\|^2)) + \delta + ab_M\|\theta\|^2 \\
\leq& -2a_m(\|z\|^2 + y^2 + \|\bar{\theta}\|^2) + \delta + ab_M\|\theta\|^2 \\
=& -2a_m\|\bar{x}_c\|^2 + \delta + ab_M\|\theta\|^2 \\
\leq& -a_m\|\bar{x}_c\|^2, \ \forall \|\bar{x}_c\| \geq \sqrt{\frac{\delta + ab_M\|\theta\|^2}{a_m}}
\end{aligned} \tag{29}
$$

which ends the proof. □

Remark 3.2: Clearly, Lemma 3.1 implies that, for any $\bar{x}_c(0)$, the state $\bar{x}_c(t)$ of the closed-loop system (17) satisfies

$$
\|\bar{x}_c(t)\| \leq \max\{\sqrt{\frac{\delta + ab_M\|\theta\|^2}{a_m}}, \underline{\gamma}^{-1}(\bar{\gamma}(\|\bar{x}_c(0)\|))\}
$$

for any $t \in [0, T_M)$.

Based on Lemma 3.1 and Remark 3.2, we obtain the main result as follows.

Theorem 3.1: Under Assumptions 2.1 and 2.2, Problem 2.1 for the system (1) is solvable by the event-triggered adaptive control law (10) under the event-triggered mechanism (13).

Proof: First, we will show that the solution $x_c(t)$ of the closed-loop system (17) will exist for all $t \geq 0$, i.e., $T_M = \infty$ and thus excluding the Zeno behavior. For this purpose, we consider the following two possible cases for the triggering time sequence $\{t_k\}$:

1) the number of the triggering time instants is finite over $t \in [0, T_M)$;
2) the number of the triggering time instants is infinite over $t \in [0, T_M)$.

For Case 1), the Zeno behavior obviously does not happen, and there is a finite time t_{k^*} such that the closed-loop system (17) becomes a time-invariant continuous-time system for all $t \in [t_{k^*}, T_M)$. Together with Remark 3.2, we conclude that $T_M = \infty$.

For Case 2), if we show $\lim_{k\to\infty} t_k = \infty$, then $T_M = \infty$. According to (13), we have

$$
\begin{aligned}
&\lim_{t\to t_{k+1}^-}(|\tilde{v}(t)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t)\|^2) \\
\geq& \lim_{t\to t_{k+1}^-}(\sigma\rho(y(t))y^2(t) + \delta) \geq \delta, \ k \in \mathbb{Z}^+.
\end{aligned} \tag{30}
$$

Besides, for any $t \in [t_k, t_{k+1})$ with $k \in \mathbb{Z}^+$, we have

$$\begin{aligned}&\frac{d\left(|\tilde{v}(t)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t)\|^2\right)}{dt}\\ =&2\tilde{v}(t)\dot{\tilde{v}}(t) + 2\frac{b_M}{a}\tilde{\psi}^T(t)\dot{\tilde{\psi}}(t)\\ =&-2\tilde{v}(t)\dot{v}(t) - 2\frac{b_M}{a}\tilde{\psi}^T(t)\dot{\psi}(t)\end{aligned} \tag{31}$$

where

$$\begin{aligned}\dot{v}(t) =& -\left(\frac{\partial\varphi(y)}{\partial y}\dot{y}\right)^T\hat{\theta} - \varphi^T(y)\dot{\hat{\theta}}\\ &-\left(\frac{\partial\rho(y)}{\partial y}y + \rho(y)\right)\dot{y}\\ \dot{\psi}(t) =& \dot{y}\varphi(y) + y\frac{\partial\varphi(y)}{\partial y}\dot{y}.\end{aligned} \tag{32}$$

From Remark 3.2, we know $\bar{x}_c(t)$ is bounded for all $t \in [0, T_M)$, which means that $z(t), y(t), \hat{\theta}(t)$ are bounded for all $t \in [0, T_M)$. Based on (9) and (12), we conclude that $\tilde{v}(t)$ and $\tilde{\psi}(t)$ are bounded for all $t \in [0, T_M)$. In addition, $d(t) \in \mathbb{D}$ is also bounded for all $t \in [0, \infty)$. Based on (15), we conclude that $\dot{z}(t), \dot{y}(t)$ and $\dot{\hat{\theta}}(t)$ are also bounded for all $t \in [0, T_M)$, and thus $\dot{v}(t)$ and $\dot{\psi}(t)$ are bounded for all $t \in [0, T_M)$. As a result, there exists a positive real number c_0 such that, for all $t \in [0, T_M)$,

$$\frac{d\left(|\tilde{v}(t)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t)\|^2\right)}{dt} \le c_0. \tag{33}$$

Inequalities (30) and (33) implies

$$\begin{aligned}c_0(t_{k+1} - t_k) \ge& |\tilde{v}(t_{k+1}^-)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t_{k+1}^-)\|^2\\ &-\left(|\tilde{v}(t_k)|^2 + \frac{b_M}{a}\|\tilde{\psi}(t_k)\|^2\right)\\ \ge& \delta - 0 = \delta\end{aligned} \tag{34}$$

for any $k \in \mathbb{Z}^+$. Therefore, we have

$$t_{k+1} - t_k \ge \frac{\delta}{c_0} \tag{35}$$

for any $k \in \mathbb{Z}^+$. Clearly, the Zeno behavior also does not happen. Also, note that, under Case 2), the number of the triggering time instants is infinite. Thus we can conclude that $\lim_{t\to\infty} t_k = \infty$, which further implies $T_M = \infty$.

Since the solution $x_c(t)$ of the closed-loop system (17) exists for all $t \in [0, \infty)$, according to Theorem 4.18 of [9] and Lemma 3.1 here, we have

$$\lim_{t\to\infty} \sup \|\bar{x}_c(t)\| \le \epsilon \tag{36}$$

with $\epsilon = \underline{\gamma}^{-1}(\bar{\gamma}(\sqrt{\frac{\delta + ab_M\|\theta\|^2}{a_m}}))$, that is to say, $\bar{x}_c(t)$ is globally ultimately bounded with the ultimate bound ϵ. □

Remark 3.3: Note that the ultimate bound ϵ depends on both the design parameters δ, a, a_1, a_2 and the system parameters b_m, b_M, θ. It is easy to see that the ultimate bound ϵ can be reduced by decreasing δ or adjusting a, a_1, a_2. In addition, from (35), we can see that lager design parameter δ leads to larger inter-execution time interval and thus leads to less triggering number.

IV. An Example

Consider the controlled Lorenz system taken from [18] as follows:

$$\begin{aligned}\dot{z}_1 &= c_1 z_1 + c_2 y\\ \dot{z}_2 &= c_3 z_2 + z_1 y\\ \dot{y} &= c_4 z_1 - z_1 z_2 + \theta y + u\end{aligned} \tag{37}$$

where $b = 1$, θ is some unknown constant parameter, $c_i(t) = \bar{c}_i + d_i(t)$, for $i = 1, 2, 3, 4$, are some time-varying parameters with $\bar{c} = \text{col}(\bar{c}_1, \cdots, \bar{c}_4) = \text{col}(-10, 10, -\frac{8}{3}, 28)$ denoting the nominal value and $d(t) = \text{col}(d_1(t), \cdots, d_4(t))$ denoting the disturbance or uncertainty. Clearly, the system (37) is in the form of (1) with

$$\begin{aligned}z &= \text{col}(z_1, z_2),\ \varphi(y) = y\\ f(z, y, d) &= \begin{bmatrix} c_1 z_1 + c_2 y\\ c_3 z_2 + z_1 y\end{bmatrix}\\ g(z, y, d) &= c_4 z_1 - z_1 z_2.\end{aligned}$$

Here we assume that $d(t) \in \mathbb{D} = \{d = \text{col}(d_1, \cdots, d_4) \in \mathbb{R}^4 \mid |d_i| \le 0.5,\ i = 1, 2, 3, 4\}$.

Clearly, Assumption 2.1 is satisfied. To verify Assumption 2.2, we let

$$V(z) = \frac{1}{2}z_1^2 + \frac{1}{4}z_1^4 + \frac{1}{2}z_2^2. \tag{38}$$

Since $V(z)$ is positive definite and radially unbounded, there exist two class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\bar{\alpha}(\cdot)$ such that (7) is satisfied. Also, it is possible to show that, along the trajectory of the z-subsystem of (37),

$$\begin{aligned}\dot{V}(z) \le& -4.25z_1^2 - 1.375z_1^4 - 1.66z_2^2\\ &+ 5.5y^2 + 2.625y^4\end{aligned} \tag{39}$$

which implies (8) is also satisfied. Thus Assumption 2.2 is satisfied.

Then, by Theorem 3.1, we can design an event-triggered adaptive control law of the form (10) and an event-triggered mechanism of the form (13) with $a = 1$, $\sigma = 0.2$, $\delta = 0.1$, $\Lambda = 1$ and $\rho(y) = 50 + 15y^2$.

Simulation is performed with the parameter $\theta = -1$, the disturbance $d(t) = \text{col}(0.2\sin(0.3t), 0.4, 0.5\cos(0.3t), -0.3)$ and the initial condition

$$[z_1(0), z_2(0), y(0), \hat{\theta}(0)] = [0.82, -1.65, 0.47, -1.42].$$

Figures 2 shows the trajectories of the states z_1, z_2, y, and $\hat{\theta}$. Figure 3 shows the control input signal $u(t)$. Figure 4 shows the event-triggered condition. It can be easily seen that all the states are globally ultimately bounded and the control input signal $u(t)$ is piecewise constant. Thus the event-triggered global robust adaptive stabilization problem for the Lorenz (37) is solved satisfactorily.

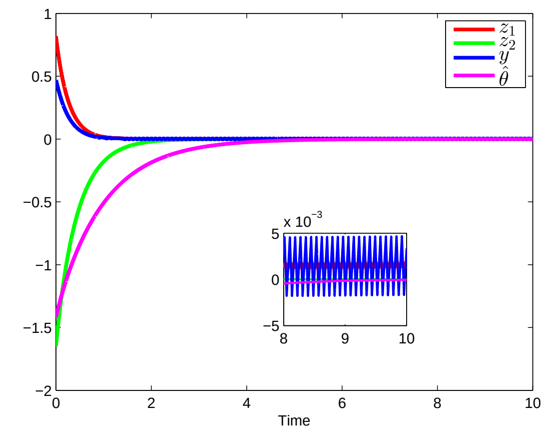

Fig. 2: Profiles of the states z_1, z_2, y and $\hat{\theta}$.

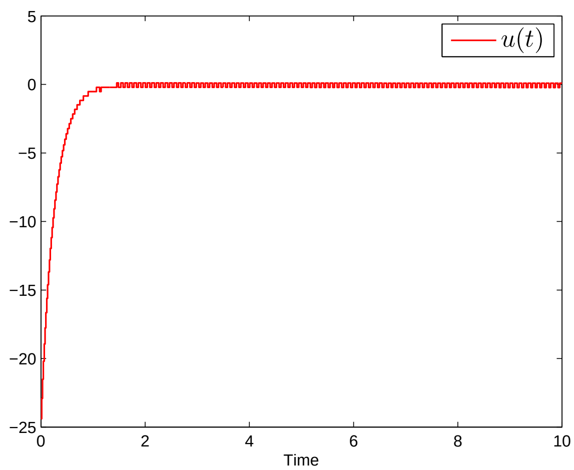

Fig. 3: Profile of the control input $u(t)$.

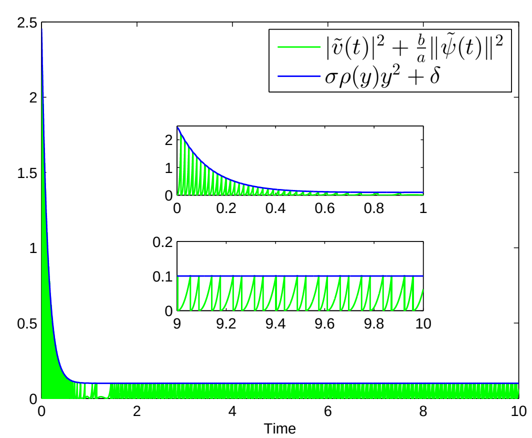

Fig. 4: Event-triggered condition.

V. Conclusion

In this paper, we have studied the event-triggered global robust adaptive stabilization problem for a class of nonlinear systems subject to disturbances, static parameter uncertainties and dynamic uncertainties. By integrating the robust control technique and the adaptive control technique, we have designed an implementable digital adaptive control law and an event-triggered mechanism to solve the problem, and shown that the Zeno behavior can be avoided. In the future, we will consider extending our result to the nonlinear systems with higher relative degree.

References

[1] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96-108, 2017.

[2] K. E. Årzén "A simple event-based PID controller," *The 14th World Congress of IFAC*, Beijing, China, 1999, pp. 8687-8692.

[3] K. J. Åström, B. Wittenmark, *Computer controlled systems*. Prentice Hall, Upper Saddle River, 1977.

[4] K. J. Åström and B. Bernhardsson, "Comparison of periodic and event based sampling for first-order stochastic systems," *The 14th World Congress of IFAC*, Beijing, China, 1999, pp. 5006-5011.

[5] Z. Chen and J. Huang, *Stabilization and regulation of nonlinear systems: a robust and adaptive approach*, Springer, 2015.

[6] V. S. Dolk, D.P. Borgers and W. P. M. H. Heemels, "Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and Zeno-freeness," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 34-49, 2017.

[7] G. F. Franklin, J. D. Powel, and A. Emami-Naeini, *Feedback control of dynamical systems*. Prentice Hall, Upper Saddle River, 2010.

[8] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," *The 51st IEEE Conference on Decision and Control*, Maul, Hawaii, USA, 2012, pp. 3270-3285.

[9] H. K. Khalil, *Nonlinear Systems-third edition*, Prentice Hall, 2002.

[10] Y. X. Li and G. H. Yang, "Event-triggered adaptive backstepping control for parametric strict-feedback nonlinear systems," *International Journal of Robust and Nonlinear Control*, DOI: 10.1002/rnc.3914, 2017.

[11] T. Liu and Z. P. Jiang, "A small-gain approach to robust event-triggered control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2072-2085, 2015.

[12] W. Liu and J. Huang, "Event-triggered global robust output regulation for a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5923-5930, 2017.

[13] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, "A framework for the event-triggered stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 982-996, 2015.

[14] E. D. Sontag and A. R. Teel, "Changing supply functions in input/state stable systems." *IEEE Transactions on Automatic Control*, vol. 40, no. 8, pp. 1476-1478, 1995.

[15] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680-1685, 2007.

[16] P. Tallapragada and N. Chopra, "On event triggered tracking for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2343-2348, 2013.

[17] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, "Event-triggered adaptive control for a class of uncertain nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 2071-2076, 2017.

[18] D. Xu and J. Huang, "Robust adaptive control of a class of nonlinear systems and its application," *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 57, no. 3, pp. 691-702, 2010.

[19] F. Zhu, M. Xia, and P. J. Antsaklis. "On passivity analysis and passivation of event-triggered feedback systems using passivity indices," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1397-1402, 2017.