Unsteady Flow of a Newtonian Fluid in a Contracting and Expanding Pipe

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Abstract— In this paper, we considered the unsteady flow of Newtonian fluid in a contracting and expanding pipe. Flow is considered to be in a circular pipe whose radius is a function of time. Further, the flow is taken to be under the influence of a negative pressure gradient that is expressed as a separable function of spatial variable and time. The function in the spatial variable is taken in two forms: linear and exponential (both decreasing) and a sinusoidal function has been assumed in terms of the time variable. Fluid flow equations have been derived under the assumptions that the fluid is incompressible and the flow is axi-symmetric. The resulting coupled system of partial differential equation in terms of the radial and axial velocity components have been solved using Homotopy perturbation method. The effect of the parameters related to the contraction and expansion of the pipe and the frequency of oscillations, on the wall shear stress and volumetric flux has been studied for the two models developed in this work and the results are presented and discussed.

Keywords— Unsteady flow, Contracting and Expanding pipe, Homotopy perturbation method, Wall shear stress, volumetric flux.

I. Introduction

Unsteady flow of viscous fluids in pipes produced by a simple contraction or expansion of the wall has applications in blood flow in coronary arteries, physiological flow pumps etc. [1, 2]. The study that is first of this kind was by Uchida and Aoki who considered the flow of Newtonian fluid in a semi- infinite contracting and expanding circular pipe [3]. They derived an exact solution to the Navier–Stokes equation using similarity transformation. Secomb [4] extended the work done by Uchida to understand the flow in a channel with pulsating walls. The wall motion was taken to be sinusoidal with the amplitude of oscillations being small. Ohki, in his work [5], discussed the flow in a semi-infinite porous pipe with expanding and contracting radius in axial direction. Recently, Si et al [6] calculated multiple solutions for the contracting or expanding porous pipe at large suction Reynolds number using singular perturbation method.

In the present study, we consider the flow of Newtonian fluid in a pipe with its radius varying with time. The expression for radius is taken to be the one considered by Uchinda and Aoki [3]. While, in [3], the flow was taken to be under a constant pressure gradient, we considered flow to be under the influence of a negative pressure gradient that is a function of both spatial variable and time. Further, we considered the flow to be axi-symmetric and the fluid to be incompressible and derived the fluid flow equations governing the radial and axial components of the velocity vector. The resulting coupled system of partial differential equations together with suitable initial- boundary conditions have been solved using Homotopy perturbation method. We, then studied the effect of the flow and other parameters on wall shear stress and volumetric flux and presented the results.

II. Problem Formulation

Consider the flow of Newtonian fluid in a pipe whose radius varies with time as described in Uchinda [3]. Cylindrical polar coordinate system (r,θ,z) (where r and z are the radial and axial coordinates respectively and θ is the azimuthal angle) is taken to describe the geometry of the problem and it is assumed that the fluid flow is in the z direction. Further, fluid is considered to be incompressible and the flow is axi-symmetric so that the velocity vector and the pressure, denoted by $\overline{q}$ and p respectively, are functions of r, z and time t only. Assume that the non-vanishing components of the velocity vector are in radial and axial directions so that the velocity vector is $\overline{q}(r,z,t)=(u(r,z,t),0,w(r,z,t))$. Also, let the thermodynamic pressure p be $p(r,z,t)$.

Now, the continuity equation and momentum equations take the form [7, 8]:

Continuity Equation: $$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

Momentum Equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{\partial}{\partial z}(\tau_{rz})\right) \qquad (2)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{\partial}{\partial z}(\tau_{zz})\right) \qquad (3)$$

Here, ρ is the density of the blood, τ_{ij} are the components of the deviatoric stress tensor.

Under the assumption that the radial flow velocity and the convective acceleration terms are respectively of smaller order of magnitude with respect to the axial flow velocity and the local acceleration terms, the radial momentum equation (2) reduces to

$$-\frac{\partial p}{\partial r}=0 \tag{4}$$

Thus, it can be seen that the pressure is independent of r. Following Shankar, Usik [9], pressure gradient is taken to be a function of z and t as $\frac{\partial p}{\partial z}=f(z)g(t)$ (5)

where, in the present study $f(z)$ is taken in two forms: one is a linear, decreasing function defined as $f(z)=(L-z)/L$, and the other as a negative exponential function as $f(z)=e^{-z/L}$ where L is the length of the pipe. Further, $g(t)$ is taken as $g(t)=A_0+A_1\,\mathrm{Cos}\,\omega t$, where A_0 is the constant amplitude of the pressure gradient, A_1 is the amplitude of its pulsatile component and ω is the angular frequency.

The fluid is taken to be Newtonian whose constitutive equation is [8]

$$\tau_{ij}=-p\delta_{ij}+2\mu e_{ij} \tag{6}$$

where p is the thermodynamic pressure, μ is the viscosity and e_{ij} is the rate of deformation tensor.

The contraction and expansion of the walls of the pipe is described as in [3] as

$$R(t)=R_0(1-\alpha t)^{1/2} \tag{7}$$

where R_0 is the radius of the pipe when $t=0$ and α is the parameter that describes the behavior of the wall. Positive values of α indicate contraction while negative values of this parameter describe expansion of the walls of the pipe.

Using the constitutive equation (6), equations (1) and (3) take the form

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{8}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right) \tag{9}$$

Now, the problem is to solve the above coupled system of partial differential equations together with the boundary conditions given by

$w=0$ on $r=R(t)$ (No slip boundary condition)

$u=\frac{dR}{dt}$ on $r=R(t)$ (Velocity of the flow matches with wall velocity)

$\frac{\partial w}{\partial r}=0$ at $r=0$ (Velocity is finite at the center of the pipe) (10)

III. Solution to the Problem

The above system of partial differential equations together with the initial and boundary conditions is solved using the homotopy perturbation method (HPM) [10]. For this, we take

$$u(r,z,t)=u_0(r,z,t)+pu_1(r,z,t)+p^2u_2(r,z,t)+\ldots$$
$$w(r,z,t)=w_0(r,z,t)+pw_1(r,z,t)+p^2w_2(r,z,t)+\ldots \tag{11}$$

and the initial approximation satisfying the conditions (10) is taken as $u_0(r,z,t)=0$, $w_0(r,z,t)=\frac{dR}{dt}$. The subsequent approximations for $u(r,z,t)$ and $w(r,z,t)$ are found by defining Homotopy functions as,

$$H_1(p)=(1-p)\left(\frac{\partial u}{\partial r}+\frac{u}{r}-\frac{\partial u_0}{\partial r}-\frac{u_0}{r}\right)+p\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}\right)$$

, for equation (8) (12)

and that for equation (9) is defined as

$$H_2(p)=(1-p)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{\partial^2 w_0}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\right)$$
$$+p\left(\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)+\frac{\partial p}{\partial z}-\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}\right)\right) \tag{13}$$

Comparing the coefficient of p, we get the equations governing $u_1(r,z,t)$ and $w_1(r,z,t)$ as

$$\frac{\partial u_1}{\partial r}+\frac{u_1}{r}=-\left(\frac{\partial w_0}{\partial z}\right) \tag{14}$$

$$\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}=-\left(\begin{array}{l}\frac{\partial p}{\partial z}+\rho\left(\frac{\partial w_0}{\partial t}+u_0\frac{\partial w_0}{\partial r}+w_0\frac{\partial w_0}{\partial z}\right)-\\ \mu\left(\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{\partial^2 w_0}{\partial z^2}\right)-\frac{\partial^2 w_0}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\end{array}\right) \tag{15}$$

And the conditions are $u_1=0$ on $r=R(t)$

$w_1=0$ on $r=R(t)$, $\frac{\partial w_1}{\partial r}=0$ at $r=0$ (16)

Solving these equations, we get expressions for the first approximation of the velocity components. Similarly, the coefficient of p^2 and higher powers have been collected to find the next approximations.

IV. Results and Discussions

A code has been developed in MATHEMATICA to compute approximations for $u(r,z,t)$ and $w(r,z,t)$ for α, both positive and negative. For a fixed set of values of A_0, A_1, μ, ρ, R_0, L and for $\omega=2\pi f$, ($f=72 beats/\min, 90 beats/\min$), wall shear stress (WSS) and the volumetric flux have been computed. Figure (1) shows the plots of WSS for different α for linear form of

$f(z)$ while Fig (2) for the exponential form, both plotted at time $t = 1/\omega$. From equation (7), we see that, for a given time t, an increase in α reduces the radius of the pipe and thus reducing the shear stress at wall. This is clearly visible for both the linear and exponential models from figures 1 and 2. Also, we see that the WSS decreases along the length of the pipe. Further, an increase in the frequency of oscillations ω increases WSS.

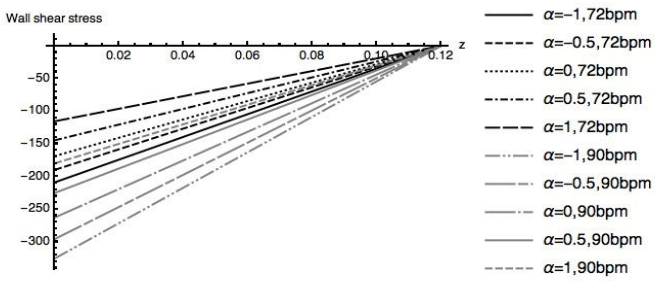

FIGURE 1. Variation of WSS with z for linear model

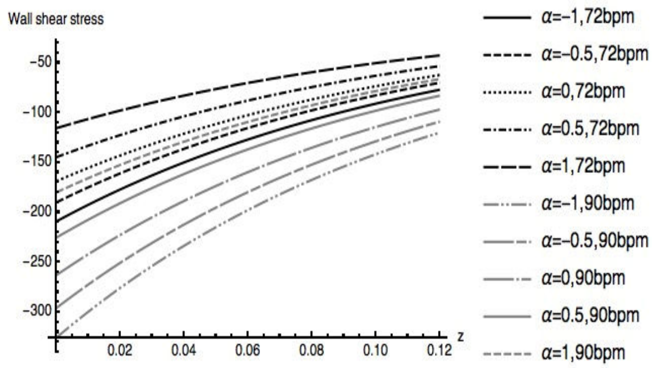

FIGURE 2. Variation of WSS with z for Exponential model

Figures 3- 8 show the plot of WSS with time for $\alpha = 0.5, 0, -0.5$. As explained before, for positive values of α, the radius of the pipe decreases with time. This results in a decrease in the WSS with time as can be seen from figures 3 and 4. Negative values of α indicates the case in which the radius of the pipe increases with time and thus WSS is likely to increase. The models developed in this paper predicted these results as shown in figures 5 and 6. Figures 7 and 8 show the plots of the variation of WSS with time when $\alpha = 0$. This is the case when the radius of the pipe is a constant.

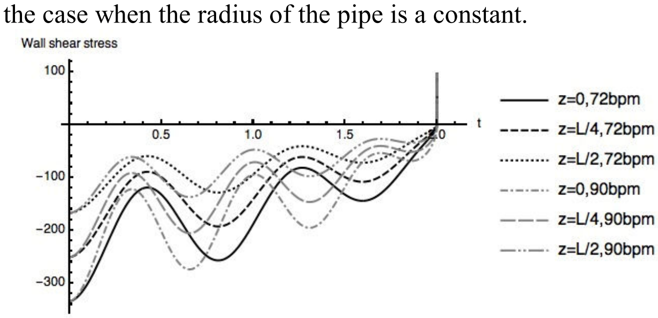

FIGURE 3. Variation of WSS with t at $\alpha = 0.5$ for Linear model

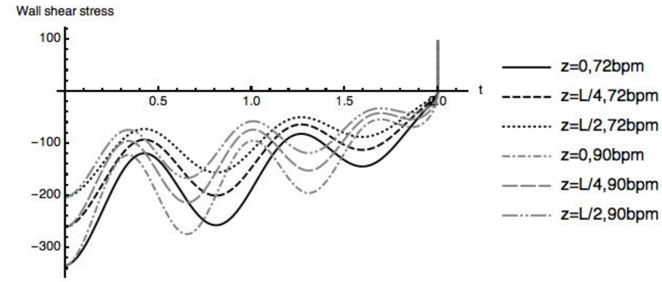

FIGURE 4. Variation of WSS with t at $\alpha = 0.5$ for Exponential model

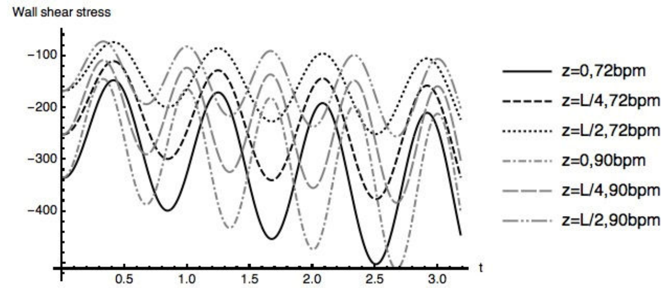

FIGURE 5. Variation of WSS with t at $\alpha = -0.5$ for Linear model

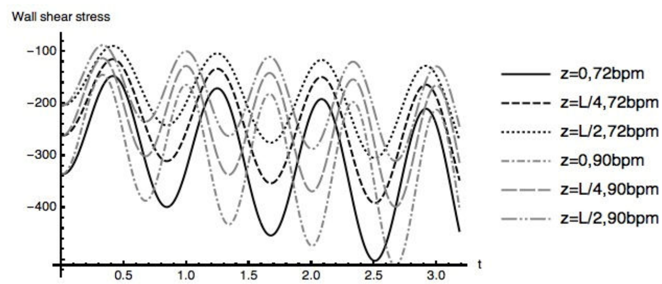

FIGURE 6. Variation of WSS with t at $\alpha = -0.5$ for Exponential model

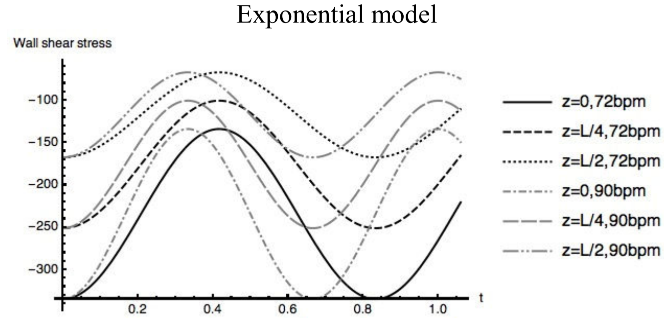

FIGURE 7. Variation of WSS with t at $\alpha = 0$ for Linear model

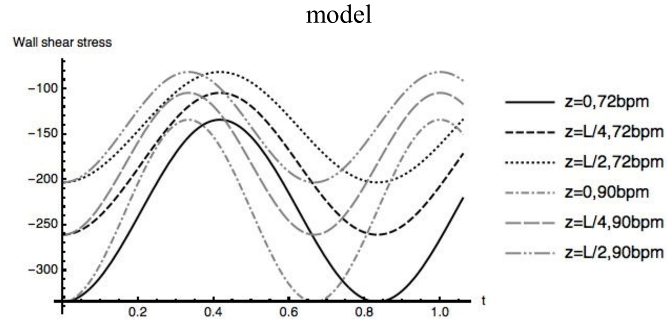

FIGURE 8. Variation of WSS with t at $\alpha = 0$ for Exponential model

We understand from equation (7) that an increase in α results in a decrease in the radius of the pipe and thus a decrease in the volumetric flux. This effect has been shown through figures 9 and 10. Further, an increase in ω increases the volumetric flux. Also, flux reduces as time elapses for positive values of α where as it increases for negative values of α as is seen from figures 11-16.

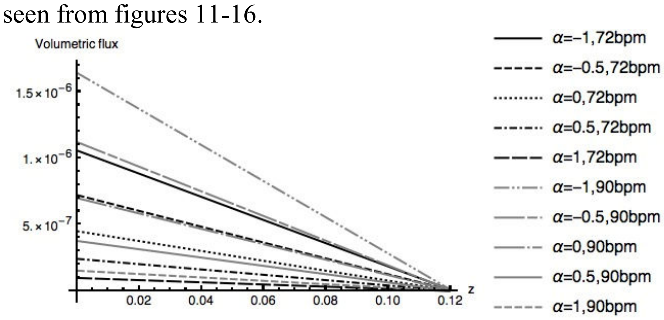

FIGURE 9. Variation of volumetric flux with z for linear model

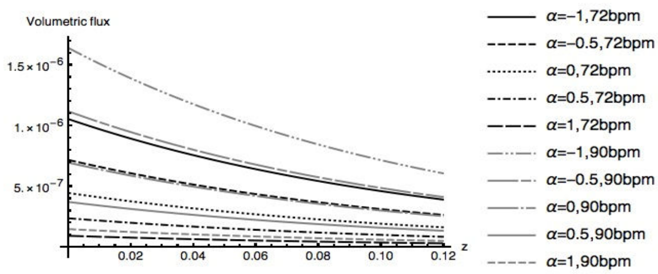

FIGURE 10. Variation of volumetric flux with z for Exponential model

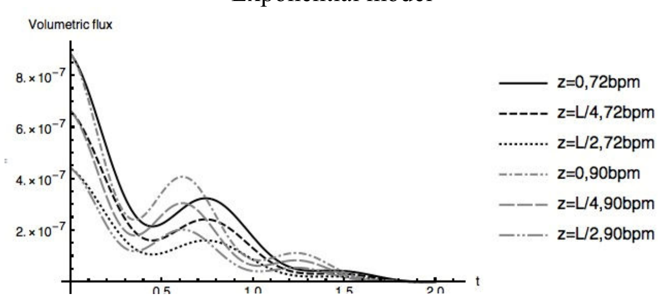

FIGURE 11. Variation of volumetric flux with t at $\alpha = 0.5$ for Linear model

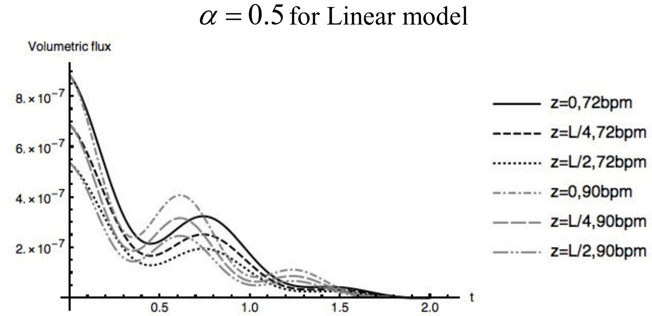

FIGURE 12 Variation of volumetric flux with t at $\alpha = 0.5$ for Exponential model

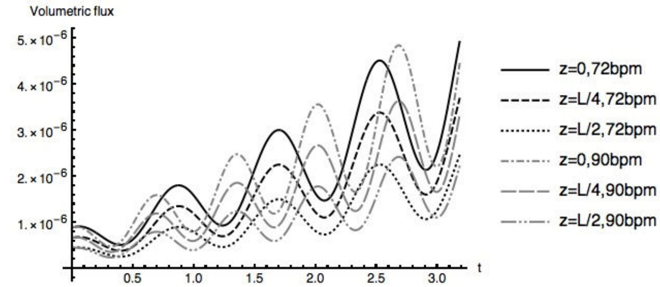

FIGURE 13. Variation of volumetric flux with t at $\alpha = -0.5$ for Linear model

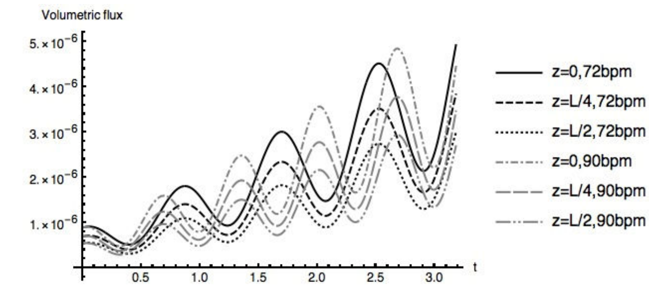

FIGURE 14. Variation of volumetric flux with t at $\alpha = -0.5$ for Exponential model

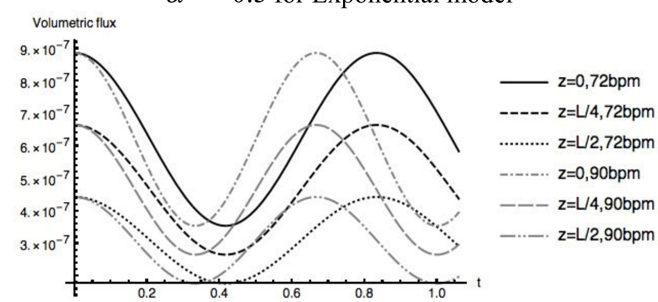

FIGURE 15. Variation of volumetric flux with t at $\alpha = 0$ for Linear model

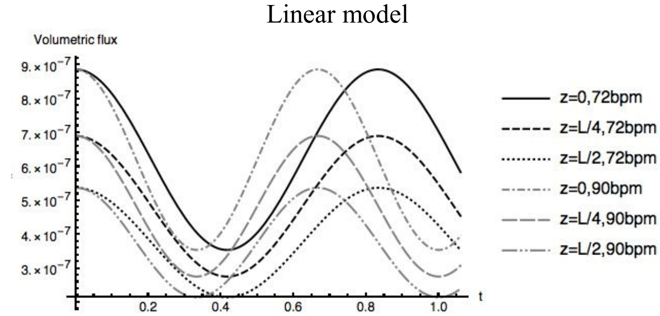

FIGURE 16. Variation of volumetric flux with t at $\alpha = 0$ for Exponential model

V. CONCLUSIONS

In this paper, we considered the flow of Newtonian fluid in an expanding and contracting pipe and studied the effect of the expansion/ contraction coefficient α on wall shear stress and volumetric flux. We considered the flow to be under a negative pressure gradient. This pressure gradient is taken to be a function of both spatial variable and time but in a separable form. Two models i.e. linear and exponential have been considered for the spatial function while a sinusoidal function is taken for the function involving time. The effect of the parameters such as the one related to the contracting

/expanding behavior of the pipe and frequency of oscillations on the flow has been studied. It was seen that, in both linear and exponential models, an increase in the expanding parameter resulted in an increase in wall shear stress and the volumetric flux. Though both the models gave similar results at different instances of time and at different points along the axial length, results predict that the exponential model could be a more appropriate form than the linear model for any natural phenomena. This is because, the latter predicted zero volumetric flux at the other end of the pipe, which is unrealistic in any natural phenomena.

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