Dual solutions in the stagnation-point flow over a shrinking sheet

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***Abstract*—**In the present study, we consider a stagnation point flow over a stretching or shrinking sheet with slip effect at the boundary. The external flow and the stretching/shrinking velocities are assumed to vary linearly from the stagnation point. Different from the previous studies, we consider both stretching and shrinking cases, as well as the slip effect at the boundary. The numerical results show that the solution is unique for the stretching case, while dual solutions are possible for the shrinking case. A stability analysis is performed for the case where dual solutions exist to determine the stability of the solutions. Applying the slip condition increases the range of solutions for the shrinking case.

***Keywords*—**Dual solutions, heat transfer, shrinking, stability.

I. Introduction

The two-dimensional stagnation flow was first considered by Hiemenz (see White [1]). Chiam [2] investigated the two-dimensional stagnation-point flow toward a stretching plate by an assumption that the plate is stretched with a velocity equal to the stagnation flow velocity in the inviscid free stream. Mahapatra and Gupta [3] reconsidered this problem to a more general velocity ratio, and found that a boundary layer is formed near the stretching surface, a contrary observation with that of Chiam.

Compared to a stretching sheet, less work has been done on the flow over a shrinking sheet. Miklavčič and Wang [4] studied the viscous flow induced by a shrinking sheet with suction effect at the boundary. The flow is unlikely to exist unless adequate suction on the boundary is imposed since the vorticity of the shrinking sheet is not confined within a boundary layer. However, with an added stagnation flow to contain the vorticity, similarity solutions may exist [5]. Wang [5] studied both the two-dimensional and axisymmetric stagnation flows toward a shrinking sheet and found that solutions do not exist for larger shrinking rates and may be non-unique in the two-dimensional case. Ishak et al. [6] studied the two-dimensional stagnation flow over a shrinking sheet in a micropolar fluid, and reported that the solution is nonunique.

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In the present study, we consider a stagnation point flow over a stretching or shrinking sheet with slip effect at the boundary. The external flow and the stretching/shrinking velocities are assumed to vary linearly from the stagnation point. Different from the previous studies, we consider both stretching and shrinking cases, as well as the slip effect at the boundary. In certain situations, the assumption of the flow field obeys the conventional no-slip condition at the boundary does no longer apply and should be replaced by partial slip boundary condition. For example, in rarefied gases, there is a slip regime where the Navier–Stokes equation is valid but slip occurs. In this case the no slip condition is replaced by Navier's partial slip condition, where the slip velocity is proportional to the local shear stress.

As reported in [4-6] mentioned above, the solutions for the flow over a shrinking sheet are not unique, multiple solutions are possible for a certain range of parameters. It is the aim of the present study to investigate, by a stability analysis, which solutions are stable and thus physically reliable.

II. Problem formulation

A steady stagnation-point flow over a linearly stretching or shrinking sheet is considered. The stretching/shrinking velocity is assumed in the form $U_w(x) = ax$ where $a > 0$ for stretching and $a < 0$ for shrinking. Further, we assume the external flow velocity in the form $U_\infty(x) = bx$, where $b > 0$. Under these assumptions, the steady governing continuity and momentum boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{2}$$

subject to the boundary conditions:

$$u = U_w(x) + L\frac{\partial u}{\partial y}, \quad v = 0 \quad \text{at} \quad y = 0$$

$$u \to U_\infty(x) \quad \text{as} \quad y \to \infty . \tag{3}$$

where u and v are the velocity components along the x- and y-axis respectively, ν is the kinematic viscosity and L denotes the slip length.

We introduce now the following similarity transformation:

$$\eta = \left(\frac{U_\infty}{\nu x}\right)^{1/2} y, \quad \psi = \left(\nu x U_\infty\right)^{1/2} f(\eta) \tag{4}$$

where η is the independent similarity variable, $f(\eta)$ is the dimensionless stream function and ψ is the stream function defined as $u = \partial\psi / \partial y$ and $v = -\partial\psi / \partial x$, which identically satisfies Eq. (1). Using Eq. (4), we obtain

$$u = bxf'(\eta) \text{ and } v = -(\nu b)^{1/2} f(\eta) \quad (5)$$

where primes denote differentiation with respect to η. The transformed ordinary differential equation is

$$f''' + ff'' + 1 - f'^2 = 0 \quad (6)$$

The boundary conditions (4) now become

$$f(0) = 0, \quad f'(0) = \varepsilon + \lambda f''(0),$$
$$f'(\eta) \to 1 \text{ as } \eta \to \infty \quad (7)$$

where $\varepsilon = a/b$ is the stretching/shrinking parameter, with $\varepsilon > 0$ for stretching and $\varepsilon < 0$ for shrinking and $\lambda = L(b/\nu)^{1/2}$ is the velocity slip parameter.

The physical quantity of interest is the skin friction coefficient C_f which is defined as

$$C_f = \frac{\tau_w}{\rho U_\infty^2 / 2} \quad (8)$$

where the surface shear stress τ_w is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad (9)$$

with μ being the dynamic viscosity. Using the similarity variables (4), we obtain

$$\frac{1}{2} C_f \, Re_x^{1/2} = f''(0) \quad (10)$$

where $\mathrm{Re}_x = U_\infty x / \nu$ is the local Reynolds number.

III. FLOW STABILITY

In order to perform a stability analysis, we consider the unsteady problem. Equation (1) holds, while (2) is replaced by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U_\infty}{\partial t} + U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (11)$$

where t denotes the time. Based on the variables (4), we introduce the following new dimensionless variables:

$$\eta = \left(\frac{U_\infty}{\nu x} \right)^{1/2} y, \quad \psi = (\nu x U_\infty)^{1/2} f(\eta, \tau), \quad \tau = at \quad (12)$$

so that (11) can be written as

$$\frac{\partial^3 f}{\partial \eta^3} + f \frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta} \right)^2 + 1 - \frac{\partial^2 f}{\partial \eta \partial \tau} = 0 \quad (13)$$

and are subjected to the boundary conditions

$$f(0,\tau) = 0, \quad \frac{\partial f}{\partial \eta}(0,\tau) = \varepsilon + \lambda \frac{\partial^2 f}{\partial \eta^2}(0,\tau),$$
$$\frac{\partial f}{\partial \eta}(\eta,\tau) \to 1 \quad \text{as} \quad \eta \to \infty \quad (14)$$

To test the stability of the steady flow solution $f(\eta) = f_0(\eta)$ satisfying the boundary-value problem (1)-(3), we write (see [9-11]),

$$f(\eta,\tau) = f_0(\eta) + e^{-\gamma\tau} F(\eta,\tau) \quad (15)$$

where γ is an unknown eigenvalue, and $F(\eta,\tau)$ is small relative to $f_0(\eta)$. Solutions of the eigenvalue problem (13)-(14) give an infinite set of eigenvalues $\gamma_1 < \gamma_2 < \cdots$; if the smallest eigenvalue is negative, there is an initial growth of disturbances and the flow is unstable; but if γ_1 is positive, there is an initial decay and the flow is stable. Introducing (15) into (13), we get the following linearized problem:

$$\frac{\partial^3 F}{\partial \eta^3} + f_0 \frac{\partial^2 F}{\partial \eta^2} + f_0'' F - \left(2 f_0' - \gamma \right) \frac{\partial F}{\partial \eta} - \frac{\partial^2 F}{\partial \eta \partial \tau} = 0 \quad (16)$$

along with the boundary conditions

$$F(0,\tau) = 0, \quad \frac{\partial F}{\partial \eta}(0,\tau) = \lambda \frac{\partial^2 F}{\partial \eta^2}(0,\tau),$$
$$\frac{\partial F}{\partial \eta}(\eta,\tau) \to 0 \quad \text{as} \quad \eta \to \infty \quad (17)$$

The solution $f(\eta) = f_0(\eta)$ of the steady equation (6) is obtained by setting $\tau = 0$. Hence $F = F_0(\eta)$ in (16) identifies initial growth or decay of the solution (15). In this respect, we have to solve the linear eigenvalue problem

$$F_0''' + f_0 F_0'' + f_0'' F_0 - \left(2 f_0' - \gamma \right) F_0' = 0 \quad (18)$$

along with the boundary conditions

$$F_0(0) = 0, \quad F_0'(0) = \lambda F_0''(0),$$
$$F_0'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \quad (19)$$

It should be stated that for particular values of γ, the stability of the corresponding steady flow solution $f_0(\eta)$ is determined by the smallest eigenvalue γ.

IV. RESULTS AND DISCUSSION

The transformed equation (6) subject to the boundary conditions (7) was solved numerically using the boundary value problem solver, bvp4c, in MATLAB software. In order to validate the numerical results obtained, we compare our results with those reported by Wang. [5] and Ishak et al. [6], which showed a favorable agreement, as presented in Tables 1 and 2.

Table 1 Values of $f''(0)$ for stretching sheet when $\lambda = 0$

ε	Wang [5]	Ishak et al. [6]	Present results
0	1.232588	1.232588	1.232588
0.1	1.14656	1.146561	1.146561

0.5	0.71330	0.713295	0.713295
0.8			0.306095
1	0	0	0
2	–1.88731	–1.887307	-1.887307
5	-10.26475	-10.264749	-10.264749

The variation of the skin friction coefficient $f''(0)$ against the stretching/shrinking parameter ε for different values of the velocity slip parameter λ is presented in Fig. 1. This figure shows the existence of dual solutions for the shrinking case $(\varepsilon < 0)$, while the solution is unique for the stretching case $(\varepsilon > 0)$. Solutions are possible for all $\varepsilon > 0$, but for $\varepsilon < 0$ (shrinking case), the solution exists up to a critical value of ε, i.e. $\varepsilon = \varepsilon_c$, beyond which no solution exist. These values of ε_c for different values of λ are presented in Figure 1. Figure 1 shows that the range of ε for which the solution exists increases as λ increases. Thus, applying the slip condition at the boundary increases the range of solutions for the shrinking case. Moreover, the skin friction coefficient is higher (in absolute sense) for the no slip condition compared to the with slip condition.

Table 2 Values of $f''(0)$ for shrinking sheet when $\lambda = 0$

ε	Wang [5]		Present results	
	First solution	Second solution	First solution	Second solution
–0.25	1.40224		1.402241	
–0.5	1.49567		1.495670	
–1.0	1.32882	0	1.328817	0
–1.1			1.186681	0.049229
–1.15	1.08223	0.116702	1.082231	0.116702
–1.2			0.932473	0.233650
–1.24			0.706605	0.435672
–1.2465			0.584295	0.554283

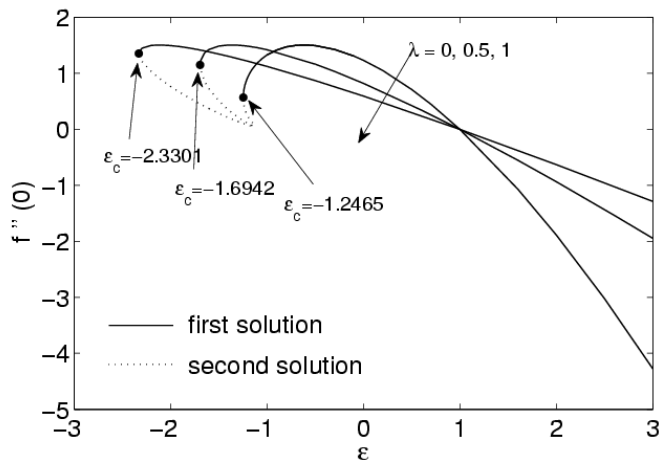

Fig 1 Variation of the skin friction coefficient $f''(0)$ with ε for different values of λ

The validity of dual solutions presented in Fig 1 is supported by the velocity profiles presented in Figure 2. It is seen in this figure that there are two different profiles for the same value of parameter ε, where both satisfy the far field boundary conditions (7) asymptotically.

To test the stability of the solutions, we perform a stability analysis and find the eigenvalues γ in (15). If the smallest eigenvalue is negative, there is an initial growth of disturbances and the flow is unstable; while when the smallest eigenvalue is positive, there is an initial decay and the flow is stable. The smallest eigenvalues γ for selected values of λ are presented in Table 3 which shows that γ is positive for the first solution and negative for the second solution. Thus, the first solution is stable, while the second solution is unstable. The transition from positive (stable) to negative (unstable) values of γ occurs at the turning points of the parametric solution curves ($\varepsilon = \varepsilon_c$), which is shown in Fig 1.

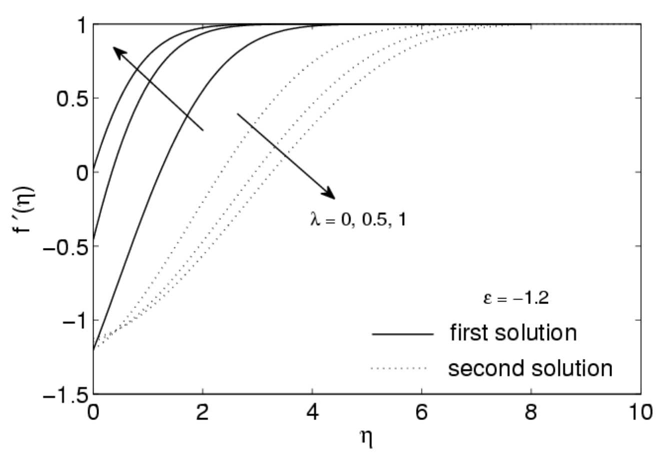

Fig 2 Velocity profiles $f'(\eta)$ for different values of λ when $\varepsilon = -1.2$

Table 3 Smallest eigenvalues γ at several values of ε when $\lambda = 1$

ε	First solution	Second solution
-1.2	2.1188	-1.5638
-1.5	1.8201	-1.4635
-2	1.1449	-1.0218
-2.3	0.3397	-0.3292
-2.3301	0.0103	-0.0103

Although the second solution is unstable and deprive of physical significant, it is still of mathematical interest since the solution is also a solution to the system of differential equation. The second solution may have more realistic meaning in other situations.

V. Conclusion

Numerical results showed that dual solutions are possible for a certain range of the shrinking strength, while for the stretching case, the solution is unique. The first and the second solutions meet at the critical point of the stretching/shrinking parameter, beyond which no solution exists. The linear stability analysis showed that there is an initial decay for the first solution, while there is an initial growth of disturbances for the second solution. Thus, the first solution is linearly stable, while the second solution is not. Applying the slip condition at the boundary increases the range of solutions for the shrinking case. Moreover, the skin friction coefficient is higher (in absolute sense) for the no slip condition compared to the with slip condition.

References

[1] F. M. White, *Viscous Fluid Flow*, Boston: McGraw Hill, 2006.

[2] T.C. Chiam, "Stagnation-point flow towards a stretching plate," *Journal of the Physical Society of Japan*, vol. 63, pp. 2443–2444, 1994.

[3] T.R. Mahapatra, A.S.Gupta, "Heat transfer in stagnation-point flow towards a stretching sheet," *Heat and Mass Transfer*, vol. 38, pp. 517–521, 2002.

[4] M. Miklavčič, and C.Y. Wang, "Viscous flow due to a shrinking sheet," *Quarterly of Applied Mathematics*, vol. 64, pp. 283–290, 2006.

[5] C.Y. Wang, "Stagnation flow towards a shrinking sheet," *International Journal of Non-linear Mechanics*, vol. 43, pp. 377–382, 2008.

[6] A. Ishak, A., Y.Y. Lok, and I. Pop, "Stagnation-point flow over a shrinking sheet in a micropolar fluid," *Chemical Engineering Communications*, vol. 197, pp. 1417–1427, 2010.

[7] N. Bachok, A. Ishak, and I. Pop, "Boundary layer stagnation-point flow and heat transfer over an exponentially stretching/shrinking sheet in a nanofluid," *International Journal of Heat and Mass Transfer*, vol. 55, pp. 8122-8128, 2012.

[8] N. Bachok, A. Ishak, and I. Pop, "Stagnation-point flow over a stretching/shrinking sheet in a nanofluid," *Nanoscale Research Letters*, vol. 6, Article Number 623, 2011.

[9] J.H. Merkin, "On dual solutions occuring in mixed convection in a porous medium," *Journal of Engineering Mathematics*, vol. 20, pp. 171–179, 1985.

[10] P.D. Weidman, D.G. Kubitschek and A.M.J. Davis, "The effects of transpiration on self-similar boundary layer flow over moving surfaces," *International Journal of Engineering Science*, vol. 44, pp. 730–737, 2006.

[11] M.H. Mat Yasin, A. Ishak, and I. Pop, "MHD stagnation-point flow and heat transfer with effects of viscous dissipation, Joule heating and partial velocity slip," *Scientific Reports*, vol. 5, Paper ID 17848 (8 pages), 2015.