An enhanced LMI approach for observer-based control design

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***Abstract*—**Enhanced linear matrix inequality form of design conditions for the observer-based state control of the continuous-time linear systems is presented in the paper. The design conditions are formulated in an enhanced bounded real lemma structure, documenting that the controller and observer parameter separation does not limit the design for systems with unknown disturbance. It is shown that the proposed design conditions provide better results considering the closed-loop system performances.

***Keywords*—**continuous-time linear systems, feedback control systems, linear matrix inequalities, observer-based control, state observers.

I. Introduction

In the control theory, the linear state feedback technique can be exploited under the assumption that all state variables are accessible. If this assumption is not valid, the feedback control law can be generated via an estimate of the system state vector, and control for systems with incomplete state measurements is equivalent to constructing observer-based state controllers.

Exploiting the separation principle, the two independent procedures are used to compute the observer and controller gain matrices for observer-based control of systems with inaccessible states. Nevertheless, for systems with unknown disturbances, there is no generic design approach [1], [8]. The more details, respecting the application conditioned distinctions in design conditions, can be found, e.g., in [10], [12].

The unknown disturbance is another important factor which could be considered in design, and the observers estimating both the system states and the disturbances can be used [7]. In such a case, also the overall stability of the system with the controller, and the enhanced observer producing the state and disturbance estimation, has to be ensured [5].

Motivated by the above facts, a new design methodology is proposed in the paper. By using the slack-matrix approach, an enhanced LMI formulation gives the possibility to design both the control law and the observer gain matrices, being out of substantially expansion of LMI dimensionality.

The work presented in this paper was supported by VEGA, the Grant Agency of Ministry of Education and Academy of Science of Slovak Republic, under Grant No. 1/0608/17. This support is very gratefully acknowledged.

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II. Problem Formulation

Through this paper the task is concerned with design of the observer-based state feedback which controls the linear system given by the set of state equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, $\boldsymbol{y}(t) \in \mathbb{R}^m$ are vectors of the state, input and output variables, respectively, and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}$, $\boldsymbol{E} \in \mathbb{R}^{n\times p}$.

The problem of the interest is to design a stable closed-loop system using the observer-based state feedback control

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}_e(t) \tag{3}$$

while $\boldsymbol{K} \in \mathbb{R}^{r\times n}$ and the Luenberger observer is defined as

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) \tag{4}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t) \tag{5}$$

where $\boldsymbol{q}_e(t) \in \mathbb{R}^n$ is the observer state vector, $\boldsymbol{y}_e(t) \in \mathbb{R}^m$ is the output vector estimate and $\boldsymbol{J} \in \mathbb{R}^{n\times m}$ is the observer gain matrix. It is considered in the following that ($\boldsymbol{A}$,$\boldsymbol{B}$) is controllable and ($\boldsymbol{A}$,$\boldsymbol{C}$) is observable.

Lemma1: *The common state-space description of the observer based state control takes the form*

$$\dot{\boldsymbol{q}}_{eo}(t) = \boldsymbol{A}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{E}_o\boldsymbol{d}(t) \tag{6}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}_o\boldsymbol{q}_{eo}(t) \tag{7}$$

where

$$\boldsymbol{q}_{eo}^T(t) = \begin{bmatrix} \boldsymbol{q}_e^T(t) & \boldsymbol{e}^T(t) \end{bmatrix}, \quad \boldsymbol{C}_o = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C} \end{bmatrix} \tag{8}$$

$$\boldsymbol{A}_o = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{JC} \\ \boldsymbol{0} & \boldsymbol{A}_e \end{bmatrix}, \quad \boldsymbol{E}_o = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{E} \end{bmatrix} \tag{9}$$

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{BK}, \quad \boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{JC} \tag{10}$$

while $\boldsymbol{A}_c, \boldsymbol{A}_e \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}_o \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{C}_o \in \mathbb{R}^{m\times 2n}$, $\boldsymbol{E}_o \in \mathbb{R}^{2n\times p}$, respectively.

Proof: Defining the error of the state estimate as

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t) \tag{11}$$

then, substituting (3) into (4) and exploiting (11), (2), (5) it can obtain from (4) that

$$\dot{\boldsymbol{q}}_e(t) = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{q}_e(t) + \boldsymbol{JC}\boldsymbol{e}(t) \tag{12}$$

and, moreover, from (1), (4) that

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{JC})\boldsymbol{e}(t) + \boldsymbol{E}\boldsymbol{d}(t) \tag{13}$$

Then, using the notation (10), it can write

$$\begin{bmatrix} \dot{\boldsymbol{q}}_e(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{JC} \\ \boldsymbol{0} & \boldsymbol{A}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_e(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{E} \end{bmatrix} \boldsymbol{d}(t) \tag{14}$$

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_e(t) \\ \boldsymbol{e}(t) \end{bmatrix} \tag{15}$$

Thus, with the notations (8), (9) then (14), (15} imply (6), (7). This concludes the proof. ■

It is evident from (9) that if $\boldsymbol{d}(t) = \boldsymbol{0}$, then the separation principle yields and the controller and the observer gains $\boldsymbol{K}$, $\boldsymbol{J}$ can be designed independently [9].

III. BASIC PRELIMINARIES

To motivate the technique used in the paper, some related results are recalled at first.

Proposition 1: [1] *(Lyapunov inequality) The autonomous part of (1) is asymptotically stable if there exists a symmetric positive definite matrix* $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, *or a symmetric positive definite matrix* $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ *such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PA} < 0 \tag{16}$$

$$\boldsymbol{V} = \boldsymbol{V}^T > 0, \quad \boldsymbol{VA}^T + \boldsymbol{AV} < 0 \tag{17}$$

Proposition 2: [2] *(Schur complement) Let* $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{R} = \boldsymbol{R}^T$, $\det \boldsymbol{R} \neq 0$, *and* $\boldsymbol{S}$ *are real matrices of appropriate dimensions, then the following inequalities are equivalent*

$$\begin{gathered} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & -\boldsymbol{R} \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} \boldsymbol{Q} + \boldsymbol{SR}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{R} \end{bmatrix} < 0 \\ \Updownarrow \\ \boldsymbol{Q} + \boldsymbol{SR}^{-1}\boldsymbol{S}^T < 0, \quad \boldsymbol{R} > 0 \end{gathered} \tag{18}$$

Proposition 3: [4] *Given a stable system (1),(2) then*

$$\int_0^\infty (\boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma^2\boldsymbol{u}^T(t)\boldsymbol{u}(t))\,\mathrm{d}\,t > 0 \tag{19}$$

where the positive scalar $\gamma \in \mathbb{R}$ *is the* H_∞ *norm of the transfer function matrix* $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B}$.

IV. CONTROL SYNTHESIS VIA ENHANCED SET OF LMIS

The enhanced technique for interaction accounting is devised in the computation in the frame of the feasible LMIs.

Theorem 1: The closed-loop system and the state observer are stable if for given positive scalar $\delta \in \mathbb{R}$ there exist symmetric positive definite matrices $\boldsymbol{R}_1, \boldsymbol{S}_1, \boldsymbol{P}_2, \boldsymbol{Q}_2 \in \mathbb{R}^{n\times n}$, matrices $\boldsymbol{Y} \in \mathbb{R}^{r\times n}$, $\boldsymbol{Z} \in \mathbb{R}^{n\times m}$ and a positive scalar $\gamma \in \mathbb{R}$ such that

$$\boldsymbol{R}_1 = \boldsymbol{R}_1^T > 0, \quad \boldsymbol{Q}_2 = \boldsymbol{Q}_2^T > 0, \quad \boldsymbol{P}_2 = \boldsymbol{P}_2^T > 0 \tag{20}$$

$$\boldsymbol{S}_1 = \boldsymbol{S}_1^T > 0, \quad \gamma > 0 \tag{21}$$

$$\begin{bmatrix} \boldsymbol{V}(1,1) & * & * & * & * & * \\ \boldsymbol{A}^T & \boldsymbol{U}(2,2) & * & * & * & * \\ \boldsymbol{V}(3,1) & \delta\boldsymbol{A} & -2\delta\boldsymbol{R}_1 & * & * & * \\ -\boldsymbol{I}_n & \boldsymbol{U}(4,2) & -\delta\boldsymbol{I}_n & -2\delta\boldsymbol{Q}_2 & * & * \\ \boldsymbol{E}^T & \boldsymbol{E}^T\boldsymbol{Q}_2 & \delta\boldsymbol{E}^T & \delta\boldsymbol{E}^T\boldsymbol{Q}_2 & -\gamma\boldsymbol{I}_p & * \\ \boldsymbol{CR}_1 & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0 \tag{22}$$

where

$$\boldsymbol{V}(1,1) = \boldsymbol{AR}_1 + \boldsymbol{R}_1\boldsymbol{A}^T - \boldsymbol{BY} - \boldsymbol{Y}^T\boldsymbol{B}^T \tag{23}$$

$$\boldsymbol{V}(3,1) = \boldsymbol{S}_1 - \boldsymbol{R}_1 + \delta\boldsymbol{AR}_1 - \delta\boldsymbol{BY} \tag{24}$$

$$\boldsymbol{U}(2,2) = \boldsymbol{Q}_2\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{Q}_2 - \boldsymbol{ZC} - \boldsymbol{C}^T\boldsymbol{Z}^T \tag{25}$$

$$\boldsymbol{U}(3,1) = \boldsymbol{P}_2 - \boldsymbol{Q}_2 + \delta\boldsymbol{Q}_2\boldsymbol{A} - \delta\boldsymbol{ZC} \tag{26}$$

When the above conditions hold, then

$$\boldsymbol{K} = \boldsymbol{YR}_1^{-1}, \quad \boldsymbol{J} = \boldsymbol{Q}_2^{-1}\boldsymbol{Z} \tag{27}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: Defining the Lyapunov function as follows

$$\begin{aligned} v(\boldsymbol{q}_{eo}(t)) &= \boldsymbol{q}_{eo}^T(t)\boldsymbol{P}_o\boldsymbol{q}_{eo}(t) + \\ &+ \gamma^{-1}\int_0^t (\boldsymbol{y}^T(v)\boldsymbol{y}(v) - \gamma^2\boldsymbol{d}^T(v)\boldsymbol{d}(v))\,\mathrm{d}\,v > 0 \end{aligned} \tag{28}$$

where $\boldsymbol{P}_o \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive definite matrix, then (19) implies there exists such a positive $\gamma \in \mathbb{R}$ that (28) is positive. Thus, evidently, the forward difference of (28) has to satisfy the condition

$$\begin{aligned} \dot{v}(\boldsymbol{q}_{eo}(t)) &= \dot{\boldsymbol{q}}_{eo}^T(t)\boldsymbol{P}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{q}_{eo}^T(t)\boldsymbol{P}_o\dot{\boldsymbol{q}}_{eo}(t) + \\ &+ \gamma^{-1}\boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma\boldsymbol{d}^T(t)\boldsymbol{d}(t) < 0 \end{aligned} \tag{29}$$

Writing (6) as follows

$$\boldsymbol{A}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{E}_o\boldsymbol{d}(t) - \dot{\boldsymbol{q}}_{eo}(t) = \boldsymbol{0} \tag{30}$$

it is evident that with arbitrary matrices $\boldsymbol{Q}_{o1}, \boldsymbol{Q}_{o2} \in \mathbb{R}^{2n\times 2n}$ it yields

$$(\boldsymbol{q}_{eo}^T(t)\boldsymbol{Q}_{o1} + \dot{\boldsymbol{q}}_{eo}^T(t)\boldsymbol{Q}_{o2})(\boldsymbol{A}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{E}_o\boldsymbol{d}(t) - \dot{\boldsymbol{q}}_{eo}(t)) = 0 \tag{31}$$

Therefore, for the sake of completeness, adding (31) and its transposition to (29), it is readily seen that the difference of the Lyapunov function takes the form

$$\begin{aligned} \dot{v}(\boldsymbol{q}_{eo}(t)) &= \dot{\boldsymbol{q}}_{eo}^T(t)\boldsymbol{P}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{q}_{eo}^T(t)\boldsymbol{P}_o\dot{\boldsymbol{q}}_{eo}(t) + \\ &+ (\boldsymbol{q}_{eo}^T(t)\boldsymbol{Q}_{o1} + \dot{\boldsymbol{q}}_{eo}^T(t)\boldsymbol{Q}_{o2})(\boldsymbol{A}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{E}_o\boldsymbol{d}(t) - \dot{\boldsymbol{q}}_{eo}(t)) + \\ &+ (\boldsymbol{A}_o\boldsymbol{q}_{eo}(t) + \boldsymbol{E}_o\boldsymbol{d}(t) - \dot{\boldsymbol{q}}_{eo}(t))^T(\boldsymbol{q}_{eo}^T(t)\boldsymbol{Q}_{o1} + \dot{\boldsymbol{q}}_{eo}^T(t)\boldsymbol{Q}_{o2})^T + \\ &+ \gamma^{-1}\boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma\boldsymbol{d}^T(t)\boldsymbol{d}(t) < 0 \end{aligned} \tag{32}$$

Grouping the variable vectors in the way that

$$\boldsymbol{q}_{eoc}^T(t) = \begin{bmatrix} \boldsymbol{q}_{eo}^T(t) & \dot{\boldsymbol{q}}_{eo}^T(t) & \boldsymbol{d}^T(t) \end{bmatrix} \tag{33}$$

then the inequality (32) can be written equivalently as

$$\boldsymbol{q}_{eoc}^T(t)\boldsymbol{P}_{oc}\boldsymbol{q}_{eoc}(t) < 0 \tag{34}$$

where

$$\boldsymbol{P}_{oc} = \begin{bmatrix} \gamma^{-1}\boldsymbol{C}_o^T\boldsymbol{C}_o + \boldsymbol{Q}_{o1}\boldsymbol{A}_o + \boldsymbol{A}_o^T\boldsymbol{Q}_{o1}^T & * & * \\ \boldsymbol{P}_o - \boldsymbol{Q}_{o1}^T + \boldsymbol{Q}_{o2}\boldsymbol{A}_o & -Q_{o2} - Q_{o2}^T & * \\ \boldsymbol{E}_o^T\boldsymbol{Q}_{o1}^T & \boldsymbol{E}_o^T\boldsymbol{Q}_{o2}^T & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \tag{35}$$

and, using the Schur complement property, then (35) implies

$$\begin{bmatrix} \boldsymbol{Q}_{o1}\boldsymbol{A}_o+\boldsymbol{A}_o^T\boldsymbol{Q}_{o1}^T & * & * & * \\ \boldsymbol{P}_o-\boldsymbol{Q}_{o1}^T+\boldsymbol{Q}_{o2}\boldsymbol{A}_o & -\boldsymbol{Q}_{o2}-\boldsymbol{Q}_{o2}^T & * & * \\ \boldsymbol{E}_o^T\boldsymbol{Q}_{o1}^T & \boldsymbol{E}_o^T\boldsymbol{Q}_{o2}^T & -\gamma\boldsymbol{I}_p & * \\ \boldsymbol{C}_o & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0 \tag{36}$$

As a matter of fact, it can be regarded that

$$\boldsymbol{P}_o=\begin{bmatrix} \boldsymbol{P}_1 & \\ & \boldsymbol{P}_2 \end{bmatrix}, \quad \boldsymbol{Q}_{o1}=\begin{bmatrix} \boldsymbol{Q}_1 & \boldsymbol{Q}_{12} \\ \boldsymbol{0} & \boldsymbol{Q}_2 \end{bmatrix} \tag{37}$$

where $\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{Q}_1,\boldsymbol{Q}_2 \in \mathbb{R}^{n x n}$ are symmetric positive definite matrices and it is evident that it yields

$$\begin{aligned} \boldsymbol{Q}_{o1}\boldsymbol{A}_o &= \begin{bmatrix} \boldsymbol{Q}_1 & \boldsymbol{Q}_{12} \\ \boldsymbol{0} & \boldsymbol{Q}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{A}-\boldsymbol{BK} & \boldsymbol{JC} \\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{JC} \end{bmatrix}= \\ &= \begin{bmatrix} \boldsymbol{Q}_1(\boldsymbol{A}-\boldsymbol{BK}) & \boldsymbol{Q}_1\boldsymbol{JC}+\boldsymbol{Q}_{12}(\boldsymbol{A}-\boldsymbol{JC}) \\ \boldsymbol{0} & \boldsymbol{Q}_2(\boldsymbol{A}-\boldsymbol{JC}) \end{bmatrix} \end{aligned} \tag{38}$$

$$\boldsymbol{Q}_{o1}\boldsymbol{E}_o=\begin{bmatrix} \boldsymbol{Q}_1 & \boldsymbol{Q}_{12} \\ \boldsymbol{0} & \boldsymbol{Q}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{E} \end{bmatrix}=\begin{bmatrix} \boldsymbol{Q}_{12}\boldsymbol{E} \\ \boldsymbol{Q}_2\boldsymbol{E} \end{bmatrix} \tag{39}$$

Thus, setting,

$$\boldsymbol{Q}_{12}=\boldsymbol{Q}_1, \quad \boldsymbol{Q}_{o2}=\delta\boldsymbol{Q}_{o1} \tag{40}$$

where a positive $\delta \in \mathbb{R}$ is the tuning parameter, then it is obtained

$$\boldsymbol{Q}_{o1}\boldsymbol{A}_o=\begin{bmatrix} \boldsymbol{Q}_1\boldsymbol{A}_c & \boldsymbol{Q}_1\boldsymbol{A} \\ \boldsymbol{0} & \boldsymbol{Q}_2\boldsymbol{A}_e \end{bmatrix}, \quad \boldsymbol{Q}_{o1}\boldsymbol{E}_o=\begin{bmatrix} \boldsymbol{Q}_1\boldsymbol{E} \\ \boldsymbol{Q}_2\boldsymbol{E} \end{bmatrix} \tag{41}$$

$$\boldsymbol{Q}_{o1}+\boldsymbol{Q}_1^T=\begin{bmatrix} 2\boldsymbol{Q}_1 & \boldsymbol{Q}_1 \\ \boldsymbol{Q}_1 & 2\boldsymbol{Q}_{o2} \end{bmatrix} \tag{42}$$

and, in consequence, the inequality (36) can be reconfigured as

$$\begin{bmatrix} \boldsymbol{U}(1,1) & * & * & * & * & * \\ \boldsymbol{A}^T\boldsymbol{Q}_1 & \boldsymbol{U}(2,2) & * & * & * & * \\ \boldsymbol{U}(3,1) & \delta\boldsymbol{Q}_1\boldsymbol{A} & -2\delta\boldsymbol{Q}_1 & * & * & * \\ -\boldsymbol{Q}_1 & \boldsymbol{U}(4,2) & -\delta\boldsymbol{Q}_1 & -2\delta\boldsymbol{Q}_2 & * & * \\ \boldsymbol{E}^T\boldsymbol{Q}_1 & \boldsymbol{E}^T\boldsymbol{Q}_2 & \delta\boldsymbol{E}^T\boldsymbol{Q}_1 & \gamma\boldsymbol{E}^T\boldsymbol{Q}_2 & -\gamma\boldsymbol{I}_p & * \\ \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0 \tag{43}$$

where

$$\begin{aligned} \boldsymbol{U}(1,1)&=\boldsymbol{Q}_1\boldsymbol{A}_c+\boldsymbol{A}_c^T\boldsymbol{Q}_1, & \boldsymbol{U}(3,1)&=\boldsymbol{P}_1-\boldsymbol{Q}_1+\delta\boldsymbol{Q}_1\boldsymbol{A}_c \\ \boldsymbol{U}(2,2)&=\boldsymbol{Q}_2\boldsymbol{A}_e+\boldsymbol{A}_e^T\boldsymbol{Q}_2, & \boldsymbol{U}(4,2)&=\boldsymbol{P}_2-\boldsymbol{Q}_2+\delta\boldsymbol{Q}_2\boldsymbol{A}_e \end{aligned} \tag{44}$$

To eliminate bilinear matrix elements in (43), the positive definite transform matrix $\boldsymbol{T} \in \mathbb{R}^{(4n+p+m)x(4n+p+m)}$ can be defined as

$$\boldsymbol{T}=\mathrm{diag}\begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{I}_n & \boldsymbol{R}_1 & \boldsymbol{I}_n & \boldsymbol{I}_n & \boldsymbol{I}_p & \boldsymbol{I}_m \end{bmatrix}, \quad \boldsymbol{R}_1=\boldsymbol{Q}_1^{-1} \tag{45}$$

Then, pre-multiplying the left side and post-multiplying the right-side, the inequality (43) gives

$$\begin{bmatrix} \boldsymbol{V}(1,1) & * & * & * & * & * \\ \boldsymbol{A}^T & \boldsymbol{U}(2,2) & * & * & * & * \\ \boldsymbol{V}(3,1) & \delta\boldsymbol{A} & -2\delta\boldsymbol{R}_1 & * & * & * \\ -\boldsymbol{I}_n & \boldsymbol{U}(4,2) & -\delta\boldsymbol{I}_n & -2\delta\boldsymbol{Q}_2 & * & * \\ \boldsymbol{E}^T & \boldsymbol{E}^T\boldsymbol{Q}_2 & \delta\boldsymbol{E}^T & \delta\boldsymbol{E}^T\boldsymbol{Q}_2 & -\gamma\boldsymbol{I}_p & * \\ \boldsymbol{C}\boldsymbol{R}_1 & \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0 \tag{46}$$

where $\boldsymbol{U}(2,2)$ and $\boldsymbol{U}(4,2)$ are given above,

$$\boldsymbol{V}(1,1)=\boldsymbol{A}_c\boldsymbol{R}_1+\boldsymbol{R}_1\boldsymbol{A}_c^T, \ \boldsymbol{V}(3,1)=\boldsymbol{S}_1-\boldsymbol{R}_1+\delta\boldsymbol{A}_c\boldsymbol{R}_1 \tag{47}$$

and

$$\boldsymbol{S}_1=\boldsymbol{R}_1\boldsymbol{P}_1\boldsymbol{R}_1 \tag{48}$$

Writing as follows

$$\boldsymbol{A}_c\boldsymbol{R}_1=(\boldsymbol{A}-\boldsymbol{GK})\boldsymbol{R}_1=\boldsymbol{A}\boldsymbol{R}_1-\boldsymbol{GY} \tag{49}$$

$$\boldsymbol{Q}_2\boldsymbol{A}_e=\boldsymbol{Q}_2(\boldsymbol{A}-\boldsymbol{JC})=\boldsymbol{Q}_2\boldsymbol{A}-\boldsymbol{ZC} \tag{50}$$

where

$$\boldsymbol{Y}=\boldsymbol{K}\boldsymbol{R}_1, \quad \boldsymbol{Z}=\boldsymbol{Q}_2\boldsymbol{J} \tag{51}$$

then (46) implies (22) and (44), (47) imply (23)-(26). This concludes the proof. ■

Remark 1: It should be emphasized that only the use of the slack matrix principle, as it is applied above, makes it possible to create a set of linear matrix inequalities for the defined synthesis task. A disadvantage in the design of observer-based control of continuous-time linear systems is the occurrence of the positive tuning parameter $\delta \in \mathbb{R}$, whereas for given $\delta=1$ solutions may be ill conditioned.

Corollary 1: If the separation principle is used, considering that $\boldsymbol{d}(t)=\boldsymbol{0}$, after applying (10) into (17) and (16), it yields

$$(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{V}+\boldsymbol{V}(\boldsymbol{A}-\boldsymbol{BK})^T<0 \tag{52}$$

$$\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{JC})+(\boldsymbol{A}-\boldsymbol{JC})^T\boldsymbol{P}<0 \tag{53}$$

Therefore, using the notations

$$\boldsymbol{Y}=\boldsymbol{KV}, \quad \boldsymbol{Z}=\boldsymbol{PJ} \tag{54}$$

the closed-loop system, as well as the state observer, are independently stabilizable if there exist symmetric positive definite matrices $\boldsymbol{P},\boldsymbol{V} \in \mathbb{R}^{n x n}$, and matrices $\boldsymbol{Y} \in \mathbb{R}^{r x n}$, $\boldsymbol{Z} \in \mathbb{R}^{n x m}$ such that

$$\boldsymbol{P}=\boldsymbol{P}^T>0, \quad \boldsymbol{V}=\boldsymbol{V}^T>0 \tag{55}$$

$$\boldsymbol{AV}+\boldsymbol{VA}^T-\boldsymbol{BY}-\boldsymbol{Y}^T\boldsymbol{B}^T<0 \tag{56}$$

$$\boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{ZC}-\boldsymbol{C}^T\boldsymbol{Z}^T<0 \tag{57}$$

When the conditions (55)-(57) hold, then the gain matrices $\boldsymbol{K}$, $\boldsymbol{J}$ can be computed as

$$\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{V}^{-1}, \quad \boldsymbol{J}=\boldsymbol{P}^{-1}\boldsymbol{Z} \tag{58}$$

It is also possible to include also the disturbance in such synthesis conditions, but the solutions is conditioned by H_∞ norm upper-bounds of two different transfer function matrices.

V. Force Mode Control

In practice, the case with $r = m$ (square plants) is often encountered. In this case with each output component can be associated the reference signal, which is expected as a desired representation of this output variable.

Definition 1: The forced regime for the system (1), (2) is given by the control policy

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}_e(t) + \boldsymbol{W}\boldsymbol{w}(t) \tag{59}$$

where $\boldsymbol{w}(t) \in \mathbb{R}^m$ is the signal vector of the desired system output values, and the matrix $\boldsymbol{W} \in \mathbb{R}^{m \times m}$ is the signal gain matrix.

Lemma 2: If the system (1), (2) is controllable and if [13]

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m \tag{60}$$

then the signal gain matrix $\boldsymbol{W}$ in (59), marking by using the static decoupling principle, can be computed as

$$\boldsymbol{W} = (\boldsymbol{C}(-\boldsymbol{A} + \boldsymbol{BKHJC})^{-1}(\boldsymbol{I}_n - \boldsymbol{BKH})\boldsymbol{B})^{-1} \tag{61}$$

where

$$\boldsymbol{H} = (-\boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{JC})^{-1} \tag{62}$$

Proof: In a steady state, which corresponds to the relations $\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{q}}_e(t) = \boldsymbol{0}$, the equality $\boldsymbol{y}_s = \boldsymbol{w}_s$ must hold. Denoting $\boldsymbol{q}_s \in \mathbb{R}^n$, $\boldsymbol{q}_{es} \in \mathbb{R}^n$, $\boldsymbol{y}_s \in \mathbb{R}^m$, $\boldsymbol{w}_s \in \mathbb{R}^m$ as the vectors of steady state values of $\boldsymbol{q}(t)$, $\boldsymbol{q}_e(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{w}(t)$, respectively, then (1), (2), (4) and (59) give

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{q}_s - \boldsymbol{BK}\boldsymbol{q}_{es} + \boldsymbol{BW}\boldsymbol{w}_s \tag{63}$$

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{q}_{es} - \boldsymbol{BK}\boldsymbol{q}_{es} + \boldsymbol{BW}\boldsymbol{w}_s + \boldsymbol{JC}\boldsymbol{q}_s - \boldsymbol{JC}\boldsymbol{q}_{es} \tag{64}$$

$$\boldsymbol{y}_s = \boldsymbol{C}\boldsymbol{q}_s \tag{65}$$

Using (64), it follows that

$$\boldsymbol{q}_{es} = (-\boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{JC})^{-1}(\boldsymbol{JC}\boldsymbol{q}_s + \boldsymbol{BW}\boldsymbol{w}_s) \tag{66}$$

and, substituting (66) into (63), it is easy to check that the following inequality yields

$$\begin{gathered}(-\boldsymbol{A} + \boldsymbol{BK}(-\boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{JC})^{-1}\boldsymbol{JC})\boldsymbol{q}_s = \\ (\boldsymbol{I}_n - \boldsymbol{BK}(-\boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{JC})^{-1})\boldsymbol{BW}\boldsymbol{w}_s\end{gathered} \tag{67}$$

Denoting that

$$\boldsymbol{H} = (-\boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{JC})^{-1} \tag{68}$$

then, evidently,

$$\boldsymbol{q}_s = (-\boldsymbol{A} + \boldsymbol{BKHJC})^{-1}(\boldsymbol{I}_n - \boldsymbol{BKH})\boldsymbol{BW}\boldsymbol{w}_s \tag{69}$$

Thus, with (65), then the steady-state value of the output vector is

$$\boldsymbol{y}_s = \boldsymbol{C}(-\boldsymbol{A} + \boldsymbol{BKHJC})^{-1}(\boldsymbol{I}_n - \boldsymbol{BKH})\boldsymbol{BW}\boldsymbol{w}_s \tag{70}$$

and from the assumption that $\boldsymbol{y}_s = \boldsymbol{w}_s$, then (70) implies (61). This concludes the proof. ■

VI. Illustrative Example

According to the proposed algorithm, the simulations are realized on the Matlab platform for the model, where the input-output dynamics is given by the state-space model parameters [6]

$$\boldsymbol{A} = \begin{bmatrix} 1.380 & -0.208 & 6.715 & -5.676 \\ -0.581 & -4.290 & 0.000 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0.693 \\ 0.397 \\ 0.872 \\ 0.458 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.000 & 0.000 \\ 5.679 & 0.000 \\ 1.136 & -3.146 \\ 1.136 & 0.000 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The state control law (3) and the state observer (4), (5) are constructed by solving (20)-(22) using SeDuMi package [11] for MATLAB. To obtain the minimal value of γ, the tuning parameter is set to $\delta = 0.09$ and conversely, since the task is feasible, then $\gamma = 50.0899$,

$$\boldsymbol{R}_1 = \begin{bmatrix} 11.9492 & 1.6723 & -7.7165 & -0.4038 \\ 1.6723 & 17.5979 & 0.4882 & 1.3114 \\ -7.7165 & 0.4882 & 23.7477 & 13.9638 \\ -0.4038 & 1.3114 & 13.9638 & 21.0697 \end{bmatrix}$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 22.2511 & -1.3243 & 10.1851 & -8.9908 \\ -1.3243 & 23.6678 & 2.8701 & 11.0029 \\ 10.1851 & 2.8701 & 14.1972 & -3.7801 \\ -8.9908 & 11.0029 & -3.7801 & 25.8786 \end{bmatrix}$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 23.0785 & -1.5555 & 7.2635 & -7.9950 \\ -1.5555 & 18.7337 & 1.9500 & 9.2708 \\ 7.2635 & 1.9500 & 10.8813 & -1.4597 \\ -7.9950 & 9.2708 & -1.4597 & 25.4612 \end{bmatrix}$$

$$\boldsymbol{S}_1 = \begin{bmatrix} 15.6153 & 0.7208 & -6.9500 & -1.0218 \\ 0.7208 & 22.1665 & 3.2004 & 1.3601 \\ -6.9500 & 3.2004 & 23.4509 & 11.5744 \\ -1.0218 & 1.3601 & 11.5744 & 23.4005 \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} -3.2186 & -8.2266 & 5.9115 & 14.4412 \\ -40.1318 & -18.5737 & 16.4921 & -1.8830 \end{bmatrix}$$

$$\boldsymbol{Z} = \begin{bmatrix} 16.7835 & -23.1722 \\ 0.4908 & 51.0450 \\ 17.5671 & 30.6450 \\ -9.1727 & 4.7659 \end{bmatrix}$$

Therefore, the feedback gain matrix is built up from (27) as

$$\boldsymbol{K} = \begin{bmatrix} -3.2186 & -8.2266 & 5.9115 & 14.4412 \\ -40.1318 & -18.5737 & 16.4921 & -1.8830 \end{bmatrix}$$

and the estimator gain matrix is computed as

$$\boldsymbol{J} = \begin{bmatrix} 0.2265 & -3.0699 \\ -0.0480 & 2.1385 \\ 1.0577 & 3.5912 \\ -0.1009 & -1.2671 \end{bmatrix}$$

The above gain matrices subsequently implies the signal gain matrix for the forced mode

$$\boldsymbol{W} = \begin{bmatrix} -0.0762 & 0.6107 \\ -0.7178 & 0.5891 \end{bmatrix}$$

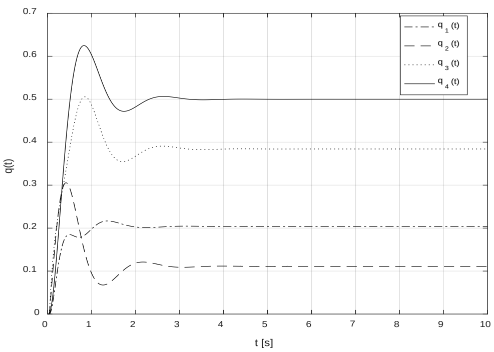

Fig. 1 Responses of the closed-loop system state variables

The matrix gain parameters also guarantee a stable discrete-time observer-based control with state feedback, where the stable eigenvalue spectra are

$$\rho(\boldsymbol{F}_c) = \{-1.6377 \pm 3.4951\mathrm{i}, -4.2157 \pm 6.6655\mathrm{i}\}$$

$$\rho(\boldsymbol{F}_e) = \{-1.7185, -5.4517, -2.5973 \pm 2.8341\mathrm{i}\}$$

Note, an aperiodic system response can be obtained if the tuning parameter is set to the value δ = 1, but the resulting value of γ is more than five times higher then the minimal value γ = 50.0899.

Simultaneous design of the state feedback and observer parameters, is rationally decidable using (55)-(57). One can verify that from the solution of (55)-(57) result the parameters

$$\boldsymbol{K} = \begin{bmatrix} -0.1714 & -0.8157 & 0.1959 & 1.0849 \\ -2.0552 & -0.0433 & 1.0208 & -0.9606 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} -0.6028 & -2.0836 \\ 0.1369 & 1.7165 \\ 2.4184 & 3.2583 \\ -0.9364 & -0.8639 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} -0.0291 & 1.1449 \\ -0.3743 & 0.5951 \end{bmatrix}$$

and the accompanying sets of eigenvalues of the system matrices are

$$\rho(\boldsymbol{F}_c) = \{-0.8983 \pm 5.9975\mathrm{i}, -1.7414 \pm 4.8274\mathrm{i}\}$$

$$\rho(\boldsymbol{F}_e) = \{-2.3868, -3.6306, -2.3969 \pm 5.6812\mathrm{i}\}$$

It is clear from these values that the closed-loop system has significantly slower dynamics and significantly less the relative damping od the closed-loop system responses. Moreover, H$_{\infty}$ norm of the closed-loop disturbance transfer function is higher then H_∞ norm of the closed-loop disturbance transfer function for system under control law designed using the optimized parameter δ.

The simulation results, obtained within the initial conditions $\boldsymbol{q}(0) = \boldsymbol{q}_e(0) = \boldsymbol{0}$ and the required system output steady-state vector $\boldsymbol{w}^T(\mathrm{t}) = [1.2 \;\; 0.5]^T$ are depicted in Fig. 1 and Fig. 2. The responses present the evolution of the controlled system states and the system outputs. These results clearly illustrate the control performances prescribed by the new synthesis strategy.

For illustration and comparison, Fig. 3 shows the response of closed-loop system output variables for the same observer-based control structure, but bulit on simultaneous synthesis of the control and estimator parameters.

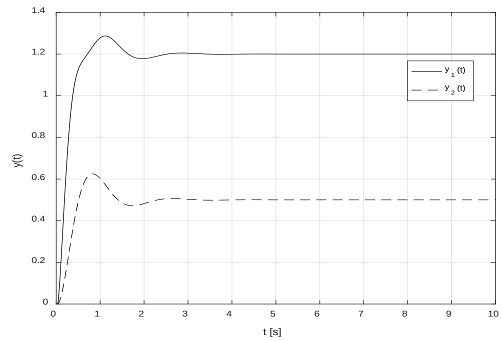

Fig. 2 Responses of the closed-loop system output variables

VII. CONCLUSION

In the paper there is presented a new design method, improving the closed-loop control performances for continuous-time linear systems under observer-based state controllers. Using an enhanced slack-matrix based approach, the widen theorem formulation is proven, adjusting with existing variable interaction also the H_∞ norm upper-bound of the closed-loop disturbance transfer function matrix. Further, it can be noted that the H_∞ norm of the closed-loop disturbance transfer function matrix is reached, although such norms are conservatively handled when transforming them into LMIs then H_∞norm upper-bounds. Moreover, applying enhanced slack-matrix based approach, the stability conditions which are inherently bilinear are simply transformed into LMIs.

From what has been presented and suggested here, it becomes evident a full characterization for the existence of observer-based state controllers. The conditions improve the state estimation consistency and guarantee the asymptotic properties of both the observer and the closed-loop structure. The methodology is completely model based, requires no iterative procedures and is smoothly convenient in use. Simulations provide evidence of effectiveness of the described algorithm and demonstrate the closed-loop system performance and disturbance robustness, pointing out the closed-loop forced mode strategy for systems with the reference output positions.

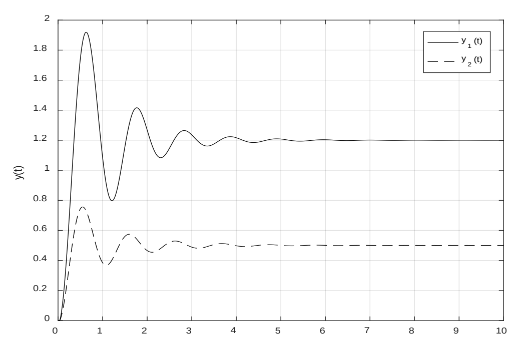

Fig. 3 Closed-loop system output response for simultaneous designed control parameters

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