Minimax extrapolation of multidimensional stationary processes with missing observations

Mikhail Moklyachuk, Oleksandr Masyutka and Maria Sidei

***Abstract*—**The problem of the mean-square optimal linear estimation of linear functionals which depend on the unknown values of a stochastic stationary stochastic process $\vec{\xi}(t)$, $t \in \mathbb{R}^+$, from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t \in \mathbb{R}^- \setminus S$ is considered. Formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are proposed under the condition of spectral certainty, where spectral densities of the processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ are exactly known. The minimax (robust) method of estimation is applied in the case of spectral uncertainty, where spectral densities are not known exactly, while sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics are proposed for some special sets of admissible densities.

***Keywords*—** mean square optimal esrtimate, minimax-robust estimate, minimax spectral characteristic, least favorable spectral density, stationary stochastic process

I. INTRODUCTION

The problem of estimation of the unknown values of stochastic processes is of constant interest in the theory of stochastic processes. Formulation of the interpolation, extrapolation and filtering problems for stationary stochastic sequences with known spectral densities and reducing of these estimation problems to the corresponding problems of the theory of functions belongs to A. N. Kolmogorov [17]. Effective methods of solution of the estimation problems for stationary stochastic sequences and processes were developed by N.Wiener [44] and A. M. Yaglom [45]-[46]. Further results are presented in the books by Yu. A. Rozanov [41]and E.J.Hannan [12]. The crucial assumption of most of the methods developed for estimating the unobserved values of stochastic processes is that the spectral densities of the involved stochastic processes are exactly known. However, in practice complete information on the spectral densities is impossible in most cases. In this situation one finds parametric or nonparametric estimate of the unknown spectral density and then apply one of the traditional estimation methods provided that the selected density is the true one. This procedure can result in significant increasing of the value of error as K.S.Vastola and H. V. Poor [43] have demonstrated with the help of some examples. To avoid this effect one can search the estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error of estimate. The paper by Ulf Grenander [11] was the first one where this approach to extrapolation problem for stationary processes was proposed. Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H.V.Poor [16]. J. Franke [8], J. Franke and H. V. Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. In the papers by M.P.Moklyachuk [25]-[28] the extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences are investigated. The estimation problems for functionals which depend on the unknown values of multivariate stationary stochastic processes is the aim of the book by M. Moklyachuk and O. Masytka [30]. I. I. Dubovets'ka, O. Yu. Masyutka and M. P. Moklyachuk [3], I. I. Dubovets'ka and M.P. Moklyachuk [4]-[7], M. P. Moklyachuk and I. I. Golichenko [29] investigated the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences. In the paper by M. M. Luz and M. P. Moklyachuk [19]-[23] results of investigation of the estimation problems for functionals which depend on the unknown values of stochastic sequences with stationary increments are described.

Results of investigations of the prediction problem for stationary stochastic sequences with missing observations are presented in the papers by P. Bondon [1], [2], Y. Kasahara, M. Pourahmadi and A. Inoue [15]. [38]. In papers by M.P.Moklyachuk and M. I. Sidei [31]-[37] results of investigations of the interpolation, extrapolation and filtering problems for stationary stochastic sequences and processes with missing observations are proposed.

Mikhail Moklyachuk is with the Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine, (phone: +380442590392; fax: +380442590392; e-mail: Moklyachuk@gmail.com).

Oleksandr Masyutka is with the Department of Mathematics and Theoretical Radiophysics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine (e-mail: masyutaAU@bigmir.net).

Maria Sidei is with the Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine (e-mail: marysidei4@gmail.com).

In this paper we deal with the problem of the mean-square optimal linear estimation of the functional $A\vec{\xi}=\int_{0}^{\infty}\vec{a}(t)^{\top}\vec{\xi}(t)dt,$ which depends on the unknown values of a stochastic multidimensional stationary stochastic process $\vec{\xi}(t)=\{\xi_k(t)\}_{k=1}^{T}$, $t\in\mathbb{R}^{+}$, from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^{-}\setminus S$, $S=\cup_{l=1}^{s}[-M_l-N_l,-M_l]$. The case of spectral certainty, as well as the case of spectral uncertainty, are considered.
Formulas for calculating the spectral characteristic and the mean-square error of the optimal linear estimate of the functional are derived in the case of spectral certainty, where the spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functional are proposed for some specific classes of admissible spectral densities.

I. Hilbert Space Projection Method of Extrapolation of Stationary Processes

Consider two uncorrelated multidimensional stationary stochastic processes $\vec{\xi}(t)=\left\{\xi_k(t)\right\}_{k=1}^{T},t\in\mathbb{R},$ and $\vec{\eta}(t)=\left\{\eta_k(t)\right\}_{k=1}^{T},t\in\mathbb{R},$ with zero first moments $E\vec{\xi}(t)=\vec{0}$, $E\vec{\eta}(t)=\vec{0}$, and correlation functions

$$R_{\xi}(n)=E\vec{\xi}(j+n)(\overrightarrow{\xi(j)})^{*}$$

and

$$R_{\eta}(n)=E\vec{\eta}(j+n)(\overrightarrow{\eta(j)})^{*}$$

respectively.
The correlation functions of the processes admit the spectral decomposition (see, for example, [10])

$$R_{\xi}(n)=\int_{-\infty}^{\infty}e^{in\lambda}W_{\xi}(d\lambda),\quad R_{\eta}(n)=\int_{-\infty}^{\infty}e^{in\lambda}W_{\eta}(d\lambda),$$

where $W_{\xi}(d\lambda)$ and $W_{\eta}(d\lambda)$ are spectral measures of the processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ respectively.

Suppose that the spectral measures $W_{\xi}(d\lambda)$ and $W_{\eta}(d\lambda)$ are absolutely continuous with respect to the Lebesgue measure. In this case the correlation functions can be represented in the form

$$R_{\xi}(n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{in\lambda}F(\lambda)d\lambda,$$

$$R_{\eta}(n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{in\lambda}G(\lambda)d\lambda,$$

where $F(\lambda)=\left\{f_{kl}(\lambda)\right\}_{k,l=1}^{T}$ and $G(\lambda)=\left\{g_{kl}(\lambda)\right\}_{k,l=1}^{T}$ are spectral densities of the processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$.

Suppose that the minimality condition holds true

$$\int_{-\infty}^{\infty}(b(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}\overline{b(\lambda)}d\lambda<\infty, \qquad (1)$$

where $b(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\alpha}(t)e^{it\lambda}dt$ is a nontrivial function of exponential type.
Under this condition the error-free extrapolation is impossible [41].

The stationary stochastic processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ admit the spectral decomposition (see, for example, [10], [14])

$$\vec{\xi}(t)=\int_{-\infty}^{\infty}e^{it\lambda}Z_{\xi}(d\lambda),\vec{\eta}(t)=\int_{-\infty}^{\infty}e^{it\lambda}Z_{\eta}(d\lambda), \qquad (2)$$

where $Z_{\xi}(d\lambda)$ and $Z_{\eta}(d\lambda)$ are orthogonal stochastic measures defined on $[-\infty,\infty)$, that correspond to the spectral measures $W_{\xi}(d\lambda)$ and $W_{\eta}(d\lambda)$, such that the following relations hold true

$$EZ_{\xi}(\Delta_1)(Z_{\xi}(\Delta_2))^{*}=W_{\xi}(\Delta_1\cap\Delta_2)=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2}F(\lambda)d\lambda,$$

$$EZ_{\eta}(\Delta_1)(Z_{\eta}(\Delta_2))^{*}=W_{\eta}(\Delta_1\cap\Delta_2)=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2}G(\lambda)d\lambda.$$

Consider the problem of the mean-square optimal linear extrapolation of the functional

$$A\vec{\xi}=\int_{0}^{\infty}\vec{a}(t)^{\top}\vec{\xi}(t)dt,$$

which depends on the unknown values of the process $\vec{\xi}(t)$, based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^{-}\setminus S$, where $S=\cup_{l=1}^{s}[-M_l-N_l,-M_l]$.

It follows from the spectral decomposition (2) of the process $\vec{\xi}(t)$ that the functional $A\vec{\xi}$ can be represented in the form

$$A\vec{\xi}=\int_{-\infty}^{\infty}(A(\lambda))^{\top}Z_{\xi}(d\lambda),$$

$$A(\lambda)=\int_{0}^{\infty}\vec{a}(t)e^{it\lambda}dt.$$

Denote by $\hat{A}\vec{\xi}$ the optimal linear estimate of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$. Let

$$\Delta(F,G)=E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^{2}$$

be the mean-square error of the estimate $\hat{A}\vec{\xi}$. Since the spectral densities of stationary processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ are known, we can use the method of orthogonal projections in the Hilbert spaces (see, for example, [17]) to find the estimate.

Consider the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ generated by random variables ξ with zero mathematical expectations, $E\xi = 0$, finite variations, $E|\xi|^2 < \infty$, and inner product $(\xi, \eta) = E\xi\overline{\eta}$. Denote by $H^s(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi_k(t)+\eta_k(t) : t \in \mathbb{R}\setminus S, k=\overline{1,T}\}$ in the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$. Let $L_2(F+G)$ be the Hilbert space of complex-valued functions $\vec{a}(\lambda) = \left\{a_k(\lambda)\right\}_{k=1}^{T}$ such that

$$\int_{-\infty}^{\infty} \vec{a}(\lambda)^{\top}(F(\lambda)+G(\lambda))\overline{\vec{a}(\lambda)}d\lambda =$$

$$= \int_{-\infty}^{\infty} \sum_{k,l=1}^{T} a_k(\lambda)\overline{a_l(\lambda)}(f_{kl}(\lambda)+g_{kl}(\lambda))d\lambda < \infty$$

Denote by $L_2^s(F+G)$ the subspace of $L_2(F+G)$ generated by functions

$$\{e^{it\lambda}\delta_k, \delta_k = \left\{\delta_{kl}\right\}_{l=1}^{T}, k=\overline{1,T}, t \in \mathbb{R}\setminus S\}.$$

The mean-square optimal linear estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ is the form

$$\hat{A}\vec{\xi} = \int_{-\infty}^{\infty} (h(\lambda))^{\top}(Z_{\xi}(d\lambda)+Z_{\eta}(d\lambda)),$$

where $h(\lambda) = \{h_k(\lambda)\}_{k=1}^{T} \in L_2^s(F+G)$ is the spectral characteristic of the estimate.

The mean-square error $\Delta(h; F, G)$ of the estimate $\hat{A}\vec{\xi}$ can be calculated by the formula

$$\Delta(h; F, G) = E\left|A\vec{\xi} - \hat{A}\vec{\xi}\right|^2 =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (A(\lambda)-h(\lambda))^{\top} F(\lambda)\overline{(A(\lambda)-h(\lambda))}d\lambda +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} (h(\lambda))^{\top} G(\lambda)\overline{h(\lambda)}d\lambda.$$

According to the Hilbert space projection method proposed by A. N. Kolmogorov [17], the optimal linear estimate of the functional $A\vec{\xi}$ is a projection of the element $A\vec{\xi}$ of the space H on the subspace $H^s(\xi+\eta)$, which can be found from the following conditions:

1). $\hat{A}\vec{\xi} \in H^s(\xi+\eta),$

2). $A\vec{\xi} - \hat{A}\vec{\xi} \perp H^s(\xi+\eta).$

It follows from the second condition that the spectral characteristic $h(\lambda)$ of the optimal linear estimate $\hat{A}\vec{\xi}$ for any $t \in \mathbb{R}^- \setminus S$ satisfies the equations

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(A(\lambda)-h(\lambda)\right)^{\top} F(\lambda)e^{-it\lambda}d\lambda -$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} (h(\lambda))^{\top} G(\lambda)e^{-it\lambda}d\lambda = \vec{0}.$$

The last relation can be represented in the form

$$\int_{-\infty}^{\infty} \left[(A(\lambda))^{\top} F(\lambda) - (h(\lambda))^{\top}(F(\lambda)+G(\lambda))\right] e^{-it\lambda}d\lambda$$

$$= 0, t \in \mathbb{R}^- \setminus S. \tag{3}$$

Consider the function

$$(C(\lambda))^{\top} = (A(\lambda))^{\top} F(\lambda) - (h(\lambda))^{\top}(F(\lambda)+G(\lambda))$$

and its Fourier transformation

$$\vec{\mathbf{c}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda)e^{-it\lambda}d\lambda, \quad t \in \mathbb{R}.$$

According to condition (3) the function $\vec{\mathbf{c}}(t)$ can be nonzero only on the set $U = S \cup [0, \infty)$. Therefore the function $C(\lambda)$ is of the form

$$C(\lambda) = \sum_{l=1}^{s} \int_{-M_l-N_l}^{-M_l} \vec{\mathbf{c}}(t)e^{it\lambda}dt + \int_{0}^{\infty} \vec{\mathbf{c}}(t)e^{it\lambda}dt,$$

and the spectral characteristic of the estimate $\hat{A}\vec{\xi}$ is of the form

$$(h(\lambda))^{\top} = (A(\lambda))^{\top} F(\lambda)(F(\lambda)+G(\lambda))^{-1} -$$

$$-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}. \tag{4}$$

It follows from the first condition, $\hat{A}\vec{\xi} \in H^s(\xi+\eta)$, which determine the optimal linear estimate of the functional $A\vec{\xi}$, that for any $t \in U$ the following relation holds true

$$\int_{-\infty}^{\infty} \left((A(\lambda))^{\top} F(\lambda)(H(\lambda))^{-1} - (C(\lambda))^{\top}(H(\lambda))^{-1}\right)$$

$$e^{-it\lambda}d\lambda = 0, \tag{5}$$

where $H(\lambda) = F(\lambda)+G(\lambda)$

In order to represent the last equations in terms of linear operators in the space $L_2(U)$ we introduce the operators

$$(\mathbf{Bx})(t) =$$

$$= \frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} (\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(H(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du +$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}(H(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{Rx})(t) =$$

$$= \frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} (\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty} F(\lambda)(H(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du +$$

$$+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(H(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{Qx})(t)=$$

$$\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(H(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du$$

$$+\frac{1}{2\pi}\int_{0}^{\infty}(\vec{\mathbf{x}}(u))^{\top}\int_{-\infty}^{\infty}F(\lambda)(H(\lambda))^{-1}G(\lambda)e^{i\lambda(u-t)}d\lambda du,$$

$$\vec{\mathbf{x}}(t)\in L_2(U),\quad t\in U.$$

Consider the function $\vec{\mathbf{a}}(t)$, $t\in U$, such that

$$\vec{\mathbf{a}}(t)=\vec{0},t\in S,\vec{\mathbf{a}}(t)=\vec{a}(t),t\geq 0.$$

Making use of the introduced operators and functions relation (5) can be represented in the form

$$(\mathbf{Ra})(t)=(\mathbf{Bc})(t),\quad t\in U. \tag{6}$$

Suppose that the operator $\mathbf{B}$ is invertible (see \cite{Salehi} for more details). Then the function $\vec{\mathbf{c}}(t)$ can be calculated by the formula

$$\vec{\mathbf{c}}(t)=(\mathbf{B}^{-1}\mathbf{Ra})(t),\quad t\in U.$$

Therefore, the spectral characteristic $h(\lambda)$ of the estimate $\hat{A}\vec{\xi}$ can be calculated by the formula

$$(h(\lambda))^{\top}=(A(\lambda))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-$$
$$-(C(\lambda))^{\top}(F(\lambda)+G(\lambda))^{-1}, \tag{7}$$

$$C(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt+\int_{0}^{\infty}(\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt.$$

The mean-square error of the estimate $\hat{A}\vec{\xi}$ can be calculated by the formula

$$\Delta(h;F,G)=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^{\top}G(\lambda)+(C(\lambda))^{\top})(F(\lambda)+G(\lambda))^{-1}F(\lambda)\times$$
$$\times(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^{\top}G(\lambda)+(C(\lambda))^{\top})^{*}d\lambda+$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^{\top}F(\lambda)-(C(\lambda))^{\top})(F(\lambda)+G(\lambda))^{-1}G(\lambda)\times$$
$$\times(F(\lambda)+G(\lambda))^{-1}((A(\lambda))^{\top}G(\lambda)+(C(\lambda))^{\top})^{*}d\lambda=$$
$$=\langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle+\langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle, \tag{8}$$

where $\left\langle\vec{a}(t),\vec{b}(t)\right\rangle$ is the inner product in the space $L_2(U)$.

The following theorem holds true.

Theorem 2.1.

Let $\vec{\xi}(t)$ and $\vec{\eta}(t)$ be uncorrelated multidimensional stationary stochastic processes with the spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(\lambda)$ and the mean-square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$, $t\in\mathbb{R}^{-}\setminus S$ can be calculated by formulas (7), (8).

Consider the problem of the mean-square optimal linear extrapolation of the functional

$$A_N\vec{\xi}=\int_{0}^{N}\vec{a}(t)^{\top}\vec{\xi}(t)dt,$$

which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ at points $t\in\mathbb{R}^{-}\setminus S$.

The linear estimate $\hat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$ is of the form

$$\hat{A}_N\vec{\xi}=\int_{-\infty}^{\infty}(h_N(\lambda))^{\top}(Z_{\xi}(d\lambda)+Z_{\eta}(d\lambda)),$$

where $h_N(\lambda)=\{h_{kN}(\lambda)\}_{k=1}^{T}\in L_2^s(F+G)$ is the spectral characteristic.

Introduce the function $\vec{\mathbf{a}}_N(t)$ such that

$$\vec{\mathbf{a}}_N(t)=\vec{a}(t),t\in[0,N],\quad \vec{\mathbf{a}}_N(t)=\vec{0},t\in S\cup R_{+}\setminus[0,N].$$

Then the spectral characteristic $h_N(\lambda)$ of the estimate $\hat{A}_N\vec{\xi}$ can be calculated by the formula

$$(h_N(\lambda))^{\top}=(A_N(\lambda))^{\top}F(\lambda)(H(\lambda))^{-1}-$$
$$-(C_N(\lambda))^{\top}(H(\lambda))^{-1}, \tag{9}$$

$$C_N(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{Ra}_N)(t)e^{it\lambda}dt+\int_{0}^{\infty}(\mathbf{B}^{-1}\mathbf{Ra}_N)(t)e^{it\lambda}dt,$$

where $A_N(\lambda)=\int_{0}^{N}\vec{a}(t)e^{-it\lambda}dt.$

The mean-square error $\Delta(h_N;F,G)$ of the estimate $\hat{A}_N\vec{\xi}$ can be calculated by the formula

$$\Delta(h_N;F,G)=$$
$$=\langle(\mathbf{Ra}_N)(t),(\mathbf{B}^{-1}\mathbf{Ra}_N)(t)\rangle+\langle(\mathbf{Qa}_N)(t),\vec{\mathbf{a}}_N(t)\rangle. \tag{10}$$

We obtain the following corollary.

Corrolary 2.1.

Let $\vec{\xi}(t)$ and $\vec{\eta}(t)$ be uncorrelated multidimensional stationary processes with the spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h_N(\lambda)$ and the mean-square error $\Delta(h_N;F,G)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$, $t\in\mathbb{R}^-\setminus S$ can be calculated by formulas (9), (10).

Consider the case where the stationary process $\vec{\xi}(t)$ is observed without noise. In this case the spectral characteristic of the estimate $\hat{A}\vec{\xi}$ is of the form

$$(h(\lambda))^\top=(A(\lambda))^\top-(C(\lambda))^\top(F(\lambda))^{-1}, \qquad (11)$$

$$C(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\mathbf{c}}(t)e^{it\lambda}dt+\int_0^\infty\vec{\mathbf{c}}(t)e^{it\lambda}dt.$$

Relation (6) in this case can be written as follows

$$\vec{\mathbf{a}}(t)=(\mathbf{Bc})(t),\quad t\in U. \qquad (12)$$

If the operator $\mathbf{B}$ is invertible, then the unknown function $\vec{\mathbf{c}}(t)$ can be found by the formula

$$\vec{\mathbf{c}}(t)=(\mathbf{B}^{-1}\mathbf{a})(t),\quad t\in U.$$

Hence, the spectral characteristic of the estimate \hat{A}\vec{\xi} can be represented by formula

$$(h(\lambda))^\top=(A(\lambda))^\top-(C(\lambda))^\top(F(\lambda))^{-1}, \qquad (13)$$

$$C(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{a})(t)e^{it\lambda}dt+\int_0^\infty(\mathbf{B}^{-1}\mathbf{a})(t)e^{it\lambda}dt.$$

The mean-square error of the estimate of the functional can be calculated by formula

$$\Delta(h;F)=\langle(\mathbf{B}^{-1}\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle. \qquad (14)$$

The following theorem holds true.

Theorem 2.2.

Let $\vec{\xi}(t)$ be a multidimensional stationary stochastic process with the spectral density $F(\lambda)$, which satisfies the minimality condition

$$\int_{-\pi}^{\pi}(b(\lambda))^\top(F(\lambda))^{-1}\overline{b(\lambda)}d\lambda<\infty \qquad (15)$$

for some nonzero vector-valued function of the exponential type

$$b(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}\vec{\alpha}(t)e^{it\lambda}dt.$$

The spectral characteristic $h(\lambda)$ and the mean-square error $\Delta(h,F)$ of the optimal linear estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)$ at time points $t\in\mathbb{R}^-\setminus S$, where $S=\cup_{l=1}^{s}[-M_l-N_l,-M_l]$, can be calculated by formulas (13),(14).

II. Minimax method of extrapolation

Formulas obtained in the section above can be applied to find a solution of the estimation problem only in the case where the spectral densities of the processes are exactly known.
In the case where the full information on spectral densities is impossible, while it is known that spectral densities belong to some specified classes of admissible densities, the minimax approach to the problem of estimation is reasonable. This method gives us a possibility to find an estimate that minimize the maximum value of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities.

Definition 3.1.

For a given class of spectral densities $D=D_F\times D_G$ the spectral densities $F^0(\lambda)\in D_F$, $G^0(\lambda)\in D_G$ are called least favorable in the class D for the optimal linear extrapolation of the functional $A\vec{\xi}$ if the following relation holds true

$$\Delta\left(F^0,G^0\right)=\Delta\left(h\left(F^0,G^0\right);F^0,G^0\right)=$$
$$=\max_{(F,G)\in D_F\times D_G}\Delta\left(h\left(F,G\right);F,G\right).$$

Definition 3.2.

For a given class of spectral densities $D=D_F\times D_G$ the spectral characteristic $h^0(\lambda)$ of the optimal linear extrapolation of the functional $A\vec{\xi}$ is called minimax-robust if there are satisfied conditions

$$h^0(\lambda)\in H_D=\bigcap_{(F,G)\in D_F\times D_G}L_2^s(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta\left(h;F,G\right)=\max_{(F,G)\in D}\Delta\left(h^0;F,G\right).$$

Making use of the definitions above and the results from the previous section, we can formulate the following lemmas.

Lemma 3.1.

Spectral densities $F^0(\lambda)\in D_F$, $G^0(\lambda)\in D_G$ satisfying the minimality condition (1) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ if the Fourier coefficients
of the functions

$$(F^0(\lambda)+G^0(\lambda))^{-1},\quad F_0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1},$$
$$F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda),$$

determine the operators $\mathbf{B}^0, \mathbf{R}^0, \mathbf{Q}^0$, which give a solution to the constrain optimization problem

$$\max_{(F,G)\in D_F\times D_G}(\langle(\mathbf{Ra})(t),(\mathbf{B}^{-1}\mathbf{Ra})(t)\rangle+\langle(\mathbf{Qa})(t),\vec{\mathbf{a}}(t)\rangle)=$$
$$=\langle(\mathbf{R}^0\mathbf{a})(t),((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)\rangle+\langle(\mathbf{Q}^0\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle. \quad (16)$$

The minimax spectral characteristic $h^0=h(F^0,G^0)$ is determined by formula (7) if $h(F^0,G^0)\in H_D$.

Corrolary 3.1.

Suppose the spectral density $F^0(\lambda)\in D_F$ satisfy the minimality condition (\ref{minimal1}). The spectral density $F^0(\lambda)\in D_F$ is the least favorable in the class D_F for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ at points $t\in\mathbb{R}^-\setminus S$, if the Fourier coefficients of the function $(F^0(\lambda))^{-1}$ determine the operator $\mathbf{B}^0$, which gives a solution to the constrain optimization problem

$$\max_{F\in D_F}\langle(\mathbf{B}^{-1}\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle=\langle((\mathbf{B}^0)^{-1}\mathbf{a})(t),\vec{\mathbf{a}}(t)\rangle. \quad (17)$$

The minimax spectral characteristic $h^0=h(F^0)$ is determined by formula (13) if $h(F^0)\in H_{D_F}$.

For more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics we observe that the least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ and the minimax spectral characteristic $h^0=h(F^0,G^0)$ form a saddle point of the function $\Delta\left(h;F,G\right)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta\left(h^0;F,G\right)\le\Delta\left(h^0;F^0,G^0\right)\le\Delta\left(h;F^0,G^0\right),$$
$$\forall h\in H_D,\forall F\in D_F,\forall G\in D_G,$$

hold true if $h^0=h(F^0,G^0)$ and $h(F^0,G^0)\in H_D$, where (F^0,G^0) is a solution to the constrained optimization problem

$$\max_{(F,G)\in D_F\times D_G}\Delta\left(h(F^0,G^0);F,G\right)=$$
$$=\Delta\left(h(F^0,G^0);F^0,G^0\right). \quad (18)$$

The linear functional $\Delta\left(h\left(F^0,G^0\right);F,G\right)$ is calculated by the formula

$$\Delta\left(h\left(F^0,G^0\right);F,G\right)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}$$
$$F(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*d\lambda$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)(F^0(\lambda)+G^0(\lambda))^{-1}$$
$$G(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}((A(\lambda))^\top G^0(\lambda)-(C^0(\lambda))^\top)^*d\lambda,$$

where

$$C^0(\lambda)=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt+$$
$$+\int_0^\infty((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt.$$

The constrained optimization problem (18) is equivalent to the unconstrained optimization problem [39]:

$$\Delta_D(F,G)=$$
$$-\Delta(h(F^0,G^0);F,G)+\delta((F,G)\big|D_F\times D_G)\to\inf, \quad (19)$$

where $\delta((F,G)\big|D_F\times D_G)$ is the indicator function of the set $D=D_F\times D_G$. Solution of the problem (19) is characterized by the condition $0\in\partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0) [40]. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities D [13], [39], [40].

Note, that the form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding solution of the problem (18). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books [28], [29], [30] for additional details).

III. Least favorable spectral densities in the class $D=D_\varepsilon\times D_V^U$

Consider the problem of minimax extrapolation of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the following classes of admissible spectral densities $D=D_\varepsilon\times D_V^U$,

$$D_\varepsilon^1=\Big\{F(\lambda)\,\Big|\,\mathrm{Tr}F(\lambda)=(1-\varepsilon)\mathrm{Tr}F_1(\lambda)+\varepsilon\mathrm{Tr}W(\lambda),$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}F(\lambda)d\lambda=p\Big\},$$
$$D_V^{U1}=\Big\{G(\lambda)\,\Big|\,\mathrm{Tr}V(\lambda)\le\mathrm{Tr}G(\lambda)\le\mathrm{Tr}U(\lambda),$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}G(\lambda)d\lambda=q\Big\},$$

$$D_\varepsilon^2=\Big\{F(\lambda)\Big|\, f_{kk}(\lambda)=(1-\varepsilon)f_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda),\ \frac{1}{2\pi}\int_{-\infty}^{\infty}f_{kk}(\lambda)d\lambda=p_k,k=\overline{1,T}\Big\},$$

$$D_V^{U2}=\Big\{G(\lambda)\Big|\, v_{kk}(\lambda)\le g_{kk}(\lambda)\le u_{kk}(\lambda),\ \frac{1}{2\pi}\int_{-\infty}^{\infty}g_{kk}(\lambda)d\lambda=q_k,k=\overline{1,T}\Big\},$$

$$D_\varepsilon^3=\Big\{F(\lambda)\Big|\,\langle B_1,F(\lambda)\rangle=(1-\varepsilon)\langle B_1,F_1(\lambda)\rangle+ +\varepsilon\langle B_1,W(\lambda)\rangle,\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_1,F(\lambda)\rangle d\lambda=p\Big\},$$

$$D_V^{U3}=\Big\{G(\lambda)\Big|\,\langle B_2,V(\lambda)\rangle\le\langle B_2,G(\lambda)\rangle\le\langle B_2,U(\lambda)\rangle,\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle B_2,G(\lambda)\rangle d\lambda=q\Big\},$$

$$D_\varepsilon^4=\Big\{F(\lambda)\Big|\,F(\lambda)=(1-\varepsilon)F_1(\lambda)+\varepsilon W(\lambda),\ \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\lambda)d\lambda=P\Big\},$$

$$D_V^{U4}=\Big\{G(\lambda)\Big|\,V(\lambda)\le G(\lambda)\le U(\lambda),\ \frac{1}{2\pi}\int_{-\infty}^{\infty}G(\lambda)d\lambda=Q\Big\},$$

where spectral densities $V(\lambda),U(\lambda),F_1(\lambda)$ are known and fixed, $W(\lambda)$ is an unknown spectral density. The class D_V^U describes the “strip” model of stochastic processes, while D_ε describes the model of “ε-contamination” of stochastic processes.

From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair $D_\varepsilon^1\times D_V^{U1}$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^*((A(\lambda))^\top G^0(\lambda)+ +(C^0(\lambda))^\top)=(\alpha^2+\gamma_1(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{20}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^*((A(\lambda))^\top F^0(\lambda)- -(C^0(\lambda))^\top)= =(\beta^2+\gamma_2(\lambda)+\gamma_3(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{21}$$

where $\gamma_1(\lambda)\le 0$ and $\gamma_1(\lambda)=0$ if $\mathrm{Tr}F^0(\lambda)>(1-\varepsilon)\mathrm{Tr}F_1(\lambda)$; $\gamma_2(\lambda)\le 0$ and $\gamma_2(\lambda)=0$ if $\mathrm{Tr}G^0(\lambda)>\mathrm{Tr}V(\lambda)$; $\gamma_3(\lambda)\ge 0$, and $\gamma_3(\lambda)=0$ if $\mathrm{Tr}G^0(\lambda)<\mathrm{Tr}U(\lambda)$.

For the second pair $D_\varepsilon^2\times D_V^{U2}$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)= =(F^0(\lambda)+G^0(\lambda))\Big\{(\alpha_k^2+\gamma_{1k}(\lambda))\delta_{kl}\Big\}_{k,l=1}^T (F^0(\lambda)+G^0(\lambda)), \tag{22}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)= =(F^0(\lambda)+G^0(\lambda))\Big\{(\beta_k^2+\gamma_{2k}(\lambda)+\gamma_{3k}(\lambda))\delta_{kl}\Big\}_{k,l=1}^T (F^0(\lambda)+G^0(\lambda)), \tag{23}$$

where $\gamma_{1k}(\lambda)\le 0$ and $\gamma_{1k}(\lambda)=0$ if $f_{kk}^0(\lambda)>(1-\varepsilon)f_{kk}^1(\lambda)$, $\gamma_{2k}(\lambda)\le 0$ and $\gamma_{2k}(\lambda)=0$ if $g_{kk}^0(\lambda)>v_{kk}(\lambda)$, $\gamma_{3k}(\lambda)\ge 0$, and $\gamma_{3k}(\lambda)=0$ if $g_{kk}^0(\lambda)<u_{kk}(\lambda)$.

For the third pair $D_\varepsilon^3\times D_V^{U3}$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)= =(\alpha^2+\gamma_{1'}(\lambda))(F^0(\lambda)+G^0(\lambda)) B_1^\top(F^0(\lambda)+G^0(\lambda)), \tag{24}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)= =(\beta^2+\gamma_2'(\lambda)+\gamma_3'(\lambda))(F^0(\lambda)+G^0(\lambda)) B_2^\top(F^0(\lambda)+G^0(\lambda)), \tag{25}$$

where $\gamma_{1'}(\lambda)\le 0$ and $\gamma_{1'}(\lambda)=0$ if $\langle B_1,F^0(\lambda)\rangle>(1-\varepsilon)\langle B_1,F_1(\lambda)\rangle$, $\gamma_2'(\lambda)\le 0$ and $\gamma_2'(\lambda)=0$ if $\langle B_2,G^0(\lambda\rangle>\langle B_2,V(\lambda)\rangle$, $\gamma_3'(\lambda)\ge 0$, and $\gamma_3'(\lambda)=0$ if $\langle B_2,G^0(\lambda\rangle<\langle B_2,U(\lambda)\rangle$.

For the fourth pair $D_\varepsilon^4\times D_V^{U4}$ we have equations

$$((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda)+(C^0(\lambda))^\top)= =(F^0(\lambda)+G^0(\lambda))(\vec{\alpha}\cdot\vec{\alpha}^*+ +\Gamma_1(\lambda))(F^0(\lambda)+G^0(\lambda)), \tag{26}$$

$$((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda)-(C^0(\lambda))^\top)= =(F^0(\lambda)+G^0(\lambda))(\vec{\beta}\cdot\vec{\beta}^*+\Gamma_2(\lambda)+ +\Gamma_3(\lambda))(F^0(\lambda)+G^0(\lambda)), \tag{27}$$

where $\Gamma_1(\lambda)\le 0$ and $\Gamma_1(\lambda)=0$ if $F^0(\lambda)>(1-\varepsilon)F_1(\lambda)$, $\Gamma_2(\lambda)\le 0$ and $\Gamma_2(\lambda)=0$ if

$G^0(\lambda) > V(\lambda)$, $\Gamma_3(\lambda) \ge 0$, and $\Gamma_3(\lambda) = 0$ if $G^0(\lambda) < U(\lambda)$.

Thus, the following statement holds true.

Theorem 4.1.
Let the minimality condition (1) hold true. The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_\varepsilon \times D_V^U$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ are determined by relations (20), (21) for the first pair $D_\varepsilon^1 \times D_V^{U1}$ of sets of admissible spectral densities; by relations (22), (23) for the second pair $D_\varepsilon^2 \times D_V^{U2}$ of sets of admissible spectral densities; by relations (24), (25) for the third pair $D_\varepsilon^3 \times D_V^{U3}$ of sets of admissible spectral densities; by relations (26), (27) for the fourth pair $D_\varepsilon^4 \times D_V^{U4}$ of sets of admissible spectral densities; constrained optimization problem (16) and restrictions on densities from the corresponding classes $D_\varepsilon \times D_V^U$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (7).

Corrolary 4.1.
Let the minimality condition (15) hold true. The least favorable spectral densities $F^0(\lambda)$ in the classes D_ε^k, $k = 1, 2, 3, 4$, for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ at points $t \in \mathbb{R}^- \setminus S$, where $S = \cup_{l=1}^s [-M_l - N_l, -M_l]$, are determined by the following equations, respectively,

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = (\alpha^2 + \gamma_1(\lambda))(F^0(\lambda))^2, \quad (28)$$

$$((C\ (\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = F^0(\lambda) \left\{ (\alpha_k^2 + \gamma_{1k}(\lambda))\delta_{kl} \right\}_{k,l=1}^T F^0(\lambda), \quad (29)$$

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = (\alpha^2 + \gamma_{1'}(\lambda)) F^0(\lambda)(B_1)^\top F^0(\lambda), \quad (30)$$

$$((C^0(\lambda))^\top)^* (C^0(\lambda))^\top = \\ = F^0(\lambda)(\vec{\alpha} \cdot \vec{\alpha}^* + \Gamma_1(\lambda)) F^0(\lambda), \quad (31)$$

constrained optimization problem (17) and restrictions on densities from the corresponding classes D_ε^k, $k = 1, 2, 3, 4$. The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (13).

Corrolary 4.2.
Let the minimality condition (15) hold true. The least favorable spectral densities $F^0(\lambda)$ in the classes D_V^{Uk}, $k = 1, 2, 3, 4$, for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ at points $t \in \mathbb{R}^- \setminus S$, where $S = \cup_{l=1}^s [-M_l - N_l, -M_l]$ are determined by the following equations, respectively,

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = (\beta^2 + \gamma_2(\lambda) + \gamma_3(\lambda))(F^0(\lambda))^2, \quad (32)$$

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = F^0(\lambda) \left\{ (\beta_k^2 + \gamma_{2k}(\lambda) + \gamma_{3k}(\lambda))\delta_{kl} \right\}_{k,l=1}^T F^0(\lambda), \quad (33)$$

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = (\beta^2 + \gamma_2'(\lambda) + \gamma_3'(\lambda)) F^0(\lambda) B_2^\top F^0(\lambda), \quad (34)$$

$$((C^0(\lambda))^\top)^* \cdot (C^0(\lambda))^\top = \\ = F^0(\lambda)(\vec{\beta} \cdot \vec{\beta}^* + \Gamma_2(\lambda) + \Gamma_3(\lambda)) F^0(\lambda), \quad (35)$$

constrained optimization problem (17}) and restrictions on densities from the corresponding classes D_V^{Uk}, $k = 1, 2, 3, 4$. The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (13).

IV. LEAST FAVORABLE SPECTRAL DENSITIES IN THE CLASS $D = D_{1\delta} \times D_{2\delta}$

Consider the problem of extrapolation of the functional $A\vec{\xi}$ in the case where spectral densities of the processes belong to the following classes of admissible spectral densities $D = D_{1\delta} \times D_{2\delta}$, where

$$D_{1\delta}^1 = \left\{ F(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \mathrm{Tr}(F(\lambda) - F_1(\lambda)) \right| d\lambda \le \delta_1 \right\},$$

$$D_{2\delta}^1 = \left\{ G(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \mathrm{Tr}(G(\lambda) - G_1(\lambda)) \right|^2 d\lambda \le \delta_2 \right\};$$

$$D_{1\delta}^2 = \\ \left\{ F(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f_{kk}(\lambda) - f_{kk}^1(\lambda) \right| d\lambda \le \delta_{1k}, k = \overline{1,T} \right\},$$

$$D_{2\delta}^2 = \\ \left\{ G(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| g_{kk}(\lambda) - g_{kk}^1(\lambda) \right|^2 d\lambda \le \delta_{2k}, k = \overline{1,T} \right\};$$

$$D_{1\delta}^3 = \left\{ F(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \left\langle B_1, F(\lambda) - F_1(\lambda) \right\rangle \right| d\lambda \le \delta_1 \right\},$$

$$D_{2\delta}^3 = \left\{ G(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \left\langle B_2, G(\lambda) - G_1(\lambda) \right\rangle \right|^2 d\lambda \le \delta_2 \right\};$$

$$D_{1\delta}^4 = \\ = \left\{ F(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f_{ij}(\lambda) - f_{ij}^1(\lambda) \right| d\lambda \le \delta_{ij}^1, i, j = \overline{1,T} \right\},$$

$D_{2\delta}^4 =$

$$\left\{ G(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|g_{ij}(\lambda) - g_{ij}^1(\lambda)\right|^2 d\lambda \le \delta_{ij}^2, i, j = \overline{1,T} \right\},$$

where $F_1(\lambda), G_1(\lambda)$ are known and fixed spectral densities. The class $D_{1\delta}$ describes the model of "δ-neighborhood" in the space L_1 of the given bounded spectral density $F_1(\lambda)$, while $D_{2\delta}$ describes the model of "δ-neighborhood" in the space L_2 of the given bounded spectral density $G_1(\lambda)$.

From the condition $0 \in \partial\Delta_D(F^0, G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair $D_{1\delta}^1 \times D_{2\delta}^1$ we have equations

$$((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top) = \alpha^2\gamma(\lambda)(F^0(\lambda) + G^0(\lambda))^2, \tag{36}$$

$$((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top) = \beta^2 \mathrm{Tr}(G^0(\lambda) - G_1(\lambda))(F^0(\lambda) + G^0(\lambda))^2, \tag{37}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathrm{Tr}(F^0(\lambda) - F_1(\lambda))\right| d\lambda = \delta_1, \tag{38}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathrm{Tr}(G^0(\lambda) - G_1(\lambda))\right|^2 d\lambda = \delta_2, \tag{39}$$

where $|\gamma(\lambda)| \le 1$ and

$$\gamma(\lambda) = \mathrm{sign}\ (\mathrm{Tr}(F^0(\lambda) - F_1(\lambda)))$$

if $\mathrm{Tr}(F^0(\lambda) - F_1(\lambda)) \ne 0.$

For the second pair $D_{1\delta}^2 \times D_{2\delta}^2$ we have equations

$$((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top) = (F^0(\lambda) + G^0(\lambda)) \left\{\alpha_k^2 \gamma_k(\lambda)\delta_{kl}\right\}_{k,l=1}^T (F^0(\lambda) + G^0(\lambda)), \tag{40}$$

$$((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top) = (F^0(\lambda) + G^0(\lambda)) \left\{\beta_k^2 (g_{kk}^0(\lambda) - g_{kk}^1(\lambda))\delta_{kl}\right\}_{k,l=1}^T (F^0(\lambda) + G^0(\lambda)), \tag{41}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| f_{kk}^0(\lambda) - f_{kk}^1(\lambda)\right| d\lambda = \delta_{1k},\ k = \overline{1,T}, \tag{42}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| g_{kk}^0(\lambda) - g_{kk}^1(\lambda)\right|^2 d\lambda = \delta_{2k},\ k = \overline{1,T}, \tag{43}$$

where $|\gamma_k(\lambda)| \le 1$ and

$$\gamma_k(\lambda) = \mathrm{sign}\ (f_{kk}^0(\lambda) - f_{kk}^1(\lambda))$$

if $f_{kk}^0(\lambda) - f_{kk}^1(\lambda) \ne 0,\ k = \overline{1,T}.$

For the third pair $D_{1\delta}^3 \times D_{2\delta}^3$ we have equations

$$((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top) = \alpha^2 \gamma'(\lambda)(F^0(\lambda) + G^0(\lambda)) B_1^\top (F^0(\lambda) + G^0(\lambda)), \tag{44}$$

$$((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top)^* ((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top) = \beta^2 \left\langle B_2, G^0(\lambda) - G_1(\lambda)\right\rangle (F^0(\lambda) + G^0(\lambda))^2, \tag{45}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\left\langle B_1, F^0(\lambda) - F_1(\lambda)\right\rangle\right| d\lambda = \delta_1, \tag{46}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\left\langle B_2, G^0(\lambda) - G_1(\lambda)\right\rangle\right|^2 d\lambda = \delta_2, \tag{47}$$

where $|\gamma'(\lambda)| \le 1$ and

$$\gamma'(\lambda) = \mathrm{sign}\ \left\langle B_1, F^0(\lambda) - F_1(\lambda)\right\rangle$$

if $\left\langle B_1, F^0(\lambda) - F_1(\lambda)\right\rangle \ne 0.$

For the fourth pair $D_{1\delta}^4 \times D_{2\delta}^4$ we have equations

$$((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top)^* ((A(\lambda))^\top G^0(\lambda) + (C^0(\lambda))^\top) = (F^0(\lambda) + G^0(\lambda)) \left\{\alpha_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^T (F^0(\lambda) + G^0(\lambda)),$$

$$((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top)^*, \tag{48}$$

$$((A(\lambda))^\top F^0(\lambda) - (C^0(\lambda))^\top) = (F^0(\lambda) + G^0(\lambda)) \left\{\beta_{ij}(g_{ij}^0(\lambda) - g_{ij}^1(\lambda))\right\}_{i,j=1}^T (F^0(\lambda) + G^0(\lambda)), \tag{49}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| f_{ij}^0(\lambda) - f_{ij}^1(\lambda)\right| d\lambda = \delta_{ij}^1,\ i, j = \overline{1,T}, \tag{50}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| g_{ij}^0(\lambda) - g_{ij}^1(\lambda)\right|^2 d\lambda = \delta_{ij}^2,\ i, j = \overline{1,T}, \tag{51}$$

where $|\gamma_{ij}(\lambda)| \le 1$ and

$$\gamma_{ij}(\lambda) = \frac{f_{ij}^0(\lambda) - f_{ij}^1(\lambda)}{\left|f_{ij}^0(\lambda) - f_{ij}^1(\lambda)\right|}$$

if

$$f_{ij}^0(\lambda)-f_{ij}^1(\lambda)\neq 0, i,j=\overline{1,T}.$$

Thus, the following theorem holds true.

Theorem 5.1.
Let the minimality condition (1) hold true. The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_{1\delta}\times D_{2\delta}$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ are determined by relations (36) - (39) for the first pair $D_{1\delta}^1\times D_{2\delta}^1$ of sets of admissible spectral densities; by relations (40) - (43) for the second pair $D_{1\delta}^2\times D_{2\delta}^2$ of sets of admissible spectral densities; by relations (44) - (47) for the third pair $D_{1\delta}^3\times D_{2\delta}^3$ of sets of admissible spectral densities; by relations (48) - (51) for the fourth pair $D_{1\delta}^4\times D_{2\delta}^4$ of sets of admissible spectral densities; constrained optimization problem (16) and restrictions on densities from the corresponding classes $D_{1\delta}\times D_{2\delta}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (7).

Corrolary 5.1.
Let the minimality condition 15) hold true. The least favorable spectral densities $F^0(\lambda)$ in the classes $D_{1\delta}^k$, $k=1,2,3,4$, for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ at points $t\in\mathbb{R}^-\setminus S$, where $S=\cup_{l=1}^s[-M_l-N_l,-M_l]$, are determined by the following equations, respectively,

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top=\alpha^2\gamma(\lambda)(F^0(\lambda))^2, \tag{52}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =F^0(\lambda)\left\{\alpha_k^2\gamma_k(\lambda)\delta_{kl}\right\}_{k,l=1}^T F^0(\lambda), \tag{53}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top=\alpha^2\gamma'(\lambda)F^0(\lambda)B_1^\top F^0(\lambda), \tag{54}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =F^0(\lambda)\left\{\alpha_{ij}(\lambda)\gamma_{ij}(\lambda)\right\}_{i,j=1}^T F^0(\lambda), \tag{55}$$

constrained optimization problem (17) and the following restrictions on densities from the corresponding classes $D_{1\delta}^k$, $k=1,2,3,4$, respectively,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}(F^0(\lambda)-F_1(\lambda))\right|d\lambda=\delta_1, \tag{56}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{kk}^0(\lambda)-f_{kk}^1(\lambda)\right|d\lambda=\delta_{1k}, \tag{57}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_1,F^0(\lambda)-F_1(\lambda)\right\rangle\right|d\lambda=\delta_1, \tag{58}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{ij}^0(\lambda)-f_{ij}^1(\lambda)\right|d\lambda=\delta_{ij}^1. \tag{59}$$

The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (13).

Corrolary 5.2.
Let the minimality condition (15) hold true. The least favorable spectral densities $F^0(\lambda)$ in the classes $D_{2\delta}^k$, $k=1,2,3,4$, for the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ at points $t\in\mathbb{R}^-\setminus S$, where $S=\cup_{l=1}^s[-M_l-N_l,-M_l]$, are determined by the following equations, respectively,

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =\beta^2\mathrm{Tr}(F^0(\lambda)-G_1(\lambda))(F^0(\lambda))^2, \tag{60}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =F^0(\lambda)\left\{\beta_k^2(f_{kk}^0(\lambda)-g_{kk}^1(\lambda))\delta_{kl}\right\}_{k,l=1}^T F^0(\lambda), \tag{61}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =\beta^2\left\langle B_2,F^0(\lambda)-G_1(\lambda)\right\rangle(F^0(\lambda))^2, \tag{62}$$

$$((C^0(\lambda))^\top)^*\cdot(C^0(\lambda))^\top= \\ =F^0(\lambda)\left\{\beta_{ij}(f_{ij}^0(\lambda)-g_{ij}^1(\lambda))\right\}_{i,j=1}^T F^0(\lambda) \tag{63}$$

constrained optimization problem (17) and the following restrictions on densities from the corresponding classes $D_{2\delta}^k$, $k=1,2,3,4$, respectively,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathrm{Tr}(F^0(\lambda)-G_1(\lambda))\right|^2d\lambda=\delta_2, \tag{64}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{kk}^0(\lambda)-g_{kk}^1(\lambda)\right|^2d\lambda=\delta_{2k}, k=\overline{1,T}, \tag{65}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left\langle B_2,F^0(\lambda)-G_1(\lambda)\right\rangle\right|^2d\lambda=\delta_2, \tag{66}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f_{ij}^0(\lambda)-g_{ij}^1(\lambda)\right|^2d\lambda=\delta_{ij}^2, i,j=\overline{1,T}. \tag{67}$$

The minimax spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (13).

V. CONCLUSION

In the article we propose methods of the mean-square optimal linear extrapolation of functionals which depend on the unknown values of the multidimensional stationary stochastic process based on observed data of the process with noise and missing values. In the case of spectral certainty where the spectral densities of the stationary processes are known we apply the method of orthogonal projection in a Hilbert space and derive formulas for calculating the spectral characteristics and the mean-square errors of the optimal estimates of the functionals. The corresponding results are obtained is the case of observations without noise. In the case of spectral uncertainty, where the spectral densities of the stationary processes are not exactly known while some special sets of admissible spectral densities are given, we apply the minimax-robust estimation method and derive relations which determine the least favourable spectral densities.

REFERENCES

[1] P. Bondon, "Influence of missing values on the prediction of a stationary time series," Journal of Time Series Analysis, vol. 26, no. 4, pp. 519–525, 2005.

[2] P. Bondon, "Prediction with incomplete past of a stationary process," Stochastic Processes and their Applications. vol.98, pp. 67–76, 2002.

[3] I. I. Dubovets'ka, O.Yu. Masyutka, and M.P. Moklyachuk, "Interpolation of periodically correlated stochastic sequences," Theory of Probability and Mathematical Statistics, vol. 84, pp. 43–56, 2012.

[4] I. I. Dubovets'ka and M. P. Moklyachuk, "Filtration of linear functionals of periodically correlated sequences," Theory of Probability and Mathematical Statistics, vol. 86, pp. 51–64, 2013.

[5] I. I. Dubovets'ka and M. P. Moklyachuk, "Extrapolation of periodically correlated processes from observations with noise," Theory of Probability and Mathematical Statistics, vol. 88, pp. 43–55, 2013.

[6] I. I. Dubovets'ka and M. P. Moklyachuk, "Minimax estimation problem for periodically correlated stochastic processes," Journal of Mathematics and System Science, vol. 3, no. 1, pp. 26–30, 2013.

[7] I. I. Dubovets'ka and M. P. Moklyachuk, "On minimax estimation problems for periodically correlated stochastic processes," Contemporary Mathematics and Statistics, vol.2, no. 1, pp. 123–150, 2014.

[8] J. Franke, "Minimax robust prediction of discrete time series," Z. Wahrscheinlichkeitstheor. Verw. Gebiete, vol. 68, pp. 337–364, 1985.

[9] J. Franke and H. V. Poor, "Minimax-robust filtering and finite-length robust predictors," in *Robust and Nonlinear Time Series Analysis.* Lecture Notes in Statistics, Springer-Verlag, vol. 26, pp. 87–126, 1984.

[10] I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes*. I. Berlin: Springer, 2004.

[11] U. Grenander, "A prediction problem in game theory," Arkiv f\"or Matematik, vol. 3, pp. 371–379, 1957.

[12] E. J. Hannan, *Multiple Time Series*, Wiley Series in Probability and Mathematical Statistics. New York etc.: John Wiley \& Sons, Inc. XI, 1970.

[13] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, Studies in Mathematics and its Applications, Vol. 6. Amsterdam, New York, Oxford: North-Holland Publishing Company. XII, 1979.

[14] K. Karhunen, "Uber lineare Methoden in der Wahrscheinlichkeitsrechnung," Annales Academiae Scientiarum Fennicae. Ser. A I, vol. 37, 1947.

[15] Y. Kasahara, M. Pourahmadi, and A. Inoue, "Duals of random vectors and processes with applications to prediction problems with missing values," Statistics and Probability Letters, vol. 79, no. 14, pp. 1637–1646, 2009.

[16] S. A. Kassam and H. V. Poor, "Robust techniques for signal processing: A survey," Proceedings of the IEEE, vol. 73, no. 3, pp. 433–481, 1985.

[17] A. N. Kolmogorov, *Selected works by A. N. Kolmogorov*. Vol. II: *Probability Theory And Mathematical Statistics*. A.N. Shiryayev, Ed. Mathematics and Its Applications. Soviet Series. 26. Dordrecht etc. Kluwer Academic Publishers, 1992.

[18] M. G. Krein and A. A. Nudelman, *The Markov Moment Problem And Extremal Problems*, Translations of Mathematical Monographs. Vol. 50. Providence, R.I.: American Mathematical Society, 1977.

[19] M. M. Luz and M. P. Moklyachuk, "Interpolation of functionals of stochastic sequences with stationary increments," Theory of Probability and Mathematical Statistics, vol. 87, pp. 117–133, 2012.

[20] M. M. Luz and M. P. Moklyachuk, "Minimax-robust filtering problem for stochastic sequence with stationary increments," Theory of Probability and Mathematical Statistics, vol. 89, pp. 127–142, 2014.

[21] M. Luz and M. Moklyachuk, "Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences," Statistics, Optimization \& Information Computing, vol. 2, no. 3, pp. 176 - 199, 2014.

[22] M. Luz and M. Moklyachuk, "Minimax interpolation problem for random processes with stationary increments," Statistics, Optimization \& Information Computing, vol. 3, no. 1, pp. 30-41, 2015.

[23] M. Luz and M. Moklyachuk, "Minimax-robust prediction problem for stochastic sequences with stationary increments and cointegrated sequences," Statistics, Optimization \& Information Computing, vol. 3, no. 2, pp. 160-188, 2015.

[24] M. P. Moklyachuk, "Minimax extrapolation and autoregressive-moving average processes," Theory of Probability and Mathematical Statistics, vol. 41, pp. 77--84, 1990.

[25] M. P. Moklyachuk, "Stochastic autoregressive sequences and minimax interpolation," Theory of Probability and Mathematical Statistics, vol. 48, pp. 95-103, 1994.

[26] M. P. Moklyachuk, "Robust procedures in time series analysis," Theory of Stochastic Processes, vol. 6, no. 3-4, pp. 127-147, 2000.

[27] M. P. Moklyachuk, "Game theory and convex optimization methods in robust estimation problems," Theory of Stochastic Processes, vol. 7, no. 1-2, pp. 253--264, 2001.

[28] M. P. Moklyachuk, *Robust Estimations Of Functionals Of Stochastic Processes*, Kyiv University, Kyiv, 2008.

[29] M. Moklyachuk and I. Golichenko, *Periodically Correlated Processes Estimates*, LAP Lambert Academic Publishing, 2016.

[30] M. Moklyachuk and O. Masyutka, *Minimax-Robust Estimation Technique for Stationary Stochastic Processes*, LAP Lambert Academic Publishing, 2012.

[31] M. P. Moklyachuk and M. I. Sidei, "Interpolation of stationary sequences observed with the noise," Theory of Probability and Mathematical Statistics, vol. 93, pp. 143-156, 2015.

[32] M. P. Moklyachuk and M. I. Sidei, "Interpolation problem for stationary sequences with missing observations," Statistics, Optimization \& Information Computing, vol. 3, no. 3, pp. 259-275, 2015.

[33] M. P. Moklyachuk and M. I. Sidei, "Extrapolation problem for functionals of stationary processes with missing observations," Bukovinian Mathematical Journal, vol.4, no. 1-2, pp. 122-129, 2016.

[34] M. P. Moklyachuk and M. I. Sidei, "Interpolation of functionals of stationary processes with missing observations," Bulletin of Taras Shevchenko National University of Kyiv, no. 1, pp. 24-30, 2016.

[35] M. P. Moklyachuk and M. I. Sidei, "Filtering problem for stationary sequences with missing observations," Statistics, Optimization \& Information Computing, vol. 4, no. 4, pp. 308 - 325 , 2016.

[36] M. Moklyachuk and M. Sidei, "Filtering Problem for functionals of stationary processes with missing observations," Communications in Optimization Theory, 2016, pp.1-18, Article ID 21, 2016.

[37] M. P. Moklyachuk and M. I. Sidei, "Extrapolation problem for stationary sequences with missing observations," Statistics, Optimization \& Information Computing, vol. 5, no. 3, pp. 212--233, 2017.

[38] M. Pourahmadi, A. Inoue and Y. Kasahara, "A prediction problem in $L^2(w)$," Proceedings of the American Mathematical Society. Vol. 135, No. 4, pp. 1233-1239, 2007.

[39] B. N. Pshenichnyj, *Necessary Conditions of an Extremum*, Pure and Applied mathematics. 4. New York: Marcel Dekker, 1971.

[40] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1997.

[41] Yu. A. Rozanov, *Stationary Stochastic Processes*, San Francisco-Cambridge-London-Amsterdam: Holden-Day, 1967.

[42] H. Salehi, "Algorithms for linear interpolator and interpolation error for minimal stationary stochastic processes," The Annals of Probability, Vol. 7, No. 5, pp. 840--846, 1979.

[43] K. S. Vastola and H. V. Poor, "An analysis of the effects of spectral uncertainty on Wiener filtering," Automatica, vol. 28, pp. 289--293, 1983.

[44] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series. With Engineering Applications*, The M. I. T. Press, Massachusetts Institute of Technology, Cambridge, Mass., 1966.

[45] A. M. Yaglom, \emph{Correlation theory of stationary and related random functions. Vol. 1: Basic results}, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.

[46] A. M. Yaglom, \emph{ Correlation theory of stationary and related random functions. Vol. 2: Supplementary notes and references}, Springer Series in Statistics, Springer-Verlag, New York etc., 1987.