A new improved technique for frequency band implementation of fractional order functions

Nitisha Shrivastava, Pragya Varshney

***Abstract*—**In this paper, a new approach for polynomial approximation of fractional order functions based on Carlson's method is presented. This novel technique allows the user to choose the frequency range (low, medium or high), in which the approximation is to be developed. In order, to obtain a numerical solution using any iterative formulae, the initial conditions play a very important role. The Newton iterative formula used by Carlson starts with an initial estimate of one. With this initial guess, the frequency versus magnitude characteristics of the approximation obtained always has 1 rad/s as the centre frequency. Moreover, the choice of frequency range is not possible in the Carlson's method. In this paper, we propose a new formula for the starting (initial) value of the approximation. The advantage of the formula is that it can be directly used for frequency band implementation of fractional order functions i.e. the approximation can be obtained in any desired frequency range with centre frequency not necessarily being one. The consistency of the novel approach has been verified for three fractional order functions. Comparisons with existing techniques have been presented and it is validated that the technique proposed in the paper shows better performance in all the frequency ranges.

***Keywords*—** fractional order functions, initial guess, polynomial approximation

I. Introduction

In recent years there has been a noticeable progress in the application areas of fractional calculus, prominently in circuits and systems, signal processing, control systems, electro-analytical chemistry, physics and biomedical engineering. The rapid growth in these areas has drawn the attention of many researchers working in the fields of science and engineering. Fractional order functions help in modeling various processes like diffusion, electrical and mechanical properties of certain materials, electrical conductance of biological systems, transmission lines etc. Fractional systems being irrational in nature cannot be implemented in its present form. As a substitute, its integer approximation is derived using the rational approximation methods. These methods are based on various concepts viz. interpolation, continued fraction expansion, weighted sum of first order filter sections and differential evolution algorithm [1]–[4]. An elaborative description of the approximation methods is given in [5] of which Charef method, Carlson method and Oustaloup method are the most popular ones. Carlson and Halijak produced an approximation formula based on regular Newton process which developed approximations to fractional capacitors $(1/s)^{1/n}$ in the form of ratios of polynomials with initial assumption one [6]–[9]. This method is widely used by researchers to obtain rational transfer function of fractional order systems [5], [10]–[17]. In all these papers, it is found that the frequency response (bode plot) of the approximation is built symmetrically only around the frequency value 1 rad/s and therefore the interval of frequencies where the approximation holds true always has one as the centre value.

The novel idea in this paper is that, with a change in the initial condition of the Newton iterative formula, we have been able to develop polynomial approximations of fractional order functions in any chosen frequency range. This means that the centre frequency is no more restricted to 1 rad/s and the approximations for low, medium and high frequency ranges can now be obtained to suit the required application. Therefore, the advantage of the novel approach is that it can be directly used for frequency band implementation of fractional order functions. The user first chooses the frequency range in which the approximation is to be developed. With this novel approach, we have overcome the limitation of the Carlson's method where choice of frequency range cannot be done. The paper is divided into five sections as follows: Section 2 describes the proposed Nitisha-Pragya-Carlson (NPC) Approximation method. In Section 3, the performance of the method is analyzed with the help of examples. The comparison of proposed method with some existing methods viz. Charef method and Oustaloup method is discussed in Section 4. Conclusions are presented in Section 5.

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II. Proposed NPC Approximation

The proposed NPC Approximation for $F(x) = a^{\pm 1/n}$ is given as

$$x_i = x_{(i-1)}\left[\frac{(\pm n-1)x_{(i-1)}^{\pm n} + (\pm n+1)a}{(\pm n+1)x_{(i-1)}^{\pm n} + (\pm n-1)a}\right] \qquad (1)$$

'a' is real variable, n ϵ N and i=1,2,3…. indicates the number of iterations performed. Equation (1) is a general version of the formula produced for $a^{1/n}$ in [6]. Here we have used it as a general form for both the cases of $a^{+1/n}$ and $a^{-1/n}$.

The first step to start the approximation is to choose the frequency range [R_1 R_2], (R_1 is the lower frequency and R_2 is the upper frequency), in which the approximation is to be developed.

In this paper, the authors have derived a formula for the initial guess as

$$x_0 = R_c^{\pm\frac{1}{n}} \tag{2}$$

Where $R_c = \sqrt{R_1 R_2}$ becomes the center frequency.

Applying the proposed approximation for the complex variable 's' we replace the real variable 'a' by the complex variable 's', and substitute the value of x_0 in (1), Thus we can develop new approximations for the fractional order function

$$F(s) = s^{\pm\frac{1}{n}} \tag{3}$$

in the specified frequency range.

The recursive formulae are as follows:

Fractional order differentiator $s^{1/n}$

$$x_1 = R_c^{\frac{1}{n}} \left[\frac{(n-1)R_c + (n+1)s}{(n+1)R_c + (n-1)s} \right] \tag{4}$$

$$x_2 = x_1 \left[\frac{(n-1)x_1^n + (n+1)s}{(n+1)x_1^n + (n-1)s} \right] \tag{5}$$

where x_1 and x_2 are the first and second iterates respectively approximating $s^{1/n}$.

Fractional order integrator $s^{-1/n}$

$$x_1 = \left(\frac{1}{R_c} \right)^{\frac{1}{n}} \left[\frac{(-n-1)\frac{1}{R_c} + (-n+1)s}{(-n+1)\frac{1}{R_c} + (-n-1)s} \right] \tag{6}$$

$$x_2 = x_1 \left[\frac{(-n-1)\left(\frac{1}{x_1}\right)^n + (-n+1)s}{(-n+1)\left(\frac{1}{x_1}\right)^n + (-n-1)s} \right] \tag{7}$$

where x_1 and x_2 are the first and second iterates respectively approximating $s^{-1/n}$.

Using these formulae, new approximation of F(s) can directly be obtained for different frequency ranges – low, medium and high, as against the indirect procedure mentioned in [10].

III. Performance Analysis and Discussion

In this section, the recursive formulae developed are verified by simulating

- a fractional differentiator of order ½ and a fractional integrator of order ¾ for different low, medium and high frequency ranges
- a fractional integrator of order 0.65 for the frequency range [10^{-2} 10^{2}] rad/s i.e. for a range having centre frequency 1 rad/s

The effectiveness of the results is validated with the help of frequency responses for all the cases. It is observed that this novel approximation holds true for fractional differentiator / integrator of any order in any specified frequency range.

A. *Example 1: Fractional differentiator of order ½ ($s^{0.5}$)*

The polynomial approximations of fractional differentiator of order ½ using NPC Approximation method (5) in the different frequency bands are

- Frequency band [10^{-2} 10^{1}] rad/s – low frequency

$$G_{1/2}\_low\_pz(s) = \left(\frac{\begin{array}{c} 5.0611(s+2.387)(s+0.4491) \\ (s+0.1054)(s+0.009832) \end{array}}{\begin{array}{c} (s+10.17)(s+0.9487) \\ (s+0.2227)(s+0.04189) \end{array}} \right) \tag{8}$$

- Frequency band [10^{2} 10^{5}] rad/s – medium frequency

$$G_{1/2}\_med\_pz(s) = \left(\frac{\begin{array}{c} 506.1072(s+2.387e4)(s+4491) \\ (s+1054)(s+98.32) \end{array}}{\begin{array}{c} (s+1.017e5)(s+9487) \\ (s+2227)(s+418.9) \end{array}} \right) \tag{9}$$

- Frequency band [10^{4} 10^{7}] rad/s – high frequency

$$G_{1/2}\_high\_pz(s) = \left(\frac{\begin{array}{c} 5061.0719(s+2.387e6)(s+4.491e5) \\ (s+1.054e5)(s+9832) \end{array}}{\begin{array}{c} (s+1.017e7)(s+9.487e5) \\ (s+2.227e5)(s+4.189e4) \end{array}} \right) \tag{10}$$

We can see that in each case ((8) – (10)) there is a recursive distribution of poles and zeros around the centre frequency and with a change in frequency range, only the position of poles and zeros vary as should be the case.

The response of any ideal fractional differentiator / integrator for a given frequency range is determined as

$$\begin{pmatrix} Ideal \\ magnitude \end{pmatrix} = +/- \begin{pmatrix} order\ of\ fractional \\ differentiator / integrator \end{pmatrix} \times 20 log\omega \text{ dB} \tag{11}$$

where ω is the frequency value in rad/s.

$$\begin{pmatrix} Ideal \\ phase \end{pmatrix} = +/- \begin{pmatrix} order\ of\ fractional \\ differentiator / integrator \end{pmatrix} \times 90 \text{ deg} \quad (12)$$

Further, slope of the magnitude characteristics is given as

$$\begin{pmatrix} Ideal \\ slope \end{pmatrix} = +/- \begin{pmatrix} order\ of\ fractional \\ differentiator / integrator \end{pmatrix} \times 20 \text{ dB/dec} \quad (13)$$

Figs. 1–3 show the magnitude and phase plots of $G_{\_1/2\_low\_pz}(s)$, $G_{\_1/2\_med\_pz}(s)$ and $G_{\_1/2\_high\_pz}(s)$ in the frequency ranges [10^{-2} 10^{1}], [10^{2} 10^{5}] and [10^{4} 10^{7}] rad/s respectively. It is observed that the magnitude response of these polynomial approximations (green dashed line) matches with the ideal response (blue bold line) in the whole frequency band. As can be seen we have been able to develop the plots for frequency ranges where centre frequency is not 1. This shows that the formula for initial estimate developed in this paper is suitable for all frequency ranges. From (11), the actual magnitude for a fractional order differentiator of order ½ becomes (½) × 20logω dB and for different values of ω, the actual magnitude is listed in Table I. A constant phase of (½) × 90° = 45° is obtained for about two and a half decades around the centre frequency for all the three frequency ranges. Also, the plots exhibit a slope of +10dB/decade.

TABLE I. Magnitude of ideal fractional differentiator of order ½ for different frequencies

ω (rad/s)	Ideal magnitude(dB)
10^{-2}	-20
10^{-1}	-10
10^{0}	0
10^{1}	10
10^{2}	20
10^{3}	30
10^{4}	40
10^{5}	50
10^{6}	60
10^{7}	70

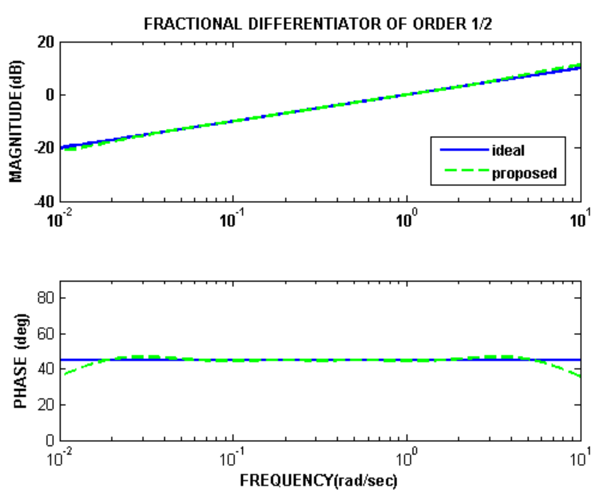

Fig. 1 Response of $s^{0.5}$ for frequency range [10^{-2} 10^{1}] rad/s

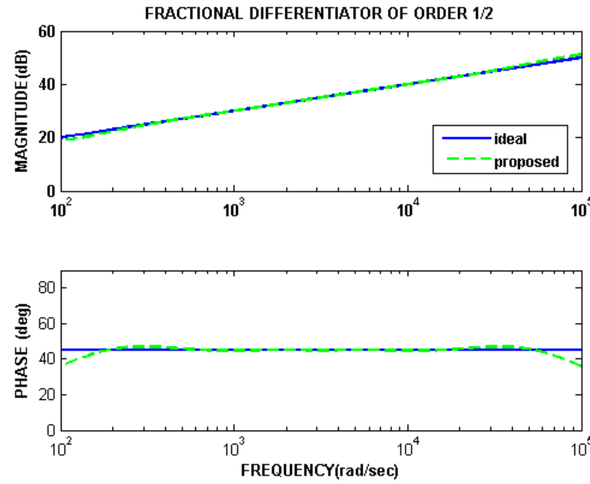

Fig. 2 Response of $s^{0.5}$ for frequency range [10^{2} 10^{5}] rad/s

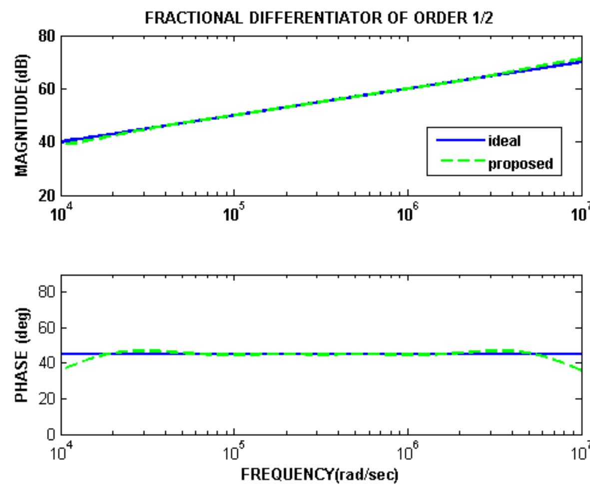

Fig. 3 Response of $s^{0.5}$ for frequency range [10^{4} 10^{7}] rad/s

B. Example 2: Fractional integrator of order ¾ ($1/s^{0.75}$)

The polynomial approximation of fractional integrator of order ¾ is obtained as $s^{-3/4} = (s^{-1/4}) \times (s^{-1/2})$. In such cases where the approximation is obtained after decomposing the fractional differentiator / integrator into further simple fractional order functions, the order of the resultant approximation is usually high. The higher order approximation can be reduced using order reduction techniques [18], [19]. The order of the second iteration approximation (7) of fractional integrator $s^{-3/4}$ obtained using NPC Approximation method for the three different frequency ranges [10^{-2} 10^{1}], [10^{1} 10^{3}] and [10^{4} 10^{7}] rad/s is ten each. It is further reduced to fourth order using the Schur Balance truncation algorithm [18]–[20]. The reduced order approximations of fractional integrator $s^{-3/4}$ for the frequency ranges :- low [10^{-2} 10^{1}]; medium [10^{1} 10^{3}]; and high [10^{4} 10^{7}] rad/s are

- Frequency band [10^{-2} 10^{1}] rad/s

$$G_{3/4}\_low\_pz(s)=\left(\frac{0.094855(s+12.83)(s+0.4962)(s+0.04565)(s+0.03429)}{(s+0.8758)(s+0.1005)(s+0.01836)(s+0.009874)}\right) \quad (14)$$

- Frequency band [10^{1} 10^{3}] rad/s

$$G_{3/4}\_med\_pz(s)=\left(\frac{0.0012649(s+4058)(s+156.9)(s+14.44)(s+10.84)}{(s+277)(s+31.77)(s+5.806)(s+3.123)}\right) \quad (15)$$

- Frequency band [10^{4} 10^{7}] rad/s

$$G_{3/4}\_high\_pz(s)=\left(\frac{2.99e-6(s+1.283e7)(s+4.962e5)(s+4.565e4)(s+3.429e4)}{(s+8.758e5)(s+1.005e5)(s+1.836e4)(s+9874)}\right) \quad (16)$$

In Fig. 4, the second iteration approximation (red dotted line) using NPC Approximation method, the reduced approximation (green dashed line) and the ideal response of fractional integrator of order ¾ are plotted for the frequency range [10^{-2} 10^{1}] rad/s. The actual magnitude of the ideal fractional integrator of order ¾ is a function of ω and is given as (-¾) × 20logω dB. So, the characteristic has a slope of (-¾) × 20 = -15 dB/decade. Table II mentions these magnitudes for different ω.

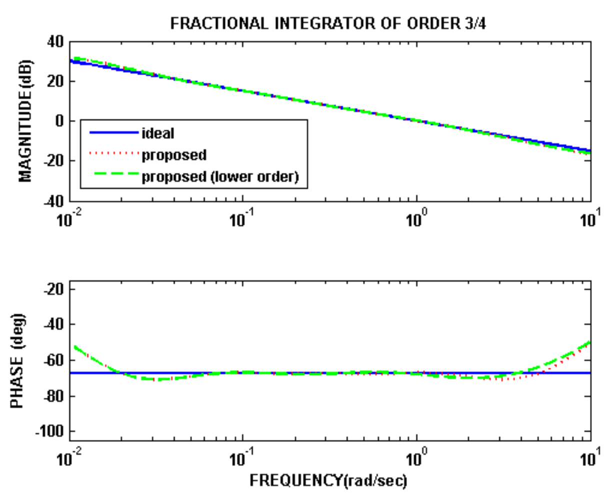

Fig. 4 Response of $1/s^{0.75}$ for frequency range [10^{-2} 10^{1}] rad/s

The phase of the ideal fractional integrator of order ¾ is constant and is equal to (-¾) × 90° = -67.5°. In Fig. 4, we see that the magnitude response obtained using NPC Approximation method is comparable to the ideal response for the whole frequency range and the phase of -67.5° is maintained for approximately two and a half decades around the centre frequency.

The frequency responses of $s^{3/4}$ for the frequency range [10^{1} 10^{3}] and [10^{4} 10^{7}] rad/s are shown in Figs. 5 & 6 respectively. These plots validate the effectiveness of the NPC Approximation method for the medium and high frequency ranges.

TABLE II. Magnitude of ideal fractional differentiator of order ¾ for different frequencies

ω (rad/s)	Ideal magnitude(dB)
10^{-2}	30
10^{-1}	15
10^{0}	0
10^{1}	-15
10^{2}	-30
10^{3}	-45
10^{4}	-60
10^{5}	-75
10^{6}	-90
10^{7}	-105

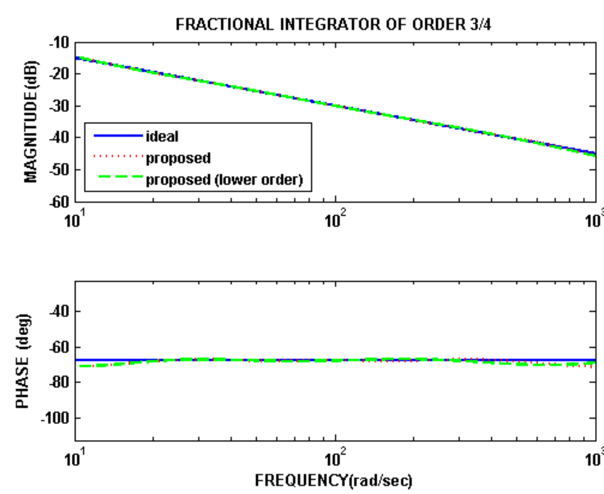

Fig. 5 Response of $1/s^{0.75}$ for frequency range [10^{1} 10^{3}] rad/s

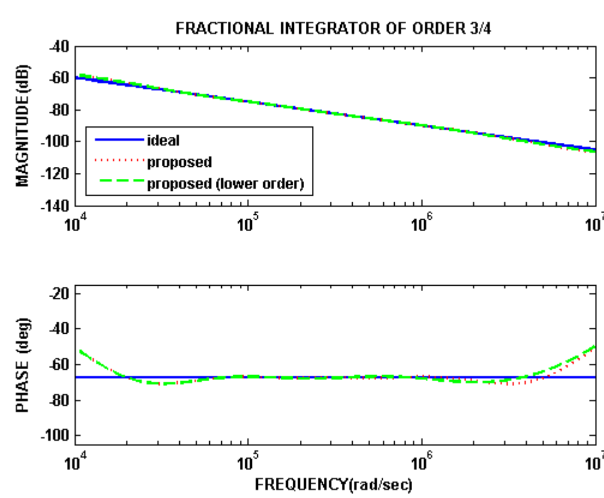

Fig. 6 Response of $1/s^{0.75}$ for frequency range [10^{4} 10^{7}] rad/s

C. Example 3: Fractional integrator of order 0.65 ($1/s^{0.65}$)

We have also simulated a fractional integrator of order 0.65 in the frequency range [10^{-1} 10^{1}] rad/s. This is done to demonstrate the suitability of the proposed NPC Approximation method for frequency ranges having centre frequency of 1 rad/s.

$s^{-0.65}$ is expanded as $s^{-0.65} = (s^{-1/10}) \times (s^{-1/2}) \times (s^{-1/20})$. The second iteration (7) approximation of fractional integrator $s^{-0.65}$ for the frequency range [10^{-1} 10^{1}] rad/s obtained using the NPC Approximation method is of order 38. It is reduced to fourth order using Schur balance truncation algorithm [18]–[20]. The fourth order transfer function of fractional integrator $s^{-0.65}$ for the frequency range [10^{-1} 10^{1}] rad/s is

$$G_{0.65}\_pz(s) = \left(\frac{0.060887(s+35.01)(s+1.714)(s+0.1917)(s+0.06402)}{(s+3.866)(s+0.4297)(s+0.05033)(s+0.03267)} \right) \qquad (17)$$

The magnitude and phase plots of fractional integrator $s^{-0.65}$ for the frequency range [10^{-1} 10^{1}] rad/s obtained using NPC Approximation method (red dotted line) and its reduced fourth order transfer function, $G\_{0.65\_pz}(s)$ (green dashed line) are compared with the ideal response of fractional integrator $s^{-0.65}$ (blue bold line) in Fig. 7. It is seen that the response of $G\_{0.65\_pz}(s)$ is in close match with the ideal characteristics in the chosen frequency range. The magnitude and phase of ideal fractional integrator of order 0.65 are (-0.65) × 20logω dB and (-0.65) × 90° = -58.5° respectively. The actual magnitudes for the different frequencies are given in Table III.

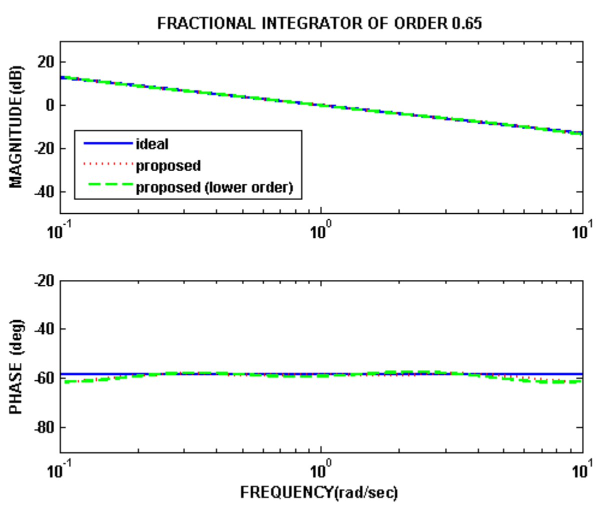

Fig. 7 Response of $1/s^{0.65}$ for frequency range [10^{-1} 10^{1}] rad/s

The results of all the three examples show that the proposed technique is true for any desired frequency range

TABLE III Magnitude of ideal fractional differentiator of order 0.65 for different frequencies

ω (rad/s)	Ideal magnitude(dB)
10^{-1}	13
10^{0}	0
10^{1}	-13

IV. Comparison with Existing Methods

We have compared NPC Approximation method with the existing popular approximation methods, viz. Charef method [21], [22] and Oustaloup method [23]. The maximum magnitude and phase errors for $s^{0.5}$, $1/s^{0.75}$ and $1/s^{0.65}$ in comparison to the ideal magnitude and phase values are given in Tables IV, V and VI and their error plots are shown in Figs. 8, 9 and 10 respectively. The comparisons are made for the following parameters:

- Order of the approximated model
- Width of the frequency range (in decades)
- Centre frequency of the frequency range

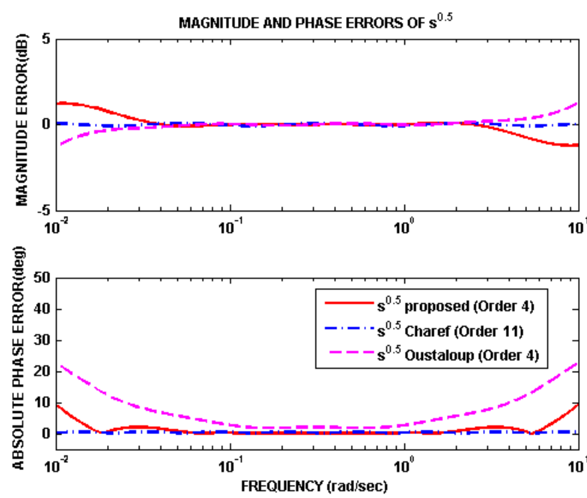

Fig. 8(a) Comparison of NPC method with existing methods for $s^{0.5}$, [10^{-2} 10^{1}] rad/s

In Figs. 8 (a), (b) & (c), the error plots for fractional differentiator $s^{0.5}$ in three different frequency ranges – low [10^{-2} 10^{1}] rad/s, medium [10^{2} 10^{5}] rad/s and high [10^{4} 10^{7}] rad/s are shown. All ranges have a width of 3 decades and in each case, the centre frequency is not 1 rad/s (unit gain frequency). From Table IV and Figs. 8 (a), (b) & (c), it can be seen that, for all the frequency ranges, the order of the NPC Approximation method based model is 4 and of the Charef method based model is 11, which is very high. For mathematical manipulation and hardware purpose, it is desirable that the orders of the system should be low. Therefore, the Charef method based models are not usable, in spite of the fact that its magnitude and phase errors are comparatively lower than those of NPC Approximation method. In terms of magnitude the errors of Oustaloup method based models are marginally higher than the NPC Approximation method based models but in terms of phase, the error values are very high (22.7149°).

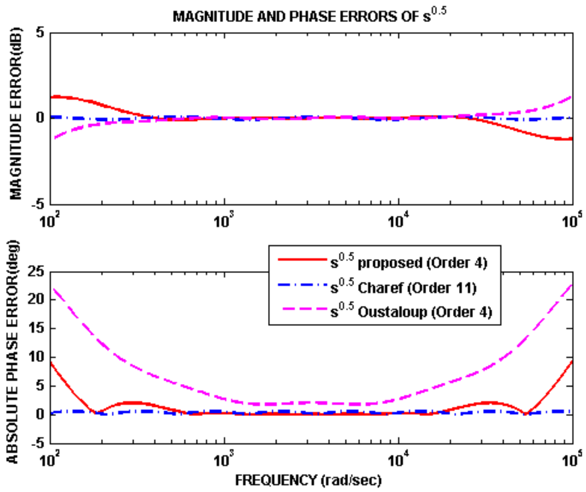

Fig. 8(b) Comparison of NPC method with existing methods for $s^{0.5}$, $[10^2\ 10^5]$ rad/s

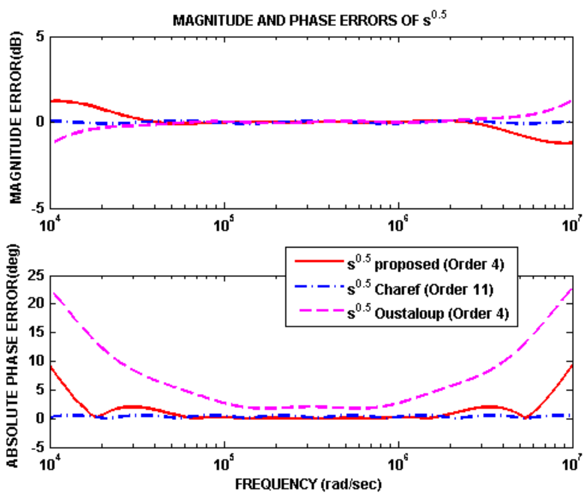

Fig. 8(c) Comparison of NPC method with existing methods for $s^{0.5}$, $[10^4\ 10^7]$ rad/s

Figs. 9 (a), (b) & (c) show the error plots for fractional integrator $1/s^{0.75}$ in the three different frequency ranges – low $[10^{-2}\ 10^1]$ rad/s, medium $[10^1\ 10^3]$ rad/s and high $[10^4\ 10^7]$ rad/s. The width of the medium frequency range is 2 decades and width of low and high frequency range is 3 decades. The centre frequency in each case is not 1 rad/s (unit gain frequency). For all the frequency ranges, the order of the NPC Approximation method based model is 4 and of the Charef method based model is 8 or 9, which is again high, and therefore not realizable. The order of Oustaloup method based model is 4, but the maximum magnitude and phase errors are higher than that of NPC Approximation method based models. Table V shows that, for the NPC Approximation method, the frequency range with width 2 decades has lower maximum magnitude and phase errors as compared to the frequency range with width 3 decades. But in case of Oustaloup method, these errors are still high even for a narrower frequency range.

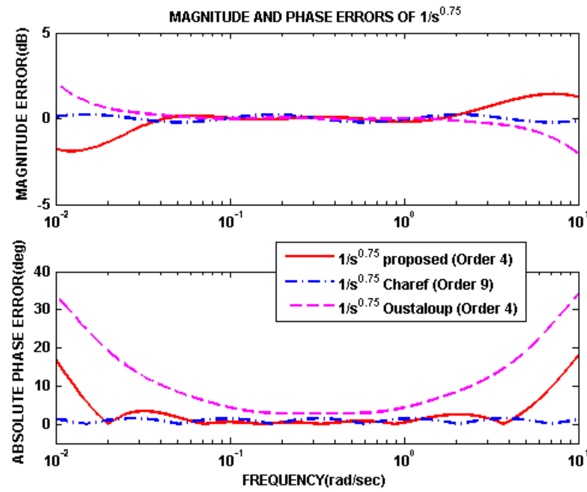

Fig. 9(a) Comparison of NPC method with existing methods for $1/s^{0.75}$, $[10^{-2}\ 10^1]$ rad/s

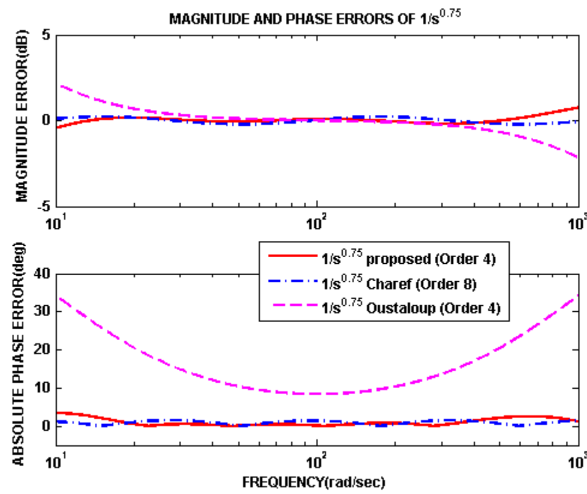

Fig. 9(b) Comparison of NPC method with existing methods for $1/s^{0.75}$, $[10^1\ 10^3]$ rad/s

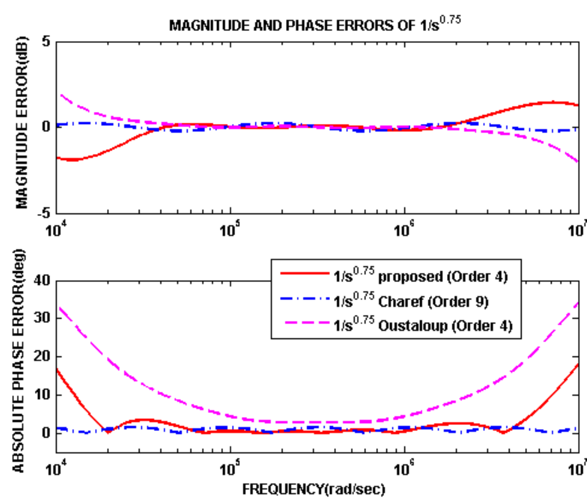

Fig. 9(c) Comparison of NPC method with existing methods for $1/s^{0.75}$, $[10^4\ 10^7]$ rad/s

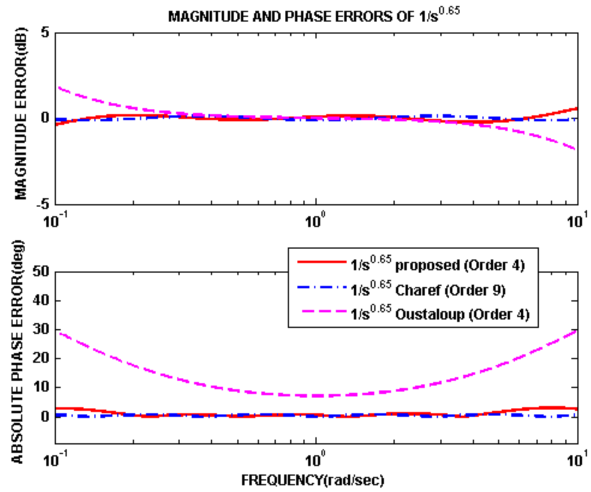

Fig. 10 Comparison of NPC method with existing methods for $1/s^{0.65}$, $[10^{-1}\ 10^1]$ rad/s

Fig. 10 shows the error plots for fractional integrator $1/s^{0.65}$ in the frequency range $[10^{-1}\ 10^1]$ rad/s. The width of the frequency range is 2 decades and centre frequency is equal to 1 rad/s, which is different from the previous two discussions. From Table VI and Fig. 10, it can be seen that, the maximum magnitude and phase error values of the Charef method based model is least, but the order of the model is very high. The order of the Oustaloup method based model is 4 and the error values are higher than that of our method.

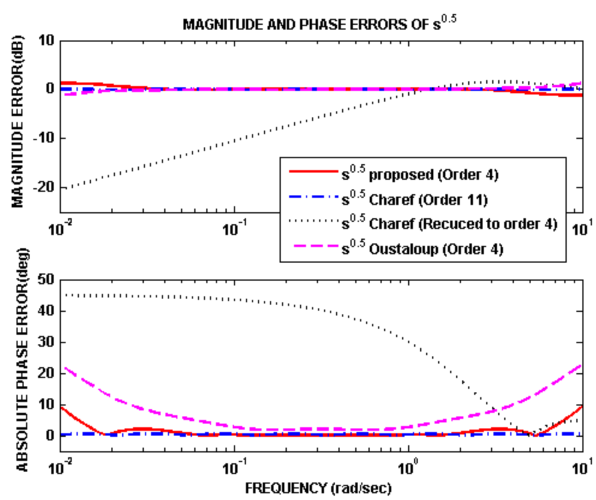

Fig. 11 Comparison of NPC method based model with reduced fourth order Charef method based model for $s^{0.5}$, $[10^{-2}\ 10^1]$ rad/s

Comparison of the proposed method is not possible with the Carlson method since in this method choice of frequency range can be done *a priori*.

From the above discussions, it can be concluded that:

- NPC Approximation method is suitable for all frequency ranges and hence is better than the existing methods. Moreover, the order of the approximated model is always 4, which can be easily realized on hardware.
- Charef method yields approximated model of very high order: For the purpose of comparison, we tried to reduce the Charef based models using existing reduction order techniques. The response of the Charef based fourth order model reduced using the Schur balance truncation algorithm [18]–[20] is shown in Fig. 11 (Fig. 8(a) revisited). It is observed that the lower order model does not match with the ideal response, and hence this work was not further pursued.
- The maximum magnitude and phase errors of approximated models obtained using Oustaloup method are comparatively higher than the proposed method.

V. Conclusion

The paper proposes an innovative formula NPC Approximation for obtaining polynomial approximations of fractional order functions. The method is based on Carlson's technique. The advantage of the formula is that it can be used in any frequency range – low, medium and high, which is a limitation of the existing Carlson's technique where frequency range cannot be chosen [10]. Three illustrative examples of fractional order functions have been considered to establish the correctness of the proposed method. Also, the proposed method is compared with two well-known methods: Charef method and Oustaloup method. It is shown through examples that Charef method yields models of very high order and the order of Oustaloup method based models is same as that of the proposed method, but with larger errors. For the method proposed in this paper by the authors, responses comparable to the ideal responses are obtained, for any fractional orders of differentiator and integrator, in any chosen frequency range.

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TABLE IV. Comparison of NPC Approximation method with existing methods for $s^{0.5}$

Approximation method	Order of the approximated model	cf	Width of frequency range	Frequency range (rad/s)	Maximum magnitude error (in dB)	Maximum phase error (in degrees)
Proposed (NPC)	4	≠1	3 decades	$[10^{-2}\ 10^{1}]$	1.2249	9.2397
				$[10^{2}\ 10^{5}]$	1.2239	9.2400
				$[10^{4}\ 10^{7}]$	1.2260	9.2381
Charef	**11 (order very high)**			$[10^{-2}\ 10^{1}]$	0.0823	0.5495
				$[10^{2}\ 10^{5}]$	0.0823	0.5495
				$[10^{4}\ 10^{7}]$	0.0823	0.5495
Oustaloup	4			$[10^{-2}\ 10^{1}]$	1.2635	22.7149
				$[10^{2}\ 10^{5}]$	1.2635	22.7149
				$[10^{4}\ 10^{7}]$	1.2635	22.7149

TABLE V. Comparison of NPC Approximation method with existing methods for $1/s^{0.75}$

Approximation method	Order of the approximated model	cf	Width of frequency range	Frequency range (rad/s)	Maximum magnitude error (in dB)	Maximum phase error (indegrees)
Proposed (NPC)	4	≠1	3 decades	$[10^{-2}\ 10^{1}]$	1.4339	17.8709
			2 decades	$[10^{1}\ 10^{3}]$	0.7559	3.3808
			3 decades	$[10^{4}\ 10^{7}]$	1.4343	17.8793
Charef	**9 (order very high)**		3 decades	$[10^{-2}\ 10^{1}]$	0.2240	1.4947
	8 (order very high)		2 decades	$[10^{1}\ 10^{3}]$	0.2209	1.5071
	9 (order very high)		3 decades	$[10^{4}\ 10^{7}]$	0.2230	1.4963
Oustaloup	4		3 decades	$[10^{-2}\ 10^{1}]$	2.0479	33.9255
			2 decades	$[10^{1}\ 10^{3}]$	2.1732	34.1777
			3 decades	$[10^{4}\ 10^{7}]$	2.0507	33.9278

TABLE VI. Comparison of NPC Approximation method with existing methods for $1/s^{0.65}$

Approximation method	Order of the approximated model	cf	Width of frequency range	Frequency range (rad/s)	Maximum magnitude error (in dB)	Maximum phase error (in degrees)
Proposed (NPC)	4	=1	2 decades	$[10^{-1}\ 10^{1}]$	0.5730	3.0860
Charef	**9 (order very high)**				0.1185	0.7896
Oustaloup	4				1.8597	29.6143