On A New Symbolic Method for Initial Value Problems for Systems of Higher-order Linear Differential Equations

Srinivasarao Thota

Abstract—This paper presents a symbolic method for solving an initial value problem (IVP) for the system of higher-order linear differential equations (HLDEs) with constant coefficients. In the proposed symbolic method, we apply the algebra of integro-differential operators for computing the Green's operator and Green's function of a fully-inhomogeneous IVP on the level of operators. Th algorithm of the proposed method will help to implement the manual calculations in commercial packages such as Mathematica, Matlab, Singular, Scilab etc. Maple implementation of the proposed method is discussed and provided sample computations. Certain examples are presented to illustrate the proposed method.

Keywords—Initial value problems, Higher order linear differential systems, Green's function, Integro-differential algebra, Interpolation method.

I. Introduction

In the last five decades, the development of relatively new field of general boundary value problems has been vigorously pursued by many researchers and engineers, noticeably by Brown, Krall and his associates [12]. Systems of HLDEs arise naturally in many applications, for example, the application of multi-body systems, models of electrical circuits, robotic modelling and mechanical systems, diffusion processes, dissipative operators, nuclear reactors, vibrating wires in magnetic fields etc.

Suppose $\mathcal{F} = C^\infty[a,b]$ and $[a,b]$ is finite interval of $\mathbb{R}$. In this paper, we are concerned with a system of s linear differential equations of order $t \geq 1$ of the form

$$A_t(x)\mathtt{D}^t u(x) + \cdots + A_1(x)\mathtt{D}u(x) + A_0(x)u(x) = f(x), \quad (1)$$

where $\mathtt{D} = \frac{d}{dx}$; $A_i(x) \in \mathcal{F}^{s\times s}$ for $i = 0,\ldots,t$; $A_t(x) \neq 0$ is the leading coefficient matrix; $f(x) = (f_1(x),\ldots,f_s(x))^T \in \mathcal{F}^s$ is a vector forcing function; $u(x) = (u_1(x),\ldots,u_s(x))^T \in \mathcal{F}^s$ is a s-dimensional unknown vector.

Definition 1. *A system of the form* (1) *satisfying the condition* $\det(A_t(x)) \neq 0$ *(determinant of* $A_t(x) \neq 0$*) is referred as the system of* first kind*, otherwise it is a* system of differential-algebraic equations*, where* $A_t(x)$ *is the leading coefficient matrix of the given system of HLDEs* (1).

In this paper, we consider a system of the first kind with constant coefficients, however the same idea can be extended to variable coefficients. For obtaining the unique solution, the given system (1) must have a set of initial conditions. The number of initial conditions is related to the dimension of the kernel of matrix differential operator (say L) of a given system. Since the system (1) is of first kind, the matrix differential operator L is regular and hence $\dim(\mathrm{Ker}(L)) = st$. Therefore, one must required st initial conditions to compute the unique solution. Suppose

$$B_1u(x) = c_1, B_2u(x) = c_2, \ldots, B_tu(x) = c_t, \quad (2)$$

are st initial conditions, where $B_i \in (\mathcal{F}^{s\times s})^*$ are initial condition operators and $c_i = (c_{i1},\ldots,c_{is})^T \in \mathbb{R}^s$, for $i = 1,\ldots,t$. Note that we can represent the given initial conditions in operator form as above. For example, the initial conditions $u_1(0) = 5, u_2'(0) = 3$ can be written in symbolic notations as $B_1u(x) = c_1$, where $B_1 = \begin{pmatrix} \mathtt{E} & 0 \\ 0 & \mathtt{ED}\end{pmatrix} \in (\mathcal{F}^{2\times 2})^*$, $c_1 = \begin{pmatrix} 5 \\ 3\end{pmatrix} \in \mathbb{R}^2$.

One can observe that the IVP (1)-(2) is fully-inhomogeneous (both the system and the initial conditions are inhomogeneous) as follows

$$\begin{aligned} &Lu = f, \\ &B_1u = c_1, \ldots, B_tu = c_t, \end{aligned} \quad (3)$$

where $L : \mathcal{F} \to \mathcal{F}$ is a matrix differential operator defined as $L = A_t\mathtt{D}^t + \cdots + A_1\mathtt{D} + A_0 \in \mathcal{F}^{s\times s}[\mathtt{D}]$; $B_1,\ldots,B_t$ are the initial condition operators; and $c_1,\ldots,c_t \in \mathbb{R}^s$ are the initial data. To treat the system (3) as an operator problem, the parameters f and $c_1,\ldots,c_t$ on right hand side of the system (3) are important. The main goal of this paper is to find an operator G such that $u = G(f; c_1,\ldots,c_t)$ and $B_1G(f; c_1,\ldots,c_t) = c_1,\ldots,B_tG(f; c_1,\ldots,c_t) = c_t$, for the given L and $B_1,\ldots,B_t$. As mentioned in [8], the operator G is called the *Green's operator*.

In the literature survey, [2], [6], [10], [11], [13], [16], [19], it is seen that there is no symbolic method/algorithm for solving the IVPs of the form (3) on the level of operators to find the matrix Green's operator and the corresponding vector Green's function via integro-differential algebras. This paper presents a new algorithm for finding the matrix Green's operator as well as the vector Green's function of a fully-inhomogeneous IVPs. From the general observation, one can always solve the fully-inhomogeneous IVP whenever the homogeneous IVP can be solved. There are familiar forms to express the general solution of a fully-inhomogeneous IVP as sum of the general solution

Srinivasarao Thota is with the Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No. 1888, Adama, Ethiopia. E-mail: srinithota@ymail.com and srinivasarao.thota@astu.edu.et

of a semi-homogeneous IVP and any one particular solution of a semi-inhomogeneous IVP. To find an explicit formula for the solution corresponding to the general linear systems (3), we use the concept of Moore-Penrose generalized inverse and the classical technique of interpolation.

The following symbols are used in present paper.

- Differential operator $\mathtt{D}$ is defined as $\mathtt{D}f(x) = \frac{df(x)}{dx}$.
- Integral operator $\mathtt{A}$ is defined as $\mathtt{A}f(x) = \int_a^x f(x)\ dx$, for a fixed initial value $a \in \mathbb{R}$.
- Evaluation operator $\mathtt{E}$ is defined as $\mathtt{E}f(x) = f(a)$, evaluates at the initialization point a, for a fixed initial value $a \in \mathbb{R}$.
- Matrix differential operator of order s is denoted by L.
- Initial condition operators are denoted by B_i, for $i = 1, \ldots, t$.
- Initial data is denoted by $c_i \in \mathbb{R}^s$.
- Vector initial data is denoted by $c = (c_1, \ldots, c_t)^T \in \mathbb{R}^{st}$.
- Matrix initial condition operator is defined as $B_i = \mathtt{E}\mathtt{D}^{i-1}I$, where I is the identity matrix.
- Initial data matrices is defined as $C_i = \text{diag}(c_i) \in \mathbb{R}^{s\times s}$, where $\text{diag}(c_i)$ denotes the diagonal matrix of c_i for $i = 1, \ldots, t$.
- Matrix initial condition operator is denoted by $B = (B_1, \ldots, B_t)^T$.
- Initial data matrix is denoted by $C = (C_1, \ldots, C_t)^T$.
- Exponential matrix X at a is denoted by X_a.

II. Operator-Based Representation of the System

In order to represent the systems of HLDEs with initial conditions in operator-based notations, we first recall the basic concepts of integro-differential algebras and operators, see, for example, [10] for further details.

A. Algebra of Integro-Differential Operators

Throughout this section $\mathbb{K}$ denotes the field of characteristic zero.

Definition 2. *[10] The algebraic structure $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ called an* integro-differential algebra *over $\mathbb{K}$ if $\mathcal{F}$ is a commutative $\mathbb{K}$-algebra with $\mathbb{K}$-linear operators $\mathtt{D}$ and $\mathtt{A}$ such that*

- *Section axiom: $\mathtt{D}(\mathtt{A}f) = f$,*
- *Leibniz axiom: $\mathtt{D}(fg) = (\mathtt{D}f)g + f(\mathtt{D}g)$,*
- *Differential Baxter axiom: $(\mathtt{AD}f)(\mathtt{AD}g) + \mathtt{AD}(fg) = (\mathtt{AD}f)g + f(\mathtt{AD}g)$*

are satisfied. Here $\mathtt{D} : \mathcal{F} \to \mathcal{F}$ and $\mathtt{A} : \mathcal{F} \to \mathcal{F}$ are two maps such that $\mathtt{D}$ is a derivation and $\mathtt{A}$ is a $\mathbb{K}$-linear right inverse of $\mathtt{D}$. The map $\mathtt{A}$ is called an integral *for $\mathtt{D}$. An integro-differential algebra over $\mathbb{K}$ is called* ordinary *if $Ker(\mathtt{D}) = \mathbb{K}$.*

The operators $\mathtt{J} = \mathtt{AD}$ and $\mathtt{E} = 1 - \mathtt{AD}$ are projectors, called the initialization and the evaluation of $\mathcal{F}$ respectively. For an ordinary integro-differential algebra, the evaluation can be translated as a multiplicative linear functional (character) $\mathtt{E} : \mathcal{F} \to \mathbb{K}$. For example [10], $\mathcal{F} = C^\infty(\mathbb{R})$ with $\mathtt{D} = \frac{d}{dx}$ and $\mathtt{A} = \int_a^x$; $\mathtt{E}f(x) = f(a)$, evaluates $f(x)$ at the initial point a, and $\mathtt{J}f(x) = f(x) - f(a)$ applies the initial condition.

The following proposition shows that the matrix ring $\mathcal{F}^{s\times s}$ is an integro-differential algebra if $\mathcal{F}$ is an integro-differential algebra.

Proposition 3. *Let $\mathcal{F}$ be an integro-differential algebra over a field $\mathbb{K}$. Then the matrix ring $\mathcal{F}^{s\times s}$ is again an integro-differential algebra over $\mathbb{K}$.*

Proof: Let $M = (m_{ij}), N = (n_{ij}) \in \mathcal{F}^{s\times s}$. Then section axiom is clearly satisfied i.e., $\mathtt{D}(\mathtt{A}M) = M$. We have $\sum_{r=1}^{s} \mathtt{D}(m_{ir}n_{rj}) = \sum_{r=1}^{s} (\mathtt{D}m_{ir})n_{rj} + \sum_{r=1}^{s} m_{ir}(\mathtt{D}n_{rj})$, for $i, j = 1, \ldots, s$, hence Liebnitz axiom is satisfied. Now, for differential Baxter axiom, we have $\sum_{r=1}^{s} (\mathtt{AD}m_{ir})(\mathtt{AD}n_{rj}) + \sum_{r=1}^{s} \mathtt{AD}(m_{ir}n_{rj}) = \sum_{r=1}^{s} (\mathtt{AD}m_{ir})n_{rj} + \sum_{r=1}^{s} m_{ir}(\mathtt{AD}n_{rj})$. Hence, the matrix ring $\mathcal{F}^{s\times s}$ is an integro-differential algebra over $\mathbb{K}$. ∎

The algebra of integro-differential operators is defined in the following definition. One can extend the definition to vector case similar to Proposition 3.

Definition 4. *[10] Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathcal{F}^*$. The* integro-differential operators *$\mathcal{F}[\mathtt{D}, \mathtt{A}]$ are defined as the $\mathbb{K}$-algebra generated by $\mathtt{D}$ and $\mathtt{A}$, the functions $f \in \mathcal{F}$, and the functionals $\mathtt{E} \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table I.*

Since we are interested in symbolic formulation, the given system of HLDEs, initial conditions and the inhomogeneity constants are denoted by triplet (L, B, C), where $L = A_t\mathtt{D}^t + \cdots + A_1\mathtt{D} + A_0$ is a surjective linear matrix differential operator [5] of order s; $B = (B_1, \ldots, B_t)^T$ is a matrix operator of initial conditions, here $B_i = \mathtt{E}\mathtt{D}^{i-1}I \in (\mathcal{F}^{s\times s})^*$; and $C = (C_1, \ldots, C_t)^T \in \mathbb{R}^{st\times s}$, here $C_i = \text{diag}(c_i) \in \mathbb{R}^{s\times s}$. Now the symbolic representation of the system (3) is

$$\begin{aligned} Lu &= f, \\ Bu &= c, \end{aligned} \tag{4}$$

where $c = (c_1, \ldots, c_t)^T$. We are not only looking for the solution of a particular system of HLDEs of the form (4) by fixing f and c on its right hand side; but also to obtain a generic expression for the solutions. Therefore, we propose a new algorithm that transform the given system and initial conditions into operator based notations on suitable spaces.

The set of independent solutions, say $\{v_1, \ldots, v_{st}\}$, of the homogeneous system $Lu = 0$ is called the *fundamental system* and the matrix $U = [v_1 \ \cdots \ v_{st}] \in \mathcal{F}^{s\times st}$ is called the *fundamental matrix*. In other words, the columns of U belongs to Ker(L). Hence we have $LU = 0$, where 0 is the zero matrix. There are several methods in literature [2], [4], [5], [6], [11], [13] to compute such fundamental matrix of a given matrix differential operator L. For example, the classical approach in which we convert the given system into a first order system. Indeed, if $\tilde{u}' = M\tilde{u}$, where M is the companion matrix, is first order homogeneous system of (1), then the fundamental matrix is obtained from the first s rows of solution $X = e^{Mx}$. The solution, $X = e^{Mx}$, of the homogeneous first order system is called the *exponential matrix* and we denote it by X; and X_a denotes the exponential matrix at a fixed point $a \in \mathbb{R}$.

Definition 5. *A matrix differential operator $L \in \mathcal{F}^{s\times s}$ is called* regular *if it has a regular (non-singular) exponential matrix.*

Definition 6. *An IVP for HLDEs (L, B, C) is called* regular *if it has a unique solution, otherwise it is called* singular.

The following proposition gives a regularity test for an IVP in terms of linear algebras.

Proposition 7. *[10], [19] Let $V = [v_1, \ldots, v_m]$ be the fundamental matrix for a matrix differential operator L, where $v_i = (v_{i1}, \ldots, v_{is})^T \in \mathcal{F}^s$; and $B = [B_1, \ldots, B_n]^T$ be a subspace of initial conditions, where $B_i = \mathtt{E}\mathtt{D}^{i-1}I \in (\mathcal{F}^{s\times s})^*$. Then the following statements are equivalent:*

- *(L, B, C) is regular, i.e. there exists a unique solution for (L, B, C).*
- *$m = sn$, and the* evaluation matrix*,*

$$\mathcal{E} = \begin{pmatrix} \sum_{i=1}^{m} B_{i1}v_{1i} & \cdots & \sum_{i=1}^{m} B_{i1}v_{mi} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} B_{im}v_{1i} & \cdots & \sum_{i=1}^{m} B_{im}v_{mi} \end{pmatrix} \in \mathbb{R}^{st\times st} \quad (5)$$

is regular.

- $\mathcal{F}^s = \mathit{Ker}(L) \oplus B^\perp$.

In order to compute the desired Green's operator and Green's function, we decompose the solution of fully-inhomogeneous IVP into the solution of semi-inhomogeneous IVP $(L, B, 0)$ and the solution of semi-homogeneous IVP (L, B, C) respectively.

III. A New Symbolic Algorithm

Recall the symbolic formulation of IVP defined in Section II as follows

$$\begin{aligned} Lu &= f, \\ Bu &= c, \end{aligned} \quad (6)$$

where $L = A_t\mathtt{D}^t + \cdots + A_1\mathtt{D} + A_0$, $B = (B_1, \ldots, B_t)^T$ and $c = (c_1, \ldots, c_t)^T$.

A. Solution of Semi-inhomogeneous Systems

In this section, we find the solution of semi-inhomogeneous IVP, i.e. for a given vector forcing function f and homogeneous initial conditions, we find $u \in \mathcal{F}^s$ such that

$$\begin{aligned} Lu &= f, \\ B_1u = \cdots &= B_tu = 0, \end{aligned} \quad (7)$$

where $B_1, \ldots, B_t \in (\mathcal{F}^{s\times s})^*$ orthogonally closed subspace of the initial conditions. The key step to find the matrix Green's operator G of the system (7) is the oblique Moore-Penrose inverse of matrix differential operator L with initial condition, i.e. $LG = I$ and $BG = 0$. Before presenting the proposed algorithm, we recall the *variation of parameters formula* for scalar differential equation in the following lemma.

Lemma 8. *[4], [5], [10] Suppose $\mathcal{L}$ is a monic linear differential operator of the following form with order $m > 0$,*

$$\mathcal{L} = \mathtt{D}^m + a_1\mathtt{D}^{m-1} + \cdots + a_m,$$

with fundamental system $\{v_1, \ldots, v_m\}$ of $\mathcal{L}$. Then the IVP

$$\begin{aligned} &\mathcal{L}y = f, \\ &\mathtt{E}y = \mathtt{E}\mathtt{D}y = \cdots = \mathtt{E}\mathtt{D}^{m-1}y = 0, \end{aligned} \quad (8)$$

has the unique solution, given by

$$y = \sum_{i=1}^{m} \frac{v_i \mathtt{A} d_i}{d} f, \quad (9)$$

where d is the determinant of Wronskian matrix W of $\{v_1, \ldots, v_m\}$; d_i is the determinant of W_i obtained from W by replacing the i-th column by m-th unit vector; and $\mathtt{A}$ is the integral operator.

For the shake of completeness and convenience to reader, we include sketch of the proof as follows.

Proof: We can transform the IVP (8) into first order linear system as follows

$$\begin{aligned} \tilde{y}' &= M\tilde{u} + \tilde{f}, \\ \mathtt{E}\tilde{y} &= 0, \end{aligned} \quad (10)$$

where M is the companion matrix and $\tilde{f} = (0, \ldots, f)^T$. Then, the solution of the first-order system (10) can be computed [2], [4], [5], [18] as

$$\tilde{y} = W\mathtt{A}W^{-1}\tilde{f},$$

where W is the Wronskian matrix of $\{v_1, \ldots, v_m\}$. Now, the solution of the given IVP (8) is obtained from the first row of $\tilde{y}$ as follows

$$y = \sum_{i=1}^{m} \frac{v_i \mathtt{A} d_i}{d} f,$$

as stated. ■

Now the following theorem presents a symbolic logarithm to obtain the solution of the semi-inhomogeneous IVP (7).

Theorem 9. *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra. For a given regular matrix differential operator $L = A_t\mathtt{D}^t + \cdots + A_1\mathtt{D} + A_0$ and a fundamental system $l_1, \ldots, l_{st}$ for scalar differential operator $\mathcal{L} = \det(L)$, the semi-inhomogeneous IVP*

$$\begin{aligned} Lu &= f, \\ Bu &= 0, \end{aligned}$$

has the unique solution

$$u = (u_1, \ldots, u_s)^T = \left(\sum_{k=1}^{s} \mathcal{D}_k^1 \mathcal{L}^\Diamond f_k, \ldots, \sum_{k=1}^{s} \mathcal{D}_k^s \mathcal{L}^\Diamond f_k \right)^T \in \mathcal{F}^s,$$

and the Green's operator is

$$G = \begin{pmatrix} \mathcal{D}_1^1\mathcal{L}^\Diamond & \ldots & \mathcal{D}_s^1\mathcal{L}^\Diamond \\ \vdots & \ddots & \vdots \\ \mathcal{D}_1^s\mathcal{L}^\Diamond & \ldots & \mathcal{D}_s^s\mathcal{L}^\Diamond \end{pmatrix} \in \mathcal{F}^{s\times s}[\mathtt{D}, \mathtt{A}],$$

where $\mathcal{D}_i^j$ is the determinant of the matrix differential operator L_i^j obtained from L by replacing j-th column by the i-th unit vector, and $\mathcal{L}^\diamondsuit$ is a fundamental right inverse of $\mathcal{L}$ computed using the variation of parameters formula (presented in Lemma 8),

$$\mathcal{L}^\diamondsuit = \sum_{i=1}^{st} l_i \mathrm{A} d^{-1} d_i \in \mathcal{F}[\mathrm{D}, \mathrm{A}],$$

here d is the determinant of the Wronskian matrix W for $l_1, \ldots, l_{st}$ and d_i the determinant of the matrix W_i obtained from W by replacing the ith column by st-th unit vector.

Proof:

The given system $Lu = f$ is regular, for the differential operator L is regular. Using the generalized Moore-Penrose inverse concept, we compute the solution u by incorporating the initial conditions. Suppose $\mathcal{L} = \det(L)$, then u is computed as

$$u = \begin{pmatrix} \mathcal{D}_1^1 & \ldots & \mathcal{D}_s^1 \\ \vdots & \ddots & \vdots \\ \mathcal{D}_1^s & \ldots & \mathcal{D}_s^s \end{pmatrix} \frac{1}{\mathcal{L}} f, \tag{11}$$

where $\mathcal{D}_i^j$ is the determinant of the matrix differential operator L_i^j, obtained from L by replacing j-th column by the i-th unit vector.

Now, we required the solution of the IVP for the scalar equation $\mathcal{L} y_i = \tilde{f}_i$ with initial conditions. Using the variation of parameters method presented in Lemma 8, we compute the solution as $y_i = \mathcal{L}^\diamondsuit \tilde{f}_i = \sum_{i=1}^{st} l_i \mathrm{A} d^{-1} d_i f_i$, where d is the determinant of the Wronskian matrix W for $l_1, \ldots, l_{st}$ and d_i the determinant of the matrix W_i obtained from W by replacing the ith column by st-th unit vector.

Therefore, from equation (11), the solution is

$$u = \begin{pmatrix} \mathcal{D}_1^1 & \ldots & \mathcal{D}_s^1 \\ \vdots & \ddots & \vdots \\ \mathcal{D}_1^s & \ldots & \mathcal{D}_s^s \end{pmatrix} \begin{pmatrix} \mathcal{L}^\diamondsuit \tilde{f}_1 \\ \vdots \\ \mathcal{L}^\diamondsuit \tilde{f}_n \end{pmatrix}.$$

On simplification, we have

$$u = (u_1, \ldots, u_s)^T = \left(\sum_{k=1}^s \mathcal{D}_k^1 \mathcal{L}^\diamondsuit f_k, \ldots, \sum_{k=1}^s \mathcal{D}_k^s \mathcal{L}^\diamondsuit f_k \right)^T. \tag{12}$$

From equation (11), we have Green's operator G is

$$G = \begin{pmatrix} \mathcal{D}_1^1 \mathcal{L}^\diamondsuit & \ldots & \mathcal{D}_s^1 \mathcal{L}^\diamondsuit \\ \vdots & \ddots & \vdots \\ \mathcal{D}_1^s \mathcal{L}^\diamondsuit & \ldots & \mathcal{D}_s^s \mathcal{L}^\diamondsuit \end{pmatrix}. \tag{13}$$

We show now that the solution (12) and the Green's operator (13) satisfy the given IVP. Indeed, we have

$$\begin{aligned} Lu = &\left(\sum_{p=1}^s L_1^p \sum_{k=1}^s \mathcal{D}_k^p \mathcal{L}^\diamondsuit f_k, \ldots, \sum_{p=1}^s L_s^p \sum_{k=1}^s \mathcal{D}_k^p \mathcal{L}^\diamondsuit f_k \right)^T \\ &= \mathcal{L}(\mathcal{L}^\diamondsuit f_1, \ldots, \mathcal{L}^\diamondsuit f_s)^T \\ &= (f_1, \ldots, f_s)^T = f, \end{aligned}$$

where L_i^j is the j-th column and i-th row of L; and $LG = \mathcal{L} I \mathcal{L}^\diamondsuit = I$. For checking initial conditions,

$$\begin{aligned} B_i u = &\left(\mathrm{E}\mathrm{D}^{i-1} \sum_{k=1}^s \mathcal{D}_k^1 \mathcal{L}^\diamondsuit f_k, \ldots, \mathrm{E}\mathrm{D}^{i-1} \sum_{k=1}^s \mathcal{D}_k^s \mathcal{L}^\diamondsuit f_k \right)^T \\ = &\left(\sum_{k=1}^s \mathrm{E}\mathrm{D}^{i-1} \mathcal{D}_k^1 \mathcal{L}^\diamondsuit f_k, \ldots, \sum_{k=1}^s \mathrm{E}\mathrm{D}^{i-1} \mathcal{D}_k^s \mathcal{L}^\diamondsuit f_k \right)^T \\ = &(0, \ldots, 0)^T = 0, \quad (\because \mathrm{E} \text{ is multiplicative and } \mathrm{E}\mathcal{L}^\diamondsuit = 0). \end{aligned}$$

The uniqueness of the solution follows from the fact that the fully-homogeneous IVP has only the trivial solution. ■

The following Examples 10 (Coupled Spring/Mass System [3, p. 186]) and 11 (steady heat conduction in a homogeneous rod) show the computation of the matrix Green's operator and the vector Green's function of the given IVP.

Example 10. *Consider a coupled spring mass system in which we suppose two masses m_1 and m_2 are connected to two springs of negligible mass having mass spring constant k_1 and k_2 respectively. Let $u_1(x)$ and $u_2(x)$ denote the vertical displacements of the masses from their equilibrium positions. Then by Newton's second law we have the motion of the coupled system as follows:*

$$\begin{aligned} m_1 \mathrm{D}^2 u_1 &= -k_1 u_1 + k_2(u_2 - u_1) \\ m_2 \mathrm{D}^2 u_2 &= -k_2(u_2 - u_1). \end{aligned} \tag{14}$$

For simplicity, we solve the system (14) *with $k_1 = 2, k_2 = 0, m_1 = 1$ and $m_2 = 1$ subject to the initial conditions $u_1(0) = 0, \mathrm{D}u_1(0) = 0, u_2(0) = 0, \mathrm{D}u_2(0) = 0$ with forcing function $f = (f_1, f_2)^T$.*

In symbolic notation, the system (14) *is $Lu = A_2 \mathrm{D}^2 u + A_1 \mathrm{D} u + A_0 u = f$, where*

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the initial condition operators are

$$B_1 = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} ED & 0 \\ 0 & ED \end{pmatrix}.$$

Now the matrix differential operator of the given system is

$$L = A_2 \mathrm{D}^2 + A_1 \mathrm{D} + A_0 = \begin{pmatrix} \mathrm{D}^2 + 2 & 0 \\ 0 & \mathrm{D}^2 \end{pmatrix},$$

and the matrix of initial condition operators $B = (B_1, B_2)^T$ is

$$B = \begin{pmatrix} E & 0 \\ 0 & E \\ ED & 0 \\ 0 & ED \end{pmatrix}.$$

The exponential matrix and fundamental matrix of L, respectively, are computed using the classical method [2], [4], [5], [6], [11], [13], given by

$$X = \begin{pmatrix} \sin(\sqrt{2}x) & \cos(\sqrt{2}x) & 0 & 0 \\ 0 & 0 & x & 1 \\ \sqrt{2}\cos(\sqrt{2}x) & -\sqrt{2}\sin(\sqrt{2}x) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} \sin(\sqrt{2}x) & \cos(\sqrt{2}x) & 0 & 0 \\ 0 & 0 & x & 1 \end{pmatrix}.$$

The evaluation matrix is computed as in Proposition 7, given by

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\det(\mathcal{E}) \neq 0$, there exists a unique solution for $(L, B, 0)$. Following the algorithm in Theorem 9, the matrix Green's operator G of the system (14) *is computed as*

$$G = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix},$$

where $G_{11} = \frac{1}{\sqrt{2}} \sin(\sqrt{2}x) \mathtt{A} \cos(\sqrt{2}x) - \frac{1}{\sqrt{2}} \cos(\sqrt{2}x) \mathtt{A} \sin(\sqrt{2}x)$, $G_{22} = x\mathtt{A} - \mathtt{A}x$ *and the Green's function $u = Gf$ is*

$$u = \begin{pmatrix} \frac{\sin(\sqrt{2}x)}{\sqrt{2}} \int_0^x \cos(\sqrt{2}x) f_1 \, dx - \frac{\cos(\sqrt{2}x)}{\sqrt{2}} \int_0^x \sin(\sqrt{2}x) f_1 \, dx \\ x \int_0^x f_2 \, dx - \int_0^x x f_2 \, dx \end{pmatrix}.$$

One can easily verify that $LG = I$, $BG = 0$ and $Lu = f$.

Example 11. *Consider the following system of equations with initial conditions.*

$$\begin{aligned} & u_1' - u_2 = 0, \\ & u_2' = f_2 \\ & \text{with} \quad u_1(0) = 0, u_2(0) = 0. \end{aligned} \tag{15}$$

Symbolic representation of the given IVP (15) *is $Lu = f$ and $Bu = 0$, where the matrix differential operator and initial condition operator are*

$$L = \begin{pmatrix} \mathtt{D} & -1 \\ 0 & \mathtt{D} \end{pmatrix}, B = \begin{pmatrix} \mathtt{E} & 0 \\ 0 & \mathtt{E} \end{pmatrix}, f = \begin{pmatrix} 0 \\ f_2 \end{pmatrix}.$$

Now from Theorem 9, the matrix Green's operator is

$$G = \begin{pmatrix} \mathtt{A} & x\mathtt{A} - \mathtt{A}x \\ 0 & \mathtt{A} \end{pmatrix},$$

and the solution $u = Gf$ is given by

$$u = \begin{pmatrix} x \int_0^x f_2 \, dx - \int_0^x x f_2 \, dx \\ \int_0^x f_2 \, dx \end{pmatrix}.$$

Here clearly $LG = I$ and $BG = 0$.

B. Solution of Semi-homogeneous Systems

In this section, we find the solution of semi-homogeneous IVP, i.e. for a given inhomogeneity constants $c_1, \ldots, c_t$ at initial point, we find $u \in \mathcal{F}^s$ such that

$$\begin{aligned} & Lu = 0, \\ & B_1 u = c_1, \ldots, B_t u = c_t. \end{aligned} \tag{16}$$

The key step to find the matrix Green's operator G for system (16) is the interpolation technique over integro-differential algebra such that $LG = 0$ and $BG = C$. The following theorem presents an algorithm to compute the solution of the given IVP (16).

Theorem 12. *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra. For a given matrix differential operator L and an exponential matrix X of L, the semi-homogeneous IVP*

$$\begin{aligned} Lu &= 0, \\ Bu &= c, \end{aligned}$$

has the unique solution

$$u = G(c_1, \ldots, c_t) = \begin{pmatrix} \sum_{k=1}^{st} v_{1,k} \mathtt{E} d^{-1} \sum_{p=1}^{st} d_p^k \tilde{c}_p \\ \vdots \\ \sum_{k=1}^{st} v_{s,k} \mathtt{E} d^{-1} \sum_{p=1}^{st} d_p^k \tilde{c}_p \end{pmatrix} \in \mathcal{F}^s,$$

where d is the determinant of the exponential matrix $X_a \in \mathcal{F}^{st \times st}$; $\{v_1, \ldots, v_{st}\}$ is a fundamental system for L; d_j^i is the determinant of X_j^i obtained from X_a by replacing the i-th column by the j-th unit vector; and $(\tilde{c}_1, \ldots, \tilde{c}_{st})^T = (c_1, \ldots, c_t)^T = c$. The matrix Green's operator is

$$G = U X_a^{-1} C.$$

Proof: Since the solution of the given IVP depends only on the inhomogeneous initial data, this amounts to an *interpolation problem* with initial conditions. Suppose

$$u = \lambda_1 v_1 + \cdots + \lambda_{st} v_{st} \tag{17}$$

is the required solution of the given IVP, where $\{v_1, \cdots, v_{st}\}$ is the fundamental system of L and $\lambda_1, \ldots, \lambda_{st}$ are the coefficients to be determined. From the given initial conditions with c, one can express the equation (17) as

$$X_a(\lambda_1, \ldots, \lambda_{st})^T = (\tilde{c}_1, \ldots, \tilde{c}_{st})^T,$$

since X_a is regular, X_a^{-1} exist. Hence

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{st} \end{pmatrix} = X_a^{-1} \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_{st} \end{pmatrix}. \tag{18}$$

Therefore, from equations (17)-(18), the solution is

$$\begin{aligned} u &= (v_1, \ldots, v_{st}) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{st} \end{pmatrix} \\ &= (v_1, \ldots, v_{st}) X_a^{-1} \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_{st} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{st} v_{1,k} \mathbb{E} d^{-1} \sum_{p=1}^{st} d_p^k \tilde{c}_p \\ \vdots \\ \sum_{k=1}^{st} v_{s,k} \mathbb{E} d^{-1} \sum_{p=1}^{st} d_p^k \tilde{c}_p \end{pmatrix}, \end{aligned} \tag{19}$$

where d is the determinant of the exponential matrix $X_a \in \mathcal{F}^{st\times st}$ and $\{v_1, \ldots, v_{st}\}$ is a fundamental system of L; d^i_j is the determinant of X^i_j obtained from X_a by replacing the i-th column by the j-th unit vector; and $(\tilde{c}_1, \ldots, \tilde{c}_{st})^T = (c_1, \ldots, c_t)^T = c$.
Again, from equations (17)-(18), the Green's operator such that $LG = 0$ and $BG = C$ is

$$G = (v_1, \ldots, v_{st})X_a^{-1}diag(c_1, \cdots, c_t) = UX_a^{-1}C. \quad (20)$$

We show now that the solution (19) and the Green's operator (20) satisfy the given IVP. Indeed, we have

$$\begin{aligned} Lu &= L\begin{pmatrix} \sum\limits_{k=1}^{st} v_{1,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \\ \vdots \\ \sum\limits_{k=1}^{st} v_{s,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{r=1}^{s} L^r_1 \sum\limits_{k=1}^{st} v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \\ \vdots \\ \sum\limits_{r=1}^{s} L^r_s \sum\limits_{k=1}^{st} v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \end{pmatrix} \\ &= LUX_a^{-1}\begin{pmatrix}\tilde{c}_1\\ \vdots \\ \tilde{c}_{st}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}, \qquad (\because LU = 0), \end{aligned}$$

where L^j_i is the j-th column and i-th row of L; and $LG = LU\mathrm{E}X_a^{-1}C = 0$. For initial conditions, we have

$$\begin{aligned} B_iu &= \begin{pmatrix} \mathrm{E}\mathrm{D}^{i-1}\sum\limits_{k=1}^{st} v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \\ \vdots \\ \mathrm{E}\mathrm{D}^{i-1}\sum\limits_{k=1}^{st} v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{k=1}^{st} \mathrm{E}\mathrm{D}^{i-1}v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \\ \vdots \\ \sum\limits_{k=1}^{st} \mathrm{E}\mathrm{D}^{i-1}v_{r,k}\mathrm{E}d^{-1}\sum\limits_{p=1}^{st} d^k_p\tilde{c}_p \end{pmatrix} \\ &= \begin{pmatrix}\tilde{c}_{(i-1)t+1}\\ \vdots \\ \tilde{c}_{(i-1)t+t}\end{pmatrix} = \begin{pmatrix}c_{i,1}\\ \vdots\\ c_{i,t}\end{pmatrix} = c_i, \ (\because \mathrm{E} \text{ is multiplicative}), \end{aligned}$$

and hence $BG = C$. The uniqueness of the solution follows from the fact that the evaluation matrix $\mathrm{E}d$ is regular. ■

C. Solution of Fully-inhomogeneous Systems

Now, the solution of fully-inhomogeneous systems of the form (3) is the composition of two solutions obtained in Section III-A and Section III-B respectively. Generalization of this fact is given in the following theorem.

Theorem 13. *Let $(\mathcal{F}, \mathrm{D}, \mathrm{A})$ be an ordinary integro-differential algebra. For a given matrix differential operator $L = A_t\mathrm{D}^t + \cdots + A_1\mathrm{D} + A_0$, an exponential matrix X and the inhomogeneity constants $c_1, \ldots, c_t$, the fully-inhomogeneous IVP*

$$\begin{aligned} Lu &= f,\\ Bu &= c \end{aligned}$$

has the unique solution

$$u = G_1(f) + G_2(c_1, \ldots, c_t) \in \mathcal{F}^s$$

and the matrix Green's operator is

$$G = G_1 + G_2 \in \mathcal{F}^{s\times s}[\mathrm{D}, \mathrm{A}],$$

where G_1 and G_2 are the matrix Green's operators of semi-inhomogeneous IVP and semi-homogeneous IVP respectively, computed as in Theorem 9 and Theorem 12.

The following Example 10 and Example 11 illustrate the computation of matrix Green's operator and vector Green's function of the given IVPs with inhomogeneous initial conditions. In [3], authors solved the Example 14 using the *method of solution by elimination*. However, we solve this example using the proposed symbolic algorithm in Case (ii) to show the simplicity of proposed algorithm.

Example 14. *Consider the system as given Example 10*

$$\begin{aligned} m_1\mathrm{D}^2u_1 &= -k_1u_1 + k_2(u_2 - u_1)\\ m_2\mathrm{D}^2u_2 &= -k_2(u_2 - u_1). \end{aligned} \quad (21)$$

Case (i): We solve system (21) *subject to the initial conditions $u_1(0) = 1, \mathrm{D}u_1(0) = 2, u_2(0) = 3, \mathrm{D}u_2(0) = -1$ for simplicity, with forcing function $f = (f_1, f_2)^T$.*

Operator notations of the given system are

$$L = \begin{pmatrix}\mathrm{D}^2+2 & 0\\ 0 & \mathrm{D}^2\end{pmatrix}, \ B = \begin{pmatrix}\mathrm{E} & 0\\ 0 & \mathrm{E}\\ \mathrm{ED} & 0\\ 0 & \mathrm{ED}\end{pmatrix}, \ C = \begin{pmatrix}1 & 0\\ 0 & 3\\ 2 & 0\\ 0 & -1\end{pmatrix}$$
$$\text{and } c = \begin{pmatrix}1\\3\\2\\-1\end{pmatrix},$$

and from Example 10, the matrix Green's operator G_1 of semi-inhomogeneous system is

$$G_1 = \begin{pmatrix}G_{111} & 0\\ 0 & G_{122}\end{pmatrix},$$

where $G_{111} = \frac{1}{\sqrt{2}}\sin(\sqrt{2}x)\mathrm{A}\cos(\sqrt{2}x) - \frac{1}{\sqrt{2}}\cos(\sqrt{2}x)\mathrm{A}\sin(\sqrt{2}x)$ and $G_{122} = x\mathrm{A} - \mathrm{A}x$. Following the algorithm in Theorem 12, the matrix Green's operator G_2 of semi-homogeneous IVP is computed as

$$G_2 = \begin{pmatrix}\cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x) & 0\\ 0 & 3-x\end{pmatrix}.$$

Now the matrix Green's operator G of the fully inhomogeneous IVP (21) *is*

$$G = \begin{pmatrix}G_1 & 0\\ 0 & G_2\end{pmatrix},$$

where $G_1 = \frac{\sin(\sqrt{2}x)}{\sqrt{2}} \mathtt{A} \cos(\sqrt{2}x) - \frac{\cos(\sqrt{2}x)}{\sqrt{2}} \mathtt{A} \sin(\sqrt{2}x) + \cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)$, $G_2 = x\mathtt{A} - \mathtt{A}x + 3 - x$ *and the Green's function* $u = G(f; c_1, c_2, c_3, c_4)$ *is*

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $u_1 = \frac{\sin(\sqrt{2}x)}{\sqrt{2}} \int_0^x \cos(\sqrt{2}x) f_1 \ dx - \frac{\cos(\sqrt{2}x)}{\sqrt{2}} \int_0^x \sin(\sqrt{2}x) f_1 \ dx + \sqrt{2}\sin(\sqrt{2}x) + \cos(\sqrt{2}x)$ *and* $u_2 = x \int_0^x f_2 \ dx - \int_0^x x f_2 \ dx - x + 3$. *One can easily check that* $LG = I$, $BG = C$, *and* $Tu = f$, $B_i G = C_i$.

Case (ii): Now we consider the system (21) *with* $k_1 = 6, k_2 = 4, m_1 = 1$ *and* $m_2 = 1$ *subject to the conditions* $u_1(0) = 0, \mathtt{D}u_1(0) = 2, u_2(0) = 0, \mathtt{D}u_2(0) = -1$ *with forcing function* $f = (0, 0)^T$ *as in [3, p. 186]. Proceeding similar to Case (i), we have*

$$L = \begin{pmatrix} \mathtt{D}^2 + 10 & -4 \\ -4 & \mathtt{D}^2 + 4 \end{pmatrix}, \ B = \begin{pmatrix} \mathtt{E} & 0 \\ 0 & \mathtt{E} \\ \mathtt{ED} & 0 \\ 0 & \mathtt{ED} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the solution of (L, B, C) *is*

$$u = \begin{pmatrix} -\frac{\sqrt{2}}{10}\sin(\sqrt{2}x) + \frac{\sqrt{3}}{5}\sin(2\sqrt{3}x) \\ -\frac{\sqrt{2}}{5}\sin(\sqrt{2}x) - \frac{\sqrt{3}}{10}\sin(2\sqrt{3}x) \end{pmatrix}.$$

Example 15. *Consider the system of equations given in Example 11 with inhomogeneous initial conditions as follows*

$$\begin{aligned} &u_1' - u_2 = 0, \\ &u_2' = f_2 \\ &u_1(0) = \alpha_1, u_2(0) = \alpha_2. \end{aligned} \tag{22}$$

Following Theorem 12, the matrix Green's operator and the solution of semi-homogeneous IVP are

$$G_2 = \begin{pmatrix} \alpha_1 & x\alpha_2 \\ 0 & \alpha_2 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} \alpha_1 + x\alpha_2 \\ \alpha_2 \end{pmatrix}.$$

Now the matrix Green's operator G *of the fully inhomogeneous IVP* (22) *is*

$$G = \begin{pmatrix} \mathtt{A} + \alpha_1 & x\mathtt{A} - \mathtt{A}x + x\alpha_2 \\ 0 & \mathtt{A} + \alpha_2 \end{pmatrix}$$

and the solution $u = G(f; \alpha_1, \alpha_2)$ *is given by*

$$u = \begin{pmatrix} x\int_0^x f_2 \ dx - \int_0^x x f_2 \ dx + \alpha_1 + x\alpha_2 \\ \int_0^x f_2 \ dx + \alpha_2 \end{pmatrix}.$$

In Section V, we provide certain real life examples using proposed algorithm.

IV. Proposed Algorithm in Maple

In this section, we discuss the Maple implementation of the proposed algorithm, `IVPforHLDEs`. We have created different data types, using the Maple package `IntDiffOp` implemented by Anja Korporal et al. [1], to express the matrix operators like matrix differential operator, matrix Green's operator etc. The Maple package `IVPforHLDEs` is available at http://www.sinivasaraothota.webs.com/research with example worksheet.

The following example gives sample computations to solve the system of differential equations with inhomogeneous initial conditions using `IVPforHLDEs` package.

Example 16. *In this example, we solve the Example 10 using Maple implementation as follows.*

```
> with(IntDiffOp):with(IVPforHLDEs):
> A2:=Matrix([[1,0],[0,1]]):
> A1:=Matrix([[0,0],[0,0]]):
> A0:=Matrix([[2,0],[0,0]]):
> L:=MatrixDiffOp(A2,A1,A0);
```

$$\begin{bmatrix} 2 + \mathtt{D}^2 & 0 \\ 0 & \mathtt{D}^2 \end{bmatrix}$$

```
> c:=Matrix([[1],[3],[2],[-1]]):
> f := Matrix([[f1(x)], [f2(x)]]):
> fm:=FundamentalMatrix(L);
```

$$\begin{bmatrix} \sin(\sqrt{2}x) & \cos(\sqrt{2}x) & 0 & 0 \\ 0 & 0 & x & 1 \end{bmatrix}$$

```
> EvMatrix(L);
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> G:=MatrixGreensOp(L,c);
```

*[[4*sin(2 ^(1/2)*x).A.1/8*cos(2 ^(1/2)*x)*2 ^(1/2)-4*cos(2 ^(1/2)*x). A. 1/8 * sin(2 ^(1/2) * x) *2 ^(1/2) + cos(2 ^(1/2)*x) + 1/2 * sin(2 ^(1/2) * x) * 2 ^(1/2), 0], [0, 4 * x . A . 1/4 * cos(2 ^(1/2) * x) ^2 + 4 * x . A . (1/4 - 1/4 * cos(2 ^(1/2) * x) ^2) - 4 . A . 1/4 * x * cos(2 ^(1/2) * x) ^2 - 4 . A . -1/4 * x * (-1 + cos(2 ^(1/2) * x) ^2) + 3 - x]]*

Matrix notation of above output G is,

$$\begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}$$

where $G_{11} = \frac{\sin(\sqrt{2}x)}{\sqrt{2}} \mathtt{A} \cos(\sqrt{2}x) - \frac{\cos(\sqrt{2}x)}{\sqrt{2}} \mathtt{A} \sin(\sqrt{2}x) + \cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)$ *and* $G_{22} = x\mathtt{A} - \mathtt{A}x + 3 - x$.

```
> u:=ApplyMatrixGreensOp(L,f,c);
```

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $u_1 = \frac{1}{\sqrt{2}} \sin(\sqrt{2}x) \left(\int_0^x \cos(\sqrt{2}x) f1 \ dx \right) - \frac{1}{\sqrt{2}} \cos(\sqrt{2}x) \left(\int_0^x \sin(\sqrt{2}x) f1 \ dx \right) + \sin(\sqrt{2}x)\sqrt{2} + \cos(\sqrt{2}x)$ *and* $u_2 = x \int_0^x f2 \ dx - \int_0^x x f2 \ dx + 3 - x$.

```
> MultiplyMatrixOp(L,G);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> fun:= Matrix([[exp(x)],[x]]):
> ApplyMatrixGreensOp(L, fun, c);
```

$$\begin{bmatrix} \frac{5}{6}\sin(\sqrt{2}x)\sqrt{2}+\frac{1}{3}e^x+\frac{2}{3}\cos(\sqrt{2}x)) \\ \frac{1}{6}x^3+3-x \end{bmatrix}$$

V. Examples

Example 17. *Suppose, we want to model the reaction path $X \rightleftharpoons_{k_1}^{k_2} Y \xrightarrow{k_3} Z$ staring with X. The reaction path can be described by the differential equations [9] as follows*

$$\begin{aligned} \frac{dC_X}{dt} &= -k_1C_X + k_2C_Y, \\ \frac{dC_Y}{dt} &= k_1C_X - (k_2+k_3)C_Y \\ \text{at} \ \ t=0; \quad C_X &= C_X^0, \\ C_Y &= 0, \end{aligned} \tag{23}$$

where C_X, t are concentration of X and time respectively. If we define $u_1 = \frac{C_X}{C_X^0}$ and $u_2 = \frac{C_Y}{C_X^0}$, then the system (23) *can be represent in symbolic notations as $Lu = f$ and $Bu = c$, where*

$$\begin{aligned} L &= \begin{pmatrix} \mathtt{D}+k_1 & -k_2 \\ -k_1 & \mathtt{D}+k_2+k_3 \end{pmatrix}; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \\ B &= \begin{pmatrix} \mathtt{E} & 0 \\ 0 & \mathtt{E} \end{pmatrix}; c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \\ C &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

here $f = 0$. In [9], authors presented the approximate solution of the system (23) *by Euler's method numerically for a special choice of $k_1 = 1000, k_2 = 1$ and $k_3 = 1$. However, the exact solution of* (23) *using the proposed method is computed similar to Example 14 as*

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

where $u_1 = \left(\frac{499\sqrt{89}+4717}{9434}\right)e^{-(501+53\sqrt{89})x} - \left(\frac{499\sqrt{89}-4717}{9434}\right)e^{(-501+53\sqrt{89})x}$ *and* $u_2 = \frac{500\sqrt{89}}{4717}\left(e^{(-501+53\sqrt{89})x} - e^{-(501+53\sqrt{89})x}\right)$.

We can easily observe that $Lu = 0$ and $Bu = c$.

Example 18. *Consider a system of* Stiff HLDEs *of the following form with initial conditions*

$$\begin{aligned} u_1' &= 9u_1 + 24u_2 + 5\cos(t) - \frac{1}{3}\sin(t), \\ u_2' &= -24u_1 - 51u_2 - 9\cos(t) - \frac{1}{3}\sin(t), \\ u_1(0) &= \frac{4}{3} \ \ \text{and} \ \ u_2(0) = \frac{2}{3}. \end{aligned} \tag{24}$$

The symbolic representation of the system (24) *is $Tu = f$ and $Bu = c$, where*

$$\begin{aligned} L &= \begin{pmatrix} \mathtt{D}-9 & -24 \\ 24 & 51+\mathtt{D} \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ f &= \begin{pmatrix} 5\cos(t) - \frac{1}{3}\sin(t) \\ -9\cos(t) + \frac{1}{3}\sin(t) \end{pmatrix}; B = \begin{pmatrix} \mathtt{E} & 0 \\ 0 & \mathtt{E} \end{pmatrix} \\ c &= \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \end{pmatrix}, C = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}. \end{aligned}$$

Using the proposed algorithm in Theorem 13, we have the matrix Green's operator

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix};$$

where
$$\begin{aligned} g_{11} &= \frac{4}{3}e^{-3t}\mathtt{A}e^{3t} - \frac{1}{3}e^{-39t}\mathtt{A}e^{39t} - \frac{4}{9}e^{-39t} + \frac{16}{9}e^{-3t}, \\ g_{12} &= \frac{2}{3}e^{-3t}\mathtt{A}e^{3t} - \frac{2}{3}e^{-39t}\mathtt{A}e^{39t} - \frac{4}{9}e^{-39t} + \frac{4}{9}e^{-3t}, \\ g_{21} &= -\frac{2}{3}e^{-3t}\mathtt{A}e^{3t} + \frac{2}{3}e^{-39t}\mathtt{A}e^{39t} + \frac{8}{9}e^{-39t} - \frac{8}{9}e^{-3t}, \\ g_{22} &= -\frac{1}{3}e^{-3t}\mathtt{A}e^{3t} + \frac{4}{3}e^{-39t}\mathtt{A}e^{39t} + \frac{8}{9}e^{-39t} - \frac{2}{9}e^{-3t}, \end{aligned}$$

and the exact solution is

$$u = \begin{pmatrix} 2e^{-3t} + \frac{1}{3}\cos(t) - e^{-39t} \\ -e^{-3t} - \frac{1}{3}\cos(t) + 2e^{-39t} \end{pmatrix}.$$

It is clear that $Lu = f$ and $Bu = c = (\frac{4}{3}, \frac{2}{3})^T$.

VI. Conclusion

In this paper, we presented a new symbolic method to solve an initial value problem for the system of higher-order linear differential equations with constant coefficients. Certain examples are discussed to illustrate the proposed symbolic algorithm. We also discussed the implementation of the proposed logarithm in Maple with sample computations. We thank Motilal Nehru National Institute of Technology Allahabad, India for supporting the Maple computational work.

TABLE I
Rewrite rules for integro-differential operators

$fg \to f \cdot g$	$\mathtt{D}f \to f\mathtt{D} + f'$	$\mathtt{A}f\mathtt{A} \to (\mathtt{A}f)\mathtt{A} - \mathtt{A}(\mathtt{A}f)$
$\chi\phi \to \phi$	$\mathtt{D}\phi \to 0$	$\mathtt{A}f\mathtt{D} \to f - \mathtt{A}f' - \mathtt{E}(f)\mathtt{E}$
$\phi f \to \phi(f)\phi$	$\mathtt{DA} \to 1$	$\mathtt{A}f\phi \to (\mathtt{A}f)\phi$

References

[1] A. Korporal, G. Regensburger, M. Rosenkranz: *A Maple package for integro-differential operators and boundary problems*. ACM Communications in Computer Algebra, 44(3) (2010), 120–122.

[2] C. E. Bacha: *Méthodes Algébriques pour la Résolution d'Équations Différentielles Matricicelles d'Ordre Arbitraire*. Ph.D. thesis, LMC-IMAG (2011).

[3] D. G. Zill, W. S. Wright: *Advanced Engineering Mathematics*, Jones & Bartlett Publishers (2014).

[4] E. A. Coddington, N. Levinson: *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London (1955).

[5] E. A. Coddington, R. Carlson: *Linear ordinary differential equations*. Society for Industrial and Applied Mathematics (1997).

[6] G. I. Kalogeropoulos, A. D. Karageorgos, A. A. Pantelous: *Higher-order linear matrix descriptor differential equations of Apostol-Kolodner type*. Electronic Journal of Differential Equations, 25 (2009), 1–13.
[7] H. Kenneth, K. Ray: *Linear Algebra*, second edition. Pearson Education, Inc., India (2008).
[8] I. Stakgold: *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York (1979).
[9] M. E. Davis: *Numerical methods and modeling for chemical engineers*. John Wiley & Sons , New York, NY. ISBN 0-471-88761-7 (1984).
[10] M. Rosenkranz, G. Regensburger, L. Tec, B. Buchberger: *Symbolic analysis for boundary problems: From rewriting to parametrized Gröbner bases* (2011).
[11] R. Ben Taher, M. Rachidi: *Linear matrix differential equations of higher-order and applications*. Electronic Journal of Differential Equations, 95 (2008), 1–12.
[12] R. C. Brown, A. M. Krall: *Ordinary differential operators under Stieltjes boundary conditions*. Trans. Amer. Math. Soc. 198 (1974), 73–92.
[13] S. Abramov: *EG - eliminations*. Journal of Difference Equations and Applications, 5 (1999), 393–433.
[14] S. Thota , S. D. Kumar: *A new method for general solution of system of higher-order linear differential equations*. International Conference on Inter Disciplinary Research in Engineering and Technology, 1 (2015), 240–243.
[15] S. Thota , S. D. Kumar: *Symbolic method for polynomial interpolation with Stieltjes conditions*. Proceedings of International Conference on Frontiers in Mathematics, 225–228 (2015).
[16] S. Thota , S. D. Kumar: *Solving system of higher-order linear differential equations on the level of operators*. International Journal of Pure and Applied Mathematics, 106(1) (2016), 11–21.
[17] S. Thota and S. D. Kumar: *On a mixed interpolation with integral conditions at arbitrary nodes*, Cogent Mathematics, 3(1) (2016), 1–10.
[18] T. M. Apostol: *Calculus*, Vol.II. John Wiley & Sons (Asia) Pte. Ltd. (2002).
[19] T. Sauer: *Polynomial interpolation, ideals and approximation order of multivariate refinable functions*. Proc. of Amer. Math. Soc. 130 (2002), 3335–3347.

Srinivasarao Thota completed his M.Sc. in Mathematics from Indian Institute of Technology Madras, India and Ph.D. in Mathematics from Motilal Nehru National institute of Technology Allahabad, India. Srinivasarao Thota's area of research interest is Computer Algebra, precisely symbolic methods for differential equations. Presentlty Working as Assistant Professor at Department of Applied Mathematics, Adama Science and Technology University, Eithiopia.