Optimal Missile Pitch Controller by using Parseval's Theorem

Nenad Popovich, Marco Ratti

***Abstract*—** This paper is focused on the optimal missile pitch control problem. Firstly, the analysis of the system by using Root Locus and Routh's stability criterion is performed. Then, Second Ziegler-Nichols and Systematic tuning methods are implemented. Different types of controllers: P, PI and PID are analyzed. PID controller has been selected as the best choice. Integral Square Error *(ISE)* criteria is chosen for the optimal system. Optimal controller parameters are found applying Parseval's theorem, based on the coefficients of the steady-state error in the frequency domain. Results are confirmed by using Matlab and Simulink. Response of the system to the unit-step function, shows a good dynamical performance. In addition, a steady-state error has been eliminated. "Derivative kick" is not significant and no needs to apply a non-linear saturation block (limiter) to the system. A disturbance is also added to check its influence on the system dynamics. Only a slight increase of the *ISE* was recorded.

***Keywords*—**Parseval's Theorem, Missile, Optimal Pitch Control, ISE Criteria, PID Controller.

I. INTRODUCTION

A missile, as well as, an airplane is characterized by six degrees of freedom of movement. To control the final trajectory, there are therefore three major control units: yaw, pitch and roll (see Fig. 1). The analysis and implementation of the missile control system was carried out on an "aerodynamic missile", as there is no coupling between longitudinal and lateral mode, thanks to roll stability [9]. Moreover, in order to simplify the analysis, the missile system was considered as a single compact body, omitting vibrations, residual fuel, etc.

The analysis was focused on the development of a pitch control system or also known as a longitudinal control. The lifts positioned at the rear make a directional change up and down, increasing or decreasing the lift [1].

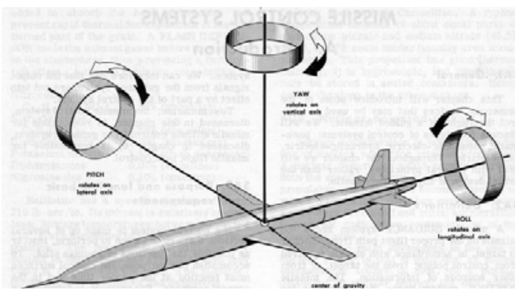

Fig. 1 Missile dynamical reference frame

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II. MATHEMATICAL MODEL

A. Open Loop Model

The open control system of the missile pitch control has been described using two transfer functions: a missile part (1) and a servo system (2), [13]:

$$\frac{-7.21}{s^2+0.12s-2.36} \tag{1}$$

$$\frac{-2750}{s^2+42.8s+2750} \tag{2}$$

By adding two transfer functions together (servo system and missile, combined) we get an open loop transfer function, G(s):

$$G(s) = \frac{19830}{s^4+42.9s^3+2753s^2+229s-6490} \tag{3}$$

A unit-step response of the open loop system is shown on Fig. 2.

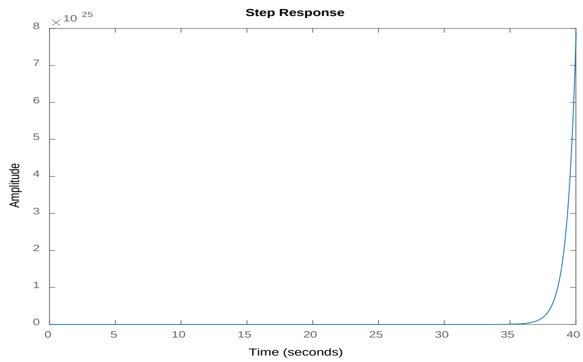

Fig. 2 Open loop step response

Obviously, the system is unstable due to unstable pole(s).

B. Closed loop system

In order to design a missile control system that can correct pitch errors, a negative unity feedback has to be applied to the structure, which will stabilize the system.
In addition, with an appropriate controller, it gives us a good dynamical behavior of the system, in terms of speed and accuracy.

Simulation model of the closed loop system is shown on Fig. 3.

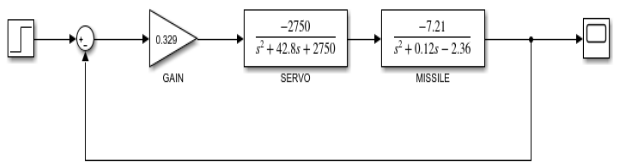

Fig. 3 Closed loop system simulation

Note: A value of the GAIN=0.329, will be discussed later.

C. Root Locus analyzes

Finding a range of stability is the first step in designing the system. One of the method applied for that range is Root Locus technique, which is shown on Fig. 4(a) and Fig. 4(b).

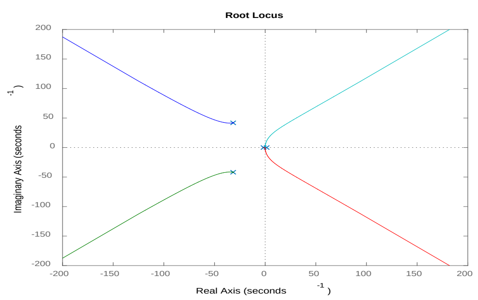

Fig. 4(a) Root Locus graph

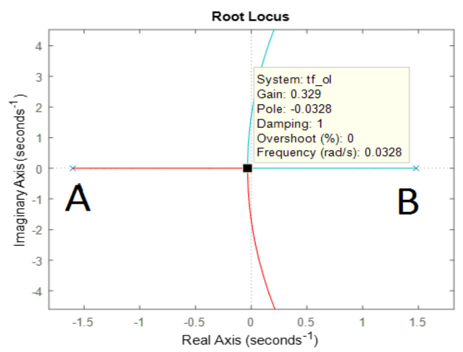

Fig. 4(b) Root Locus graph (enlarged)

By analyzing the graphs is evident the unstable nature of the system [4, 8, 11].

The open loop transfer function is characterized by a constant number in the numerator (no zeros) and the 4th order polynomial in the denominator (4 poles): 2 conjugate complex poles on the left-hand side of the "s-plane" (stable), and 2 poles (A and B) on the real axis: A is positive (unstable) and B is negative (stable).
The dynamic of the system reveals that the stability occurs just for a small range of the gain values.

By neglecting two poles at the far left (they are not of primary importance), it is necessary to analyze the path of the poles closer to the imaginary axis: starting from gain equal zero to $K \to \infty$. The two poles (A and B) will move towards each other and overlap at $S_{1,2}$=-0. 0328 (negative double pole). That means: all four poles will be stable. To keep the system stable, it is possible to force it to work with a value of gain, K= 0.329 (in Fig. 3).

With a further increasing of K, the two poles will cross the imaginary axis, producing an instability in the system.

D. Routh's Stability Criteria

Routh's stability criteria [11] is an analytical method for determining a stability range, purely based on the coefficients of the characteristic equation, C.E.:

$$\text{C.E.} = 1+G(s) = 0 \quad (4)$$

By using this criterion, it is found that the system will be stable in a small range of the gain (5):

$$0.327 < K < 1.065 \quad (5)$$

That means: if the gain is smaller than 0.327 and bigger than 1.065, the system will become unstable.
Note: GAIN =0.329 (in Fig. 3) is in that range of stability, which shows the correctness of Root Locus method.

A closed loop response to the unit step function, based on simulation model is given in Fig. 5. A proper reducer transmission gear has been introduced, which will reduce a large rotation from one of the gears, as well as to avoid mechanical breakdown.

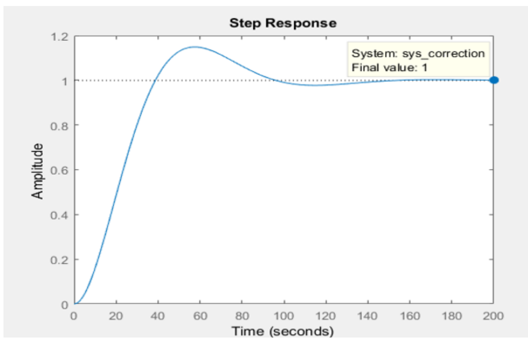

Fig. 5 Closed loop response

The closed loop system response does not have a steady-state error, which is one of the key requirements of the missile control system. On the other hand, system is relatively slow (settling time around 80 secs), which is not a huge disadvantage in the case of ballistic intercontinental missile, which travels for a long period. For other types of missiles speed is also an important requirement and the reaction of the system must be fast (i.e. a correction to the right path), to avoid a collision point of the missile far away from the target.

Anyway, it is just a first step for designing appropriate system, including an "optimal" controller.

III. Designing Controller

A. *Ziegler-Nichols Second Tuning Method*

One of the most common tuning methods is the Second Ziegler-Nichols tuning method that gives us a range of parameters in which we are trying to find the best ("optimal") controller parameters. This method can be used in our case because our system is not always unstable, nor always stable for any proportional gain [11].

The starting point for that method is to disable an integral and derivative controller part and try to achieve sustain oscillations with only proportional parts, which occur when system is on the "edge of stability". In other words, that happens when system is "neutral" or starting to be unstable (i.e. when dominant system's poles are on the imaginary axis), Fig. 4b.

Using simulation model in Fig. 3, (with a "low" proportional gain=0.5, in the range of stability), Fig. 6, till obtaining sustain oscillations, Fig. 7(a) and Fig. 7(b).

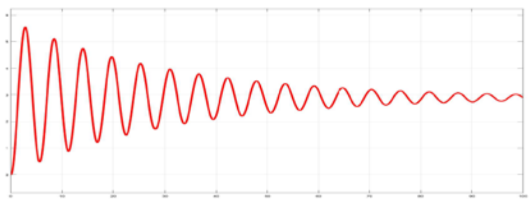

Fig. 6 Step response when gain=0.5

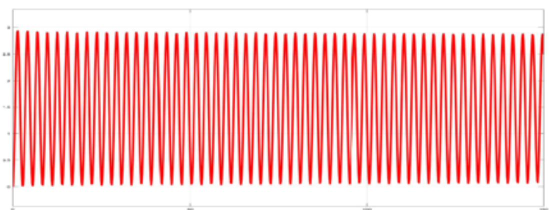

Fig. 7(a) Sustain oscillations when gain=1.0234

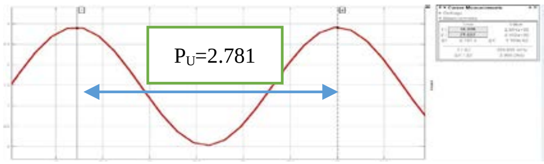

Fig. 7(b) Sustain oscillations (enlarged), gain=1.0234

From the above response, it is possible to calculate an ultimate period (P_U=21.007-18.226=2.781) and record gain when those oscillations occur, as an ultimate gain (G_U=1.0234).

Based on those two parameters (P_U, G_U) and using Table 1,
it is possible to calculate P (proportional), PI (Proportional + Integral) and PID (Proportional + Integral + Derivative) controller parameters:

- K_P -Proportional gain constant
- T_I -Integral time constant
- T_D -Derivative time constant
- K_I -Integral gain constant, where $K_I = K_P/T_I$
- K_D -Derivative gain constant, where $K_D = K_P * T_D$

Table 1

Type of controller	K_P	T_I	T_D
P	0.5* G_U		
PI	0.45* G_U	P_U /1,2	
PID	0.65* G_U	P_U /2	P_U /8

Note: Ziegler-Nichols Second tuning method is based on empirical formula and it is not so occurate. That means, Ziegler-Nichols Second tuning method, gives us only guidance regarding to the controller selection, as well as the initial settings for cotrollers parameters [4].

Responses of the system with P, PI and PID controllers calculated from Table 1, are shown on Fig. 8, Fig. 9 and Fig. 10, respectively.

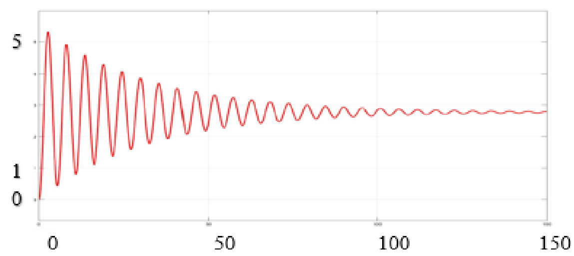

Fig. 8 System response with P controller (K_P=0.5117)

Note: Very big overshoot and steady-state error, high Oscillations and long settling time.

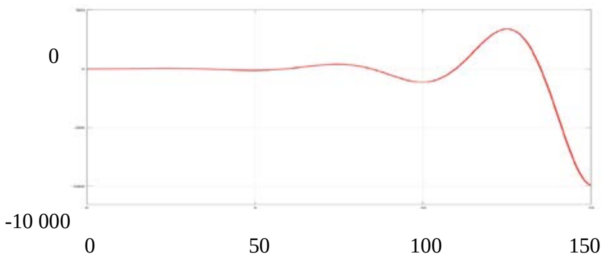

Fig. 9 System response with PI controller (K_P=0.4065, K_I=0.1754)

Note: unstable system!

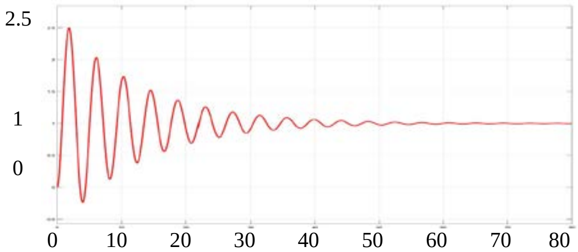

Fig. 10 System response with PID controller (K_P=0.6652, K_I=1.3905, K_D=0.2312)

Note: Big overshoot, high oscillations, no steady-state error and long settling time.

Obviously, PI controller is the worst of those types, which was expected, because both P and I part make more oscillatory and less stability. P controller is stable, but not acceptable, mainly because of a big steady-state error.

PID controller is not "ideal", but definitely better than the other two types. Unfortunately, overshoot and settling time are still high, but at least without a steady-state error (e_{ss}=0).

Calculating controller's parameters using Zieglar-Nichols Second tuning method does not lead us to an "optimal" system, but gives us a range of the controller's parameters for a fine tuning. That range will be used to determine the "optimal" parameters by using Parseval's theorem.

IV. Optimal Control by Using Parseval's Theorem

A. Parseval's Theorem (basic concept)

Parseval's theorem is also known as Rayleigh's energy theorem, it"connects" a time domain with a frequency domain [6], in a general form:

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |E(2\pi f)|^2 df \qquad (6)$$

where: $E(\omega) = F_\omega\{e(t)\}$ represents the continuous Fourier transform of *e(t),* and $\omega = 2\pi f$ is frequency in radians per second. The LHS side is energy in time space while the RHS is energy in frequency (spectral) space.

In mathematics, Parseval's theorem usually refers to the results that the Fourier transformation is unitary and that the sum (or integral) of the square of the function is equal to the sum of the square of its transformation [5]. By using Parceval's theorem it is possible to calculate the integral from the LHS through the frequency spectrum (positive and negative) in "s" or Laplace's domain, using formulae:

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(s)E(-s)ds \qquad (7)$$

$$E(s) = \frac{a(s)}{b(s)} \qquad (8)$$

Combining those two formulae gives:

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a(s)a(-s)}{b(s)b(-s)} ds \qquad (9)$$

where: "n" is an order of the system, and "a" and "b" are the coefficients of the polynomes in the descending order of "s", from equation (8):

$$a(s) = a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \cdots + a_0$$

$$b(s) = b_n s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0$$

B. Optimal criteria (ISE and Cost Function)

The "best" controller parameters: K_P, K_D and K_I could be calculated by using an Integral Squared Error (*ISE*) criteria (10a), i.e. when *ISE* has a minimum value.

$$ISE = \int_0^t e(t)^2 dt \to \min \qquad (10a)$$

Note: In control systems we consider that system start at *t=0*. Also, integration could go till settling time, $t = t_s$.

Based on the losses through a deviation (dynamic error) from the desired pitch, we can call *ISE* as a "Cost Function-C.F."

$$\text{C.F.} = \int_{-\infty}^{\infty} |e(t)|^2 dt \to \min. \qquad (10b)$$

The Cost Function, (10b) is used to find the most efficient values for K_P, K_D and K_I. This criteria will not necessarily produce the "best" output response with the smallest overshoot, nor the fastest system. It is simply used to determine gain values that will make the control more efficient. In the industry, this criteria is used mostly to lower fuel consumption. The name: "Cost Function" is derived from the meaning of the least cost as possible (i.e. less deviation less C.F) [3].

It is very hard to calculate deviation from the desired pitch in time domain (dynamic error), *e(t)*, even harder its quadratic form, as well as integral by using it in (6). That means, deviation can be easier found in a frequency domain, *E(s)* from Fig. 11 (in appendix) combining transfer functions of the whole system: PID controller, servo system and missile, it gives us numerator and denominator of the equation (9), for n=5:

$$a(s) = s^4 + 42.9s^3 + 2753s^2 + 229s - 6490$$

$$b(s) = s^5 + 42.9s^4 + 2753s^3 + (229 + 19827.5K_D)s^2 + \\ +(19827.5\,K_P - 6490)s + 19827.5\,K_I$$

The minimum value of the calculated Cost Function by using Parseval's theorem, gives the optimal controller's parameters.

C. Implementing Parseval's Theorem on ISE or C.F.

Integral (9) , for the fifth order system may be tranformed as a "table integral" (I_5**)** in the form:

$$I_5 = \frac{1}{2\Delta_5}[a_4^2 m_0 + (a_3^2 - 2a_2a_4)m_1 + \\ +(a_2^2 - 2a_1a_3 + 2a_0a_4)m_2 + (a_1^2 - 2a_0a_2)m_3 + a_0^2 m_4]$$

where:

$$\Delta_5 = b_0(b_1m_4 - b_3m_3 + b_5m_2) \qquad m_0 = \frac{1}{b_5}(b_3m_1 - b_1m_2)$$

$$m_1 = -b_0b_3 + b_1b_2 \qquad m_2 = -b_0b_5 + b_1b_4$$

$$m_3 = \frac{1}{b_0}(b_2m_2 - b_4m_1) \qquad m_4 = \frac{1}{b_0}(b_2m_3 - b_4m_2)$$

Taking the whole range of the PID controller's parameters obtained by Ziegler-Nichols Second tuning method and putting derivative gain constant and K_D unchanged (smaller K_D leads us to more oscillations and overshoot, ultimately a bigger C.F),

C.F has been calculated (using Microsoft Excel) and tabulated in Table 2:

Table 2

		K_I						
		0,1	0,2	0,25	0,3	0,35	0,4	0,47
Kp	0,1	-3,65	-2,36	-2,18	-2,14	-2,20	-2,39	-3,10
	0,2	4,03	1,64	1,20	0,93	0,74	0,61	0,46
	0,3	0,70	0,32	0,25	0,20	0,16	0,14	0,11
	0,4	0,19	0,08	0,06	0,04	0,03	0,02	0,02
	0,5	0,004	-0,10	-0,01	-0,01	-0,01	-0,01	-0,01
	0,6	-0,08	-0,05	-0,05	-0,04	-0,04	-0,04	-0,03
	0,66	-0,12	-0,07	-0,06	-0,05	-0,05	-0,05	-0,04

A minimum value (shaded, in yellow) of the "table integral", gives us "optimal" PID controller parameters: K_P=0.5, K_I=0.1, K_D=0.2312.

It should be emphasized that the values of C.F. presented in Table 2 do not correspond to the values obtained from an integral calculation, but from the "table integral". Of course, negative integral values cannot exist, but just indicate that system is not stable. Those values are irrelevant, because the system stability is a paramount of all criteria. As you can see from Table 2, they occur mainly where K_P is out of stable range in equation (5), including some negative influence of K_I on stability.

In addition, Parseval's theorem includes frequencies from negative infinity to positive infinity, while in control system engineering, frequencies go from zero to positive infinite (as it was mentioned before).

Results of the "optimal" PID values have been checked by using a simulation model (Fig. 11) on the *ISE* scope (not presented in this paper).

Output of the system with those "optimal" parameters is shown on Fig. 12:

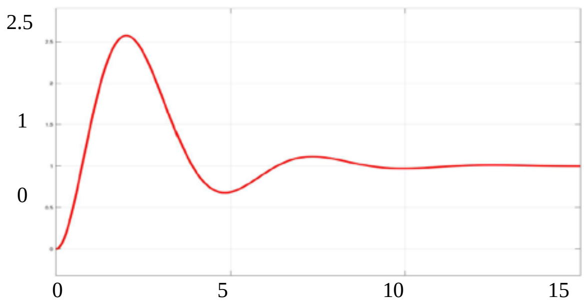

Fig. 12 Output of the system with "optimal"parameters

Obviously, output has significately better dynamics (smaller: rise, peak and settling times) with less oscillations, while maintains a zero steady-state error.

V. Disturbance

So far, simulation of the missile control system has been carried out in a closed system with no external noise.

In a real life, every control system is subjected to unwanted signal(s), referred as the disturbance(s). Those external disturbances can cause the system to have steady state error, as well as more oscillation of the output. Very high values of disturbances can cause instability of the system [8].

A missile is subjected to many actions: wind, pressure difference, vibration, etc. We assume that wind is the main disturbance, with known strength and direction (realistically: 10% of the input or wanted signal), [2]. The most likely position of the wind is before the servo and missile, where it has the biggest influence on the system (represented by 0.1 step block in Fig. 11).

By manipulating the values of the PID gain constants, it was possible to obtain the "best" *ISE* value, when select: K_P=0.6652, K_I=0.2 and K_D=0.2312 (it remains the same).

Also, *ISE* value slightly increased compared to the one with no disturbance, which is understandable, because dynamic error increases too in the presence of disturbance.

Note: This result is based on simulations (from the *ISE* block), not from calculation by using Parseval's theorem, which becomes more complicated with two inputs than only one.

VI. Conclusion

The analysis of a missile control system has developed though Root Locus method and Routh's criteria for stability.
A very narrow range of stability was determined.
Subsequent to open-loop analysis, closed-loop system was created in Simulink.
Ziegler-Nichols Second tuning method was used to find an initial range of the controller's parameters.
Different controller's types have been examined, then PID controller selected as the best choice.
Criteria for "optimal" controller's parameters was defined through the Cost Function (C.F.).
Parseval's theorem for calculating "optimal" parameters was applied to *ISE* (Integral Square Error) criteria.
Disturbance was also added into the system which suggests that the object will experience 10% extra force in the direction of travel.
The *ISE* value was slightly increased compared to the one with no disturbance.
A "derivative kick" was not significant and it was not necessary to add an extra protection for the steering gear, which could lead us to a non-linear system.
Further research could be focused on testing a real physical object to see if it does react as it was suggested in the simulation.

Acknowledgment

We would like to extent our sincere gratitude to Jay Patel and Shivneet Chand who were part of the Final Year Project group that carried out the research "Missile Pitch Control System", AUT, Auckland, New Zealand, 2017.

Appendix

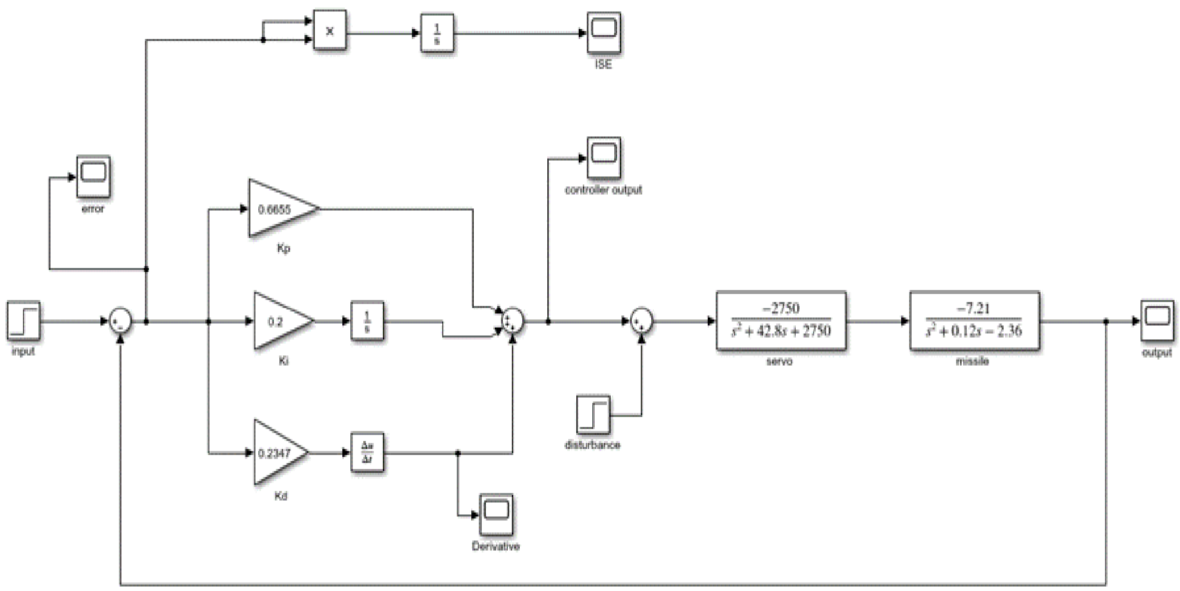

Fig. 11 Simulation model for the whole system

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