Group consensus analysis for first order collective model with spatial coordinates coupling

Fen Nie,Xiaojun Duan,Yicheng Liu

Abstract—In this paper, we studied some consensus and group consensus algorithms for the collective rotating motions of a team of agents, which has been widely studied in different disciplines ranging from physics, networks and engineering. discrete group consensus algorithm when delay are free and consensus algorithm with processing delays are investigated. Based on algebraic matrix theories, graph theories and the properties of Kronecker product, some necessary and sufficient criteria for the consensus and group consensus are derived, where we show that both the eigenvalue distribution of the Laplacian matrix and the Euler angle of the rotation matrix play an important role in achieving group consensus and consensus. Finally, simulation examples are presented to validate the effectiveness of the theoretical results.

Keywords—Multi-agent system (MAS), Fixed topology, Consensus, Group consensus, Processing delay.

I. INTRODUCTION

AS we know, a multi-agent systems consists of a number of agents who communicate with each other via some pairwise links and aims to accomplish various control objectives by local interactions of designated agents. The consensus problems derive from all agents eventually reach an agreement of interest generally determined by their initial stats, first appear in distributed computation and automata theory in computer science [1]. It is important to understand the way these subsystems manage to accomplish a collective behaviour, as such phenomena are observed in nature. These collective behaviours such as flocking, herding, and schooling have been observed in many self-organized systems including fish swimming in schools, birds flying in flocks for the purpose of enhancing the foraging success, and the flight guidance in honeybee swarms. See, for example, Vicsek, Czirok, Ben-Jacob, Cohen, and Shochet[2]; Vicsek [3]; Strogatz [4]; Couzin, Krause, Franks and Levin(2005)[5].Note that the above references all consider such consensus where the states of all agents converge to the same consensus value. However, Due to the changes of situations or cooperative tasks, the consensus values may be different for agents from different sub-networks, in [6] group consensus was introduced, where the states of all agents in the same sub-network reach the same consistent value while there is no agreement between any two sub-networks. group consensus problems for dynamic multi-agent systems were investigated by many researches [7]-[9], in [7], By introducing double-tree-form transformations, solve the first-order group consensus problems in networks of dynamic agents under switching topologies and when exist time-varying communication delays. in [8], By graph theories and matrix theories, solve the first-order group consensus problems of multi-agent systems with directed information exchange. in [9], By algebraic matrix theories, graph theories and the properties of Kronecker product, necessary and sufficient condition for the group consensus of multi-agent systems is established.

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The rest of this paper is organized as follows. In Section 2, some preliminaries and the problem formulation are introduced. group consensus protocol is proposed and the eigenvalues as well as the corresponding eigenvectors of the system matrix are analyzed in Section 3. consensus protocol with delay is proposed in Section 4. Simulation results are presented in Section 5 to demonstrate the effectiveness of the theoretical results. Section 6 concludes this paper with a discussion.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. *Preliminaries*

In this subsection, we will review some basic concepts in graph theory and introduce some lemmas which will be used in this paper.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph with a finite nonempty set of nodes $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$; a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A}$. An edge $e_{ij} = (v_j, v_i) \in \mathcal{E}$ means that node v_i can receive information from node v_j. $A = [a_{ij}]_{n\times n}$ is defined as $a_{ij} \neq 0$ if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$. Moreover, $a_{ii} = 0$ is assumed for all i. A graph is called undirected if $a_{ij} = a_{ji}$. The neighbor set of node v_i is denoted by $N_i = \{v_j | e_{ij} \in \mathcal{E}\}$. For a given network system of dynamic agents, a directed graph $\mathcal{G}$ will be used to model the information communication among all agents.

Lemma 2.1: [10] Let L be the Laplacian matrix of the directed graph $\mathcal{G}$. Then L has a simple zero eigenvalue and all other eigenvalues have positive real parts if and only if $\mathcal{G}$ has a directed spanning tree. let μ_i is eigenvalue of L for $i = 1, 2, \cdots, n$, $\mu_1 = 0$, Moreover, 0 is an eigenvalue of L with an associated right eigenvector $\mathbf{1}_n$, left eigenvector $\mathbf{p}$(nonnegative vector) satisfying $L\mathbf{1}_n = \mathbf{0}_{n\times 1}$, $\mathbf{p}^T L = \mathbf{0}_{1\times n}$ and $\mathbf{p}^T \mathbf{1}_n = 1$.

Lemma 2.2: [11] Given a rotation matrix $C \in \mathbb{R}^{3\times 3}$. Let $\vec{a} = [a_1, a_2, a_3]^T$ be a unit vector in the direction of rotation and let $\theta \in (0, 2\pi)$ be the rotation angle. Then eigenvalues of C are $c_1 = 1, c_2 = e^{\mathbf{j}\theta}, c_3 = e^{-\mathbf{j}\theta}$. If a_2, a_3 not all zero, then we may choose the right eigenvectors of C to be $\varrho_1 = \vec{a}, \varrho_2 = [a_2^2 + a_3^2, -a_1a_2 + a_3\mathbf{j}, -a_1a_3 - a_2\mathbf{j}]^T, \varrho_3 = \bar{\varrho}_2$, left eigenvectors is $\rho_1 = \varrho_1, \rho_2 = \bar{\varrho}_2/|\varrho_2|^2, \rho_3 = \bar{\varrho}_3/|\varrho_3|^2$, where $\mathbf{j} = \sqrt{-1}$ is the imaginary unit , $\bar{\cdot}$ is the conjugate of a complex number. Moreover, $\varrho_l^T\rho_l = 1$, $l = 1, 2, 3$.

Lemma 2.3: [12],[13] Suppose that $U \in \mathbb{R}^{p\times p}, V \in \mathbb{R}^{q\times q}$, U has the eigenvalues β_i with associated eigenvectors $f_i \in \mathbb{C}^p, i = 1, \cdots, p$,and V has the eigenvalues α_j with associated eigenvectors $g_i \in \mathbb{C}^q, i = 1, \cdots, q$, then the pq eigenvalues of $U \otimes V$ are $\beta_i\alpha_j$ with associated eigenvectors of $f_i \otimes g_i$,$i = 1, \cdots, p, j = 1, \cdots, q$.

Lemma 2.4: ([14]) Given a complex coefficient polynomial of order two as follows:

$$h(s) = s^2 + c_1s + c_2, \tag{1}$$

where $c_k = a_k + \mathbf{j}b_k$, a_k, b_k are real constants for $k = 1, 2$. then,$h(s)$ is stable if and only if $a_1 > 0$ and $a_1^2a_2 + b_2(a_1b_1 - b_2) > 0$.

B. Problem formulation

Suppose that the network system under consideration consists of $n + m$ agents. Each agent is regarded as a node in a directed graph $\mathcal{G}$. To analyze the group consensus problem, without loss of generality, we divide the communication network into two sub-networks, where the first n agents belong to the first sub-network and the remaining m agents belong to the second sub-network. $\mathcal{G}_1$ and $\mathcal{G}_2$ are used to model the information communication of these two subnetworks. Suppose that each agent is described by the following dynamics:

$$R_i(k + 1) = R_i(k) + T\mu_i(k).i = 1, 2, \cdots, n + m \tag{2}$$

where $R_i(k) \in \mathbb{R}^3$ and $\mu_i(k) \in \mathbb{R}^3$ are the position and control input of agent i at time instant kT, respectively; T is the sampling period.

Denote

$$R^1(k) = [R_1(k), R_2(k), \cdots, R_n(k)]^T$$

$$R^2(k) = [R_{n+1}(k), R_{n+2}(k), \cdots, R_{n+m}(k)]^T$$

$$\mathcal{L}_1 = \{1, 2, \cdots, n\}, \mathcal{L}_2 = \{n + 1, n + 2, \cdots, n + m\},$$

$$\mathcal{V}_1 = \{v_1, v_2, \cdots, v_n\}, \mathcal{V}_2 = \{v_{n+1}, v_{n+2}, \cdots, v_{n+m}\},$$

$$\mathcal{L} = \mathcal{L}_1 \bigcup \mathcal{L}_2, R(k) = [R^1(k)^T, R^2(k)^T]^T, \mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2$$

$$\mathcal{N}_{i,1} = \{v_j \in \mathcal{V}_1 | e_{ij} \in \mathcal{E}\}, \mathcal{N}_{i,2} = \{v_j \in \mathcal{V}_2 | e_{ij} \in \mathcal{E}\}.$$

then $\mathcal{N}_i = \mathcal{N}_{i,1} \bigcup \mathcal{N}_{i,2}, \mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k, \mathcal{A}_k)$, where $\mathcal{E}_k = \{e_{ij} | i, j \in \mathcal{L}_k\}$ and A_k inherit A, $k = 1, 2$ Therefore, $\mathcal{N}_{i,k}$ can been seen as the neighbor set of agent i in $\mathcal{G}_k$, $k = 1, 2$. Note that $\mathcal{E}_1 \bigcup \mathcal{E}_2$ is a subset of $\mathcal{E}$ as information transition exists not only among agents in the same sub-network but also among agents from different sub-networks, where $\mathcal{E}$ represents the set of edges corresponding to the information communication among all agents in the communication network.

Definition 2.1: System (2) is said to reach consensus asymptotically if for any initial conditions, we have:

$$\lim_{k\to\infty} |R_i(k) - R_j(k)| = 0, \quad \forall i, j \in \mathcal{V}.$$

Definition 2.2: System (2) is said to reach couple-group consensus asymptotically if for any initial conditions, we have:

$$\lim_{k\to\infty} |R_i(k) - R_j(k)| = 0, \quad \forall i, j \in \mathcal{V}_k, k = 1, 2.$$

Assumption 2.1: we make the following assumption as in ([6],[8])

$$(a) \sum_{j=n+1}^{n+m} a_{ij} = 0, \forall i \in \mathcal{L}_1; (b) \sum_{j=1}^{n} a_{ij} = 0, \forall i \in \mathcal{L}_2.$$

These assumptions mean that the interaction between the two subgroups is balanced.

The problem to be addressed in this paper is to design consensus protocol and establish conditions under which couple-group consensus can be achieved by applying proposed protocol.

III. First-order group consensus in directed networks

In this section, group consensus protocol will be designed and the spectrum of the system matrix will be analyzed.

A. Group consensus protocol

To solve the group consensus problem for multi-agent system (2), the following consensus protocol:

$$u_i(k) = \begin{cases} \sum\limits_{\forall v_j \in \mathcal{N}_{i,1}} a_{ij}C(R_j(k) - R_i(k)) + \sum\limits_{\forall v_j \in \mathcal{N}_{i,2}} a_{ij}CR_j(k) \\ \qquad \forall i \in \mathcal{L}_1 \\ \sum\limits_{\forall v_j \in \mathcal{N}_{i,1}} a_{ij}CR_j(k) + \sum\limits_{\forall v_j \in \mathcal{N}_{i,2}} a_{ij}C(R_j(k) - R_i(k)) \\ \qquad \forall i \in \mathcal{L}_2 \end{cases} \tag{3}$$

is proposed, where C denotes the 3×3 rotating matrices. $a_{ij} \geq 0$, for all $i, j \in \mathcal{L}_k, k = 1, 2$, and $a_{ij} \in \mathbb{R}$ for all $i \in \mathcal{L}_1, j \in \mathcal{L}_2$ or $i \in \mathcal{L}_2, j \in \mathcal{L}_1$.

By applying protocol (3), the dynamics of system (2) can be re-written as

$$R(k + 1) = \Phi R(k) \tag{4}$$

where

$$\Phi = I + T\Psi \tag{5}$$

with $\Psi = -L \bigotimes C$,$L = [l_{ij}]$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j=1,j\neq i}^{n+m} a_{ij} & otherwise, \end{cases} \tag{6}$$

$i, j = 1, 2, \cdots, n + m$. From (5), one gets

$$\lambda_i(\Phi) = 1 + T\lambda_i(\Psi), i = 1, 2, \cdots, 3(n + m). \tag{7}$$

where $\lambda_i(\Phi)$ and $\lambda_i(\Psi)$ are the i-th eigenvalues of Φ and Ψ , respectively; and Φ has the same eigenvectors as Ψ. To proceed with the analysis, we first present a necessary lemma about L and restate its proof to derive two important vectors, which will be used in this paper.

Lemma 3.1: ([6]) Under Assumption 21, L has a zero eigenvalue whose geometric multiplicity is at least two.

Proof The definition of L in (6) ensures the property that $\sum_{j=1}^{n+m} l_{ij} = 0$ for all $i = 1, \cdots, n+m$. So, 0 is an eigenvalue of L. Furthermore, one can verify that

$$Lq_i = 0 \cdot q_i, i = 1, 2,$$

where

$$q_1 = [\mathbf{1}_n^T, \mathbf{0}_m^T]^T, q_2 = [\mathbf{0}_n^T, \mathbf{1}_m^T]^T. \qquad (8)$$

q_1 and q_2 are two linearly independent right eigenvectors of L associated with zero eigenvalues.

B. Eigenvalues and eigenvectors of the system matrix

To find out the eigenvalues of Ψ, one needs to solve its characteristic equation

$$\begin{aligned} det(\lambda I_{3(n+m)} - \Psi) &= det(\lambda I_{3(n+m)} + L \bigotimes C) \\ &= \prod_{j=1}^{3} \prod_{i=1}^{n+m} (\lambda + \mu_i c_j) = 0 \qquad (9) \end{aligned}$$

$$\lambda_{i,j}(\Psi) = -\mu_i c_j, i = 1, 2, \cdots, n+m, j = 1, 2, 3. \qquad (10)$$

where $\lambda_{i,j}(\Psi)$ are the eigenvalues of Ψ corresponding to $\mu_i c_j$. From Eqs. (9), we know that $\mu_i = 0$ is equivalent to $\lambda_{i,j}(\Psi) = 0$. So, we have the following result about the relationship of the zero eigenvalues of L and Ψ.

Lemma 3.2: Ψ has an eigenvalue 0 of multiplicity six if and only if L has an eigenvalue 0 of multiplicity two. the right eigenvector of Ψ associated with eigenvalue 0 is given by $q_1 \bigotimes \varrho_l$,$q_2 \bigotimes \varrho_l$, and the left eigenvector given by $p_1^T \bigotimes \rho_l$, $p_2^T \bigotimes \rho_l$, where $l = 1, 2, 3$. $p_1 = [p_{11}^T, p_{12}^T]^T$ and $p_2 = [p_{21}^T, p_{22}^T]^T$ are left eigenvectors of L associated with zero eigenvalues with $p_1^T q_1 = 1$ and $p_2^T q_2 = 1 (p_{11}, p_{21} \in \mathbb{R}^n, p_{12}, p_{22} \in \mathbb{R}^m)$; q_1 and q_2 are defined in Eq. (8).

proof By Eq. (10), if L has an eigenvalue 0 of multiplicity two, then when $j = 1, 2, 3$, Ψ has an eigenvalue 0 of multiplicity six, and the converse is also true.

Let w be a right eigenvector of Ψ corresponding to zero eigenvalues,we have

$$\Psi w = \mathbf{0}_{3(n+m)}$$

$$L \bigotimes C w = \mathbf{0}_{3(n+m)}$$

by the properties of Kronecker product, $w_{2l-1,r} = q_1 \bigotimes \varrho_l, w_{2l,r} = q_2 \bigotimes \varrho_l, l = 1, 2, 3$. Suppose that p_1 and p_2 are left eigenvectors of L corresponding to zero eigenvalues which satisfy $p_1^T q_1 = 1$ and $p_2^T q_2 = 1$. Similarly, it can be proved that $w_{2l-1,l} = p_1 \bigotimes \rho_l, w_{2l,l} = p_2 \bigotimes \rho_l, l = 1, 2, 3$.

Remark 3.1: From Eq. (5), Φ has an eigenvalue 1 of multiplicity six if and only if L has an eigenvalue 0 of multiplicity two.

Remark 3.2: From Eq. (5), Φ has an eigenvalue 1 of multiplicity six if and only if L has an eigenvalue 0 of multiplicity two.

C. Main results

In this section, necessary and sufficient conditions will be derived to solve the couple-group consensus problem for multi-agent system (2) with fixed communication topology.

Theorem 3.1: By applying consensus protocol (3), multi-agent system (2) achieves couple-group consensus asymptotically if and only if Φ has exactly an eigenvalue 1 of multiplicity six and all the other eigenvalues lie inside the unit circle. Furthermore, we have the following results about the final consensus values:

$$|R_i(k) - p_{11}^T R^1(0) + p_{12}^T R^2(0)| \to 0 \quad \forall i \in \mathcal{L}_1$$
$$|R_i(k) - p_{21}^T R^1(0) + p_{22}^T R^2(0)| \to 0 \quad \forall i \in \mathcal{L}_2$$

as $k \to \infty, p_{11}, p_{12}, p_{21}$and $p_{22}$are defined in Eq.(8)

Proof (Sufficiency) Let J be the Jordan canonical form of Φ, then there exists an invertible matrix$P = [w_{1,r}, w_{2,r}, w_{3,r}, w_{4,r}, w_{5,r}, w_{6,r}, ...]$ such that

$$P^{-1} \Phi P = J$$

where

$$J = \begin{bmatrix} J_1 & \mathbf{0}_{6 \times (3(n+m)-6)} \\ \mathbf{0}_{(3(n+m)-6) \times 6} & J_2 \end{bmatrix}, J_1 = I_6$$

J_2 represents the Jordan block corresponding to the other eigenvalues of Φ. To facilitate our analysis, we partition P into block form such as $P = [P_1, P_2]$,where $P_1 = [w_{1,r}, w_{2,r}, w_{3,r}, w_{4,r}, w_{5,r}, w_{6,r}]$, P_2 is the block composed of right eigenvectors or generalized right eigenvectors corresponding to the other eigenvalues of Φ. In a similar way, P^{-1} will be partitioned into $P^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ where $Q_1 = \begin{bmatrix} w_{1,l} & w_{2,l} & w_{3,l} & w_{4,l} & w_{5,l} & w_{6,l} \end{bmatrix}^T$, Q_2 is the block composed of left eigenvectors or generalized left eigenvectors corresponding to the other eigenvalues of Φ. Then, we have

$$\Phi^k = P_1 J_1^k Q_1 + P_2 J_2^k Q_2. \qquad (11)$$

where

$$\begin{aligned} P_1 J_1^k Q_1 &= P_1 I_6 Q_1 \\ &= \sum_{i=1}^{3} (q_1 p_1 + q_2 p_2) \bigotimes (\varrho_i \rho_i) \\ &= \begin{bmatrix} 1_n p_{11}^T & 1_n p_{12}^T \\ 1_m p_{21}^T & 1_m p_{22}^T \end{bmatrix} \bigotimes \sum_{i=1}^{3} (\varrho_i \rho_i) \\ &= \Gamma \bigotimes I_3 \end{aligned}$$

where $\Gamma = \begin{bmatrix} 1_n p_{11}^T & 1_n p_{12}^T \\ 1_m p_{21}^T & 1_m p_{22}^T \end{bmatrix}$ and J_2 satisfies

$$\lim_{k \to \infty} J_2^k = \mathbf{0}_{3(n+m)-6}.$$

When the communication topology is fixed, system (15) will evolve according to the following dynamics:

$$R(k+1) = \Phi^k R(0)$$

Based on Eq. (11), we obtain that

$$\lim_{k \to \infty} \left\| \begin{bmatrix} R^1(k) \\ R^2(k) \end{bmatrix} - \Gamma \bigotimes I_3 \begin{bmatrix} R^1(0) \\ R^2(0) \end{bmatrix} \right\|$$

$$= \lim_{k\to\infty} \left\| \Phi^k \begin{bmatrix} R^1(0) \\ R^2(0) \end{bmatrix} - \Gamma \bigotimes I_3 \begin{bmatrix} R^1(0) \\ R^2(0) \end{bmatrix} \right\|$$

$$= \lim_{k\to\infty} \left\| P_1 J_1^k Q_1 \begin{bmatrix} R^1(0) \\ R^2(0) \end{bmatrix} - \Omega \bigotimes I_3 \right\| = \mathbf{0}$$

where $\Omega = \begin{bmatrix} 1_n p_{11}^T R^1(0) + 1_n p_{12}^T R^2(0) \\ 1_m p_{21}^T R^1(0) + 1_m p_{22}^T R^2(0) \end{bmatrix}$ Therefore, couple-group consensus is achieved asymptotically.

(Necessity) We know from Lemma 3.2 and Remark 3.1 that Φ has an eigenvalue 1 of multiplicity at least six. If the sufficient condition does not hold, Φ has at least seven eigenvalues whose modulus are greater than or equal to 1. Thus, rank$(J^k) > 6$ holds as $k \to \infty$. From the proof of the sufficiency, it is obvious that couple-group consensus is achieved if and only if

$$\lim_{k\to\infty} \Phi^k \to \begin{bmatrix} 1_n p^T \\ 1_m q^T \\ 1_n s^T \\ 1_m t^T \\ 1_n e^T \\ 1_m f^T \end{bmatrix}$$

where p, q, s, t, e and f are arbitrary column vectors with appropriate dimensions. This implies that rank$(\Phi^k) \leq 6$ as $k \to \infty$. Note the fact that rank(Φ^k)=rank(J^k), a contrary is resulted in.

We notice that the algebraic condition in Theorem 3.1 is not straightforward to be checked. Now, for a given communication topology, the following theorem is proposed for choosing appropriate control parameters and sampling period to ensure couple-group consensus.

Theorem 3.2: By applying consensus protocol (3), multi-agent system (2) reaches couple-group consensus asymptotically if and only if L has exactly an eigenvalue 0 of multiplicity two and all the other eigenvalues have positive real parts, meanwhile for $i = 3, 4, \cdots, n+m$,

$$f(T, \theta, \mu_i) = 2\cos(\theta + \arg(\mu_i)) - T|\mu_i| > 0.$$

where μ_i are the non-zero eigenvalues of L.

proof From lemma 3.2, theorem 3.1, we just need to prove $\lambda_{i,j}(\Phi)$ lie inside the unit circle for $i = 3, 4, \cdots, n+m, j = 1, 2, 3.$. By (5) and (10), we can get:

$$\lambda_{i,j}(\Phi) = 1 - T\mu_i c_j, i = 3, 4, \cdots, n+m, j = 1, 2, 3.$$

$$|\lambda_{i,j}(\Phi)| < 1 \Leftrightarrow |1 - T\mu_i c_j| < 1,$$

Equivalent to

$$(1 - T\mu_i c_j)(1 - T\overline{\mu}_i \overline{c}_j) < 1.$$

$$1 - T\mu_i c_j - T\overline{\mu}_i \overline{c}_j + T^2|\mu_i|^2 < 1$$

$$T|\mu_i|^2 < \mu_i c_j + \overline{\mu}_i \overline{c}_j = 2|\mu_i|\cos(\theta + \arg(\mu_i))$$

This completes the proof.

IV. First-order consensus in delayed directed networks

A. Consensus protocol

Because of the time delay, group Consensus discussion becomes very difficult. so, in this section, we assume $m = 0$, discussing the consensus of Multi-Agent Systems, The starting point for our discussion is a continuous framework with delay effects, which embeds both processing delay and transmission delay describing consensus dynamics. Mathematically, we consider the discrete evolution of n agents, x_i denotes the position of ith agent, and each agent adjusts its position according to the position of its neighbors:

$$\frac{d}{dt}x_i(t) = T \sum_{\forall v_j \in \mathcal{N}_i} a_{ij} C(x_j(t-\tau_T-\tau_P) - x_i(t-\tau_P)) \quad (12)$$

where $x_i \in \mathbb{R}^3, i = 1, 2, \cdots, n$, $a_{ij} \geq 0$ are constants for all i, j; τ_P is *processing delay* (i.e., the time it takes agents to process the packet data), and τ_T is *transmission delay* (i.e., the amount of time required to push the information from one agent to another). In general, it costs more time for an agent to process information than to transmit it. That is, $\tau_P > \tau_T$. To normalize the processing delay, set $t = \tau_P s$, $y_i(s) = x_i(\tau_P s)$, we have

$$\begin{aligned} \frac{d}{ds}y_i(s) &= \frac{d}{dt}x_i(\tau_P s) \times \tau_P \\ &= T\tau_P \sum_{\forall v_j \in \mathcal{N}_i} a_{ij} C[x_j(\tau_P s - \tau_T - \tau_P) - x_i(\tau_P s - \tau_P)] \\ &= T\tau_P \sum_{\forall v_j \in \mathcal{N}_i} a_{ij} C[y_j(s - 1 - \frac{\tau_T}{\tau_P}) - y_i(s-1)]. \end{aligned}$$

Then the corresponding discretization equation with unit step size is given by following

$$R_i(k+1) = R_i(k) + T\tau_P \sum_{\forall v_j \in \mathcal{N}_i} a_{ij} C[R_j(k-1-\frac{\tau_T}{\tau_P}) - R_i(k-1)), \quad (13)$$

where $i = 1, 2, \cdots, n, R_i(k) = (r_{1i}(k), r_{2i}(k), r_{3i}(k))$, $k = 1, 2, \cdots$.

In this work, we ignore the effects of transmission delay and consider the effects of processing delay. To this end, let $\tau_P = \tau$ and $\tau_T = 0$ in (13), then we obtain the following first-order difference system with processing delay:

$$R_i(k+1) = R_i(k) + T\tau \sum_{\forall v_j \in \mathcal{N}_i} a_{ij} C(R_j(k-1) - R_i(k-1)). \quad (14)$$

equipping with the initial value $R_i(0) = R_i^0$ and $R_i(1) = R_i^1, i = 1, 2, \cdots, n$.

B. Eigenvalues and eigenvectors of the system matrix

Setting $R^1(k) = (R_1(k), R_2(k), \ldots, R_n(k))^T$, then the system (2) with protocol (14) can be written as

$$R^1(k+1) = R^1(k) - T\tau(L \otimes C)R^1(k-1). \quad (15)$$

Let $X(k) = (R^1(k), R^1(k-1))^T$, then the system would be transmitted as follows:

$$X(k+1) = MX(k), \quad (16)$$

where M is a $6n \times 6n$ matrix which is given by

$$M = \begin{bmatrix} I_{3n} & -T\tau L \otimes C \\ I_{3n} & \mathbf{0}_{3n} \end{bmatrix}.$$

On the other hand, reset $\hat{X}(k) = (R_2(k) - R_1(k), R_3(k) - R_1(k), \ldots, R_n(k) - R_1(k))^T$, then the system (2) with protocol (14) can further be written as

$$\hat{X}(k+1) = \hat{X}(k) - T\tau\tilde{L} \otimes C\hat{X}(k-1), \tag{17}$$

where

$$\tilde{L} = \begin{bmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n2} - l_{12} & \cdots & l_{nn} - l_{1n} \end{bmatrix}$$

Let $\bar{X}(k) = (\hat{X}(k)^T, \hat{X}(k-1)^T)^T$, then the system (17) can be written as follows

$$\bar{X}(k+1) = E\bar{X}(k), \tag{18}$$

where E is a $6(n-1) \times 6(n-1)$ matrix as

$$E = \begin{bmatrix} I_{3(n-1)} & -T\tau\tilde{L} \otimes C \\ I_{3(n-1)} & \mathbf{0}_{3(n-1)} \end{bmatrix}.$$

At this stage, we require some key lemmas.

Lemma 4.1: Let M be given in (16). Then 0 is an eigenvalue of L with algebraic multiplicity m if and only if 1 is an eigenvalues of M with algebraic multiplicity $3m$.

Proof. Compute

$$\begin{aligned} \det(\sigma I_{6n} - M) &= \det \begin{bmatrix} (\sigma-1)I_{3n} & T\tau L \otimes C \\ -I_{3n} & \sigma I_{3n} \end{bmatrix} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{3} m_{ij}(\sigma) = 0 \end{aligned} \tag{19}$$

where

$$m_{ij}(\sigma) = \sigma^2 - \sigma + T\tau\mu_i c_j.$$

$$\sigma_{ij,1} = \frac{1 + \sqrt{1 - 4T\tau\mu_i c_j}}{2}, \sigma_{ij,2} = \frac{1 - \sqrt{1 - 4T\tau\mu_i c_j}}{2}$$

for $i = 1, 2, \ldots, n, j = 1, 2, 3$. Therefore, 1 is an eigenvalues of M with algebraic multiplicity $3m$ if and only if L has a zero eigenvalue with algebraic multiplicity m.

Lemma 4.2: The eigenvalues of the reduced Laplacian matrix $\widetilde{L}$ consist of the rest eigenvalues of Laplacian matrix L except a zero eigenvalue, Moreover, M has three more 1 and 0 eigenvalues than E , and the rest eigenvalues are the same.

Proof. The first part of this lemma can be obtained from the proof of Lemma 1 in [**?**]. now we prove the second part of this lemma. By the proof of Lemma 4.1, we get that

$$\det(\sigma I_{6n} - M) = \det \begin{bmatrix} (\sigma-1)I_{3n} & T\tau L \otimes C \\ -I_{3n} & \sigma I_{3n} \end{bmatrix}$$
$$= \prod_{i=1}^{n}\prod_{j=1}^{3} \sigma^2 - \sigma + T\tau\mu_i c_j$$

$$\begin{aligned} \det(\sigma I_{6n-6} - E) &= \det \begin{bmatrix} (\sigma-1)I_{3(n-1)} & T\tau\widetilde{L} \otimes C \\ -I_{3(n-1)} & \sigma I_{3(n-1)} \end{bmatrix} \\ &= \prod_{i=2}^{n}\prod_{j=1}^{3} \sigma^2 - \sigma + T\tau\mu_i c_j \end{aligned} \tag{20}$$

This implies that M has three more eigenvalues 1 and 0 than E, and the algebraic multiplicity of the other eigenvalues is the same.

It is evident from the previous two lemmas that the system (16) achieves consensus asymptotically if and only if the system (18) is asymptotically stable.

Lemma 4.3: If 0 is a simple eigenvalue of the matrix L, then zero is an eigenvalue of the matrix $L \bigotimes C$ with algebraic multiplicity 3, and 1 is an eigenvalue of the matrix M with algebraic multiplicity 3. Meanwhile, the right eigenvector of M associated with eigenvalue 1 is given by $\begin{pmatrix} \mathbf{1}_n^T \bigotimes \varrho_l^T & \mathbf{1}_n^T \bigotimes \varrho_l^T \end{pmatrix}^T$, and the left eigenvector given by $\begin{pmatrix} \mathbf{p}^T \bigotimes \rho_l & \mathbf{0}_n^T \bigotimes \rho_l \end{pmatrix}$, where $l = 1, 2, 3$.

Proof By Lemma 2.3 and Lemma 4.1, it is clear that if L has a simple zero eigenvalue, then $L \bigotimes C$ has a zero eigenvalue with algebraic multiplicity 3 and the matrix M has an eigenvalue 1 with algebraic multiplicity 3.

Next, we calculate the eigenvector of the eigenvalue 1. we assume $w = \begin{pmatrix} w_a^T, & w_b^T \end{pmatrix}^T$ is the right eigenvector of M, then

$$Mw = \begin{bmatrix} I_{3n} & -T\tau L \bigotimes C \\ I_{3n} & \mathbf{0}_{3n\times 3n} \end{bmatrix} \begin{bmatrix} w_a \\ w_b \end{bmatrix} = \begin{bmatrix} w_a \\ w_b \end{bmatrix}.$$

Thus, we have

$$\begin{cases} I_{3n}w_a - T\tau L \bigotimes C w_b = w_a, \\ I_{3n}w_a = w_b. \end{cases}$$

So w_b is the right eigenvectors of $L \bigotimes C$ associated with the zero eigenvalue, and the right eigenvectors of M associated with the eigenvalue 1 is given by

$$\begin{pmatrix} \mathbf{1}_n^T \bigotimes \varrho_l^T, & \mathbf{1}_n^T \bigotimes \varrho_l^T \end{pmatrix}^T.$$

The left eigenvectors can found similarly.

C. Main results

In this section, necessary and sufficient conditions will be derived to solve the consensus problem for multi-agent system (2) with fixed communication topology.

Theorem 4.1: By applying consensus protocol (14),multi-agent system (2) achieves consensus if and only if the matrix M has exactly an eigenvalue 1 with multiplicity 3 and all the other eigenvalues lie inside the unit circle. In addition, if the consensus is reached, we have

$$\lim_{k\to\infty} \|R_i(k) - \mathbf{p}^T R^1(0)\| = 0. \quad i = 1, 2, \cdots, n.$$

where $\mathbf{p} = (p_1, p_2, \cdots, p_n)^T$ satisfying $\mathbf{p}^T 1_n = 1$ is the unique nonnegative left eigenvector of L associated with zero eigenvalue.

Proof (Necessity) noting that 1 is the eigenvalue of matrix M with algebraic multiplicity 3, by lemma 4.3, we see that the corresponding right eigenvectors associated with the eigenvalue 1 are $(\mathbf{1}_{2n} \otimes \varrho_1)$, $(\mathbf{1}_{2n} \otimes \varrho_2)$ and $(\mathbf{1}_{2n} \otimes \varrho_3)$, the corresponding left eigenvectors associated with the eigenvalue 1 are $(\mathbf{p}^T \otimes \rho_1, \mathbf{0}_n \otimes \rho_1)$, $(\mathbf{p}^T \otimes \rho_2, \mathbf{0}_n \otimes \rho_2)$ and $(\mathbf{p}^T \otimes \rho_3, \mathbf{0}_n \otimes \rho_3)$. They are obviously linear independent. So the geometric multiplicity of the eigenvalue 1 of matrix M is 3 too. There exists a nonsingular matrix $P \in \mathbb{R}^{6n\times 6n}$, such that

$$P^{-1}MP = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0} & \tilde{J} \end{bmatrix},$$

where $\tilde{J}$ is the diagonal matrix composed of Jordan blocks associated with the other eigenvalues of matrix M. Thus

$$M = \begin{bmatrix} \zeta_1, \ldots, \zeta_{6n} \end{bmatrix} \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0} & \tilde{J} \end{bmatrix} \begin{bmatrix} \eta_1^T \\ \vdots \\ \eta_{6n}^T \end{bmatrix}$$

where ζ_j and $\eta_j (j = 1, 2, \ldots, 6n)$ are columns and rows of P and P^{-1}, respectively. Since the eigenvalues of matrix M satisfy $|\sigma| < 1$ except for the eigenvalue $\sigma_{1,2,3} = 1$. Thus $\lim\limits_{k\to+\infty} \tilde{J}^k = \mathbf{0}_{(6n-3)\times(6n-3)}$.

noting that

$$\begin{aligned}
& \lim_{k\to+\infty} X(k) = \lim_{k\to+\infty} M^k X(0) \\
= & \begin{pmatrix} \mathbf{1}_{2n} \otimes \varrho_1, \mathbf{1}_{2n} \otimes \varrho_2, \mathbf{1}_{2n} \otimes \varrho_3, \ldots \end{pmatrix} \\
& \begin{pmatrix} I_3 & \mathbf{0} \\ \mathbf{0} & \lim_{k\to+\infty} \tilde{J}^k \end{pmatrix} \begin{pmatrix} \mathbf{p}^T \otimes \rho_1 & \mathbf{0}_n^T \otimes \rho_1 \\ \mathbf{p}^T \otimes \rho_2 & \mathbf{0}_n^T \otimes \rho_2 \\ \mathbf{p}^T \otimes \rho_3 & \mathbf{0}_n^T \otimes \rho_3 \\ & \vdots \end{pmatrix} X(0) \\
= & \sum_{i=1}^{3} ((\mathbf{1}_n^T \quad \mathbf{1}_n^T)^T \otimes \varrho_i)((\mathbf{p}^T \quad \mathbf{0}_n^T) \otimes \rho_i) X(0) \\
= & \begin{pmatrix} \mathbf{1}_n \mathbf{p}^T & \mathbf{1}_n \mathbf{0}_n^T \\ \mathbf{1}_n \mathbf{p}^T & \mathbf{1}_n \mathbf{0}_n^T \end{pmatrix} \otimes \sum_{i=1}^{3} \varrho_i \rho_i X(0) \\
= & \begin{pmatrix} \mathbf{1}_n \mathbf{p}^T & \mathbf{1}_n \mathbf{0}_n^T \\ \mathbf{1}_n \mathbf{p}^T & \mathbf{1}_n \mathbf{0}_n^T \end{pmatrix} \otimes I_3 X(0).
\end{aligned}$$

Thus

$$\lim_{k\to+\infty} R^1(k) = \mathbf{1}_n \mathbf{p}^T \otimes I_3 R^1(0).$$

so the consensus value is $\mathbf{p}^T R^1(0)$, we have

$$\lim_{k\to\infty} \|R_i(k) - \mathbf{p}^T R^1(0)\| = 0. \quad i = 1, 2, \cdots, n.$$

(Sufficiency) Suppose to the contrary, if the matrix M has exactly an eigenvalue 1 with multiplicity 3 and all the other eigenvalues are stay in the unit disk is not satisfied, then by Lemma 4.1, the multiplicity of 1 eigenvalue in M is at least 3 since L has a zero eigenvalue at least. Hence, there are three cases needed to be discussed:

Case I: The multiplicity of 1 eigenvalue in M is 3, and there exists at least an eigenvalue which is not in the unit disk;

Case II: The multiplicity of 1 eigenvalue in M is more than 3, and the rest eigenvalues are in the unit disk;

Case III: The multiplicity of 1 eigenvalue in M is more than 3, and there exists at least an eigenvalue which is not in the unit disk.

For Case I, by Lemma 4.2, if M has an eigenvalue which is not in the unit circle, then E also has an eigenvalue which is not in the unit circle. Therefore, the stability of system (18) cannot be achieved, which means that the consensus of system (16) cannot be achieved. Similarly, we can prove Case II and Case III. This completes the proof.

As the same with Theorem 3.1 and Theorem 3.2. Now, for a given communication topology, the following theorem is proposed for choosing appropriate control parameters and sampling period to ensure consensus.

Theorem 4.2: By applying consensus protocol (14),multi-agent system (2) achieves consensus asymptotically if and only if the digraph $\mathcal{G}$ has a directed spanning tree and all the other eigenvalues have positive real parts, meanwhile for $i = 2, 3, \cdots, n$,

$$\begin{cases} g_1(T, \tau, \theta, \mu_i) = \cos(\theta + \arg \mu_i) - T\tau|\mu_i| > 0 \\ g_2(T, \tau, \theta, \mu_i) = \cos(\theta + \arg \mu_i) - \frac{3T\tau|\mu_i| - T^3\tau^3|\mu_i|^3}{2} > 0 \end{cases} \tag{21}$$

Proof (Sufficiency) It follows from Theorem 4.1 that if multi-agent system (2) with the protocol (14) achieves asymptotical consensus, then 1 is an eigenvalue of matrix M with algebraic multiplicity three and and all the other eigenvalues lie inside the unit circle. By Lemma 4.1 and Lemma 2.1, matrix L has a simple zero eigenvalue, which implies that $\mathcal{G}$ has a directed spanning tree.

Meanwhile, considering the characteristic equation (20), by applying the bilinear transformation $s = \frac{\sigma+1}{\sigma-1}$ to m_{ij}, we get a series of new polynomials

$$\begin{aligned} f_i(s) &= (s-1)^2((\frac{s+1}{s-1})^2 - \frac{s+1}{s-1} + T\tau\mu_i c_j) \\ &= s^2 T\tau\mu_i c_j + 2s(1 - T\tau\mu_i c_j) + 2 + T\tau\mu_i c_j. \end{aligned}$$

Define $\gamma_i(s)$ $(i = 2, 3, \cdots, n)$ as

$$\gamma_i(s) = \frac{f_i(s)}{T\tau\mu_i c_j} = s^2 + (\frac{2}{\iota} - 2)s + (\frac{2}{\iota} + 1), \tag{22}$$

where $\iota = T\tau\mu_i c_j$.

noting that the properties of bilinear function, we see that all roots of (20) are inside the unit disk if and only if all roots of $\gamma_i(s) = 0$ have negative real parts for $i = 2, 3, \cdots, n$. Let $a_1 + b_1\mathbf{j} = \frac{2}{\iota} - 2, a_2 + b_2\mathbf{j} = \frac{2}{\iota} + 1$, then we see that $b_1 = b_2, a_1 = a_2 - 3$. It follows from Lemma 2.4 that all roots of $\gamma_i(s) = 0$ have negative real parts if and only if

$$a_1 > 0, a_1^2(a_1 + 3) + a_1 b_1^2 - b_1^2 > 0.$$

noting the fact that

$$\begin{aligned} a_1 &= \Re(\frac{2}{T\tau\mu_i c_j} - 2) = \Re(\frac{2\overline{\mu_i c_j}}{T\tau|\mu_i|^2} - 2) \\ &= \frac{2\cos(\theta + \arg \mu_i)}{T\tau|\mu_i|} - 2 > 0 \end{aligned}$$

if and only if

$$\cos(\theta + \arg \mu_i) - T\tau|\mu_i| > 0$$

Also, by direct calculation, we get

$$\begin{aligned} a_1^2 + b_1^2 &= (\frac{2}{\iota} - 2)(\frac{2}{\bar{\iota}} - 2) = 4(\frac{1}{|\iota|^2} + 1 - \frac{2\Re(\iota)}{|\iota|^2}) \\ &= 4(\frac{1}{T^2\tau^2|\mu_i|^2} + 1 - \frac{2\cos(\theta + \arg \mu_i)}{T\tau|\mu_i|}) \end{aligned}$$

and

$$a_1^2 = 4(\frac{\cos(\theta + \arg \mu_i)}{T\tau|\mu_i|} - 1)^2.$$

It follows from

$$a_1^2(a_1 + 3) + a_1 b_1^2 - b_1^2 > 0$$

that

$$(a_1 - 1)(a_1^2 + b_1^2) + 4a_1^2 > 0.$$

This implies that

$$\cos(\theta + \arg \mu_i) - \frac{3T\tau|\mu_i| - T^3\tau^3|\mu_i|^3}{2} > 0$$

Hence the sufficiency.

(necessity)if θ satisfy (21), we have that all the roots of (20) stay inside the unit disk for each $i = 2, 3, \cdots, n$. It implies that the eigenvalues of M are lie inside the unit circle except eigenvalue 1. Since the digraph $\mathcal{G}$ contains a directed spanning tree, we have that the Laplacian matrix L has a simple zero-eigenvalue. By Lemma 4.2, 1 is not the eigenvalue of matrix E, but 1 is the eigenvalue of matrix M with algebraic multiplicity three. By Theorem 4.1, system (16) achieves consensus asymptotically. This completes the proof of Theorem 4.2.

V. Simulation Examples

A. Couple-group consensus of a multi-agent system with directed topology

Consider a multi-agent system (2) applying consensus protocol (3) with directed topology, where

$$L = \begin{bmatrix} 1 & -1 & 0 & 5 & 0 & -4 & -1 \\ -2 & 3 & -1 & -1 & 0 & 0 & 1 \\ 0 & -2 & 2 & 0 & -3 & 0 & 3 \\ 1 & -1 & 0 & 2 & 0 & 0 & -2 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -2 & 0 & -1 & 3 \end{bmatrix}$$

numerical computation shows that L has eigenvalues $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0.4840 + 2.6710i, \mu_4 = 0.4840 - 2.6710i, \mu_5 = 6.5046, \mu_6 = 3.5427, \mu_7 = 1.9847$,from Theorem 3.1,$\Phi$ has an eigenvalue 1 of multiplicity six and the rest eigenvalues modulus are less than 1,the final consensus values $(1.5341, 2.5341, 3.5341)$ for $i \in \mathcal{L}_1$,$(13.4273, 14.4273, 15.4273)$ for $i \in \mathcal{L}_2$,from Theorem 3.2,$f(0.01, 5, \mu_i) > 0$, for $i = 3, 4, \cdots, 7$. Evolutions of the position states of all agents are shown in Fig. 1, where $R^1(0) = [1, 2, 3, 4, 5, 6, 7, 8, 9]$,$R^2(0) = [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]$, Couple-group consensus is achieved as guaranteed by theory. The position states of all agents are shown in Fig. 2, where $f(0.01, 10, \mu_3) < 0$ and the initial conditions are the same as which in Fig. 1. Couple-group consensus of the multi-agent system cannot be reached.

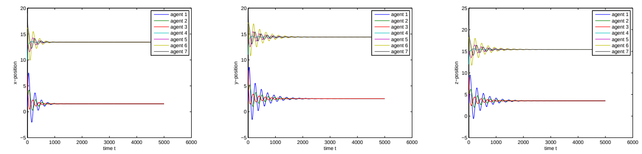

Fig. 1. x, y, z position states of all agents, where $\theta = 5\ degree, T = 0.01$.

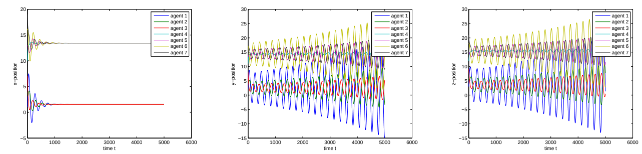

Fig. 2. x, y, z position states of all agents, where $\theta = 10\ degree, T = 0.01$.

B. consensus of a multi-agent system with delay directed topology

Consider a multi-agent system (2) applying consensus protocol (14) with directed topology, where

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3 & -1 & -2 \\ -1 & -4 & 5 & 0 \\ -1 & 0 & -3 & 4 \end{bmatrix} \tag{23}$$

numerical computation shows that L has eigenvalues $\mu_1 = 0, \mu_2 = 5.7869 + 2.1051\mathbf{j}, \mu_3 = 5.7869 - 2.1051\mathbf{j}$ and $\mu_4 = 2.4262$, from Theorem 3.1,M has an eigenvalue 1 of multiplicity six and the rest eigenvalues modulus are less than 1,the final consensus values $(2.4565 \quad 4.5326 \quad 4.2500)$, for $i \in \mathcal{L}$, from Theorem 3.2, $g_1(0.04, 3, \frac{\pi}{4}, \mu_i) > 0, g_2(0.04, 3, \frac{\pi}{4}, \mu_i) > 0$ for $i = 2, 3, 4$. Evolutions of the position states of all agents are shown in Fig. 3, where $R^1(0) = [2, 3, 5, 2, 8, 2, 2, 2, 2, 4, 3, 9]$, consensus is achieved as guaranteed by theory. The position states of all agents are shown in Fig. 4, where $g_2(0.043, 3, \frac{\pi}{4}, \mu_2) < 0$ and the initial conditions are the same as which in Fig. 3. consensus of the multi-agent system cannot be reached.

VI. Conclusion

In this paper, couple-group consensus and dalay consensus problem for discrete-time first-order multi-agent systems is

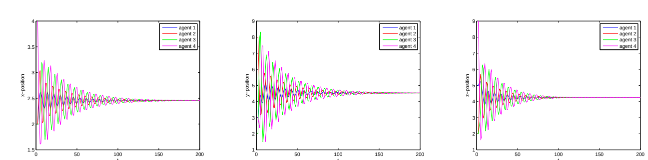

Fig. 3. x, y, z position states of all agents, where $\theta = \frac{\pi}{4}, T = 0.04$.

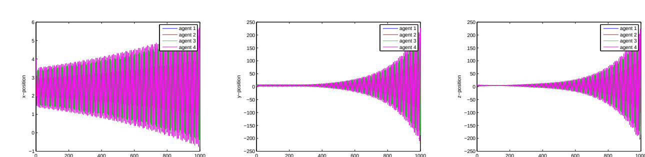

Fig. 4. x, y, z position states of all agents, where $\theta = \frac{\pi}{4}, T = 0.043$.

investigated for networks with fixed communication topology. Consensus protocol is designed and some necessary and sufficient conditions are established to ensure first-order couple-group consensus and consensus with delay. It is found that couple-group consensus and dalay consensus will be reached only if the nonzero eigenvalues of the Laplacian matrix all have positive real parts. Simulation examples are presented to demonstrate the effectiveness of the theoretical results.

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