Transient analysis of a repairable single server queue with working vacations and system disasters

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***Abstract*—This study investigates the repairable single server queue with working vacations and system disasters. The server allows to take a working vacation if there is no any customers in the system. There is a possibility of breakdowns happening in a system. When the system occurs server breakdowns, the server goes to the failure state and all customers in the queue are flushed away. The repairing process starts immediately, when the server comes to the failure state. The explicit expression for system size probabilities of the queueing system is derived in terms of the modified Bessel function of first kind using the probability generating function method, Laplace transform and continued fractions. Additionally, the mean and variance for number of jobs in the system at time t are derived as the performance measures.**

***Index Terms*—$M/M/1$ queue, system break-downs, working vacations, transient solution**

I. Introduction

Applications of queueing model with vacations exist in various fields such as network service, web service, file transfer service and mail service. Working vacation is one type of the vacation policies and Servi and Finn [19] introduced this concept generalizing the classical single server vacation model. They derived the explicit expressions for the mean, variance of the number of customers in the queue. In working vacation duration, the server serves the customers with a lower rate than the normal service rate. This may be a reason to reduce the leaving of customers from the system during the vacation period. Wu and Takagi [25] have derived the expressions for the number of jobs in the queue and the response time for an arbitrary customer extending the $M/M/1/WV$ model to new $M/G/1/WV$ model. A $GI/M/1$ queue with multiple working vacations was analyzed by Baba [3] to obtain the steady state result for the system size in the queue both at arrival and arbitrary epochs. Banik et al. [5] discussed the finite buffer single server $GI/M/1$ queue with multiple working vacations and they presented the distribution for number of customers in the system at pre-arrival and arbitrary epoch. Do [8] obtained time independent expression for the retrial $M/M/1$ queue with working vacations. Yang et al. [26] applied the matrix-analytic method to derive the steady-state probabilities and some system characteristics of the F -policy $M/M/1/K$ queueing system with working vacation. Baba [4] has discussed about $M^X/M/1$ queue with multiple working vacation and he obtained the probability distribution for system size and some of the performance measures for the queueing system considering that the server is in its equilibrium state. Arivudainambi et al. [2] have derived stationary result for a single server retrial queue introducing the concept of single working vacation. The $M/G/1$ queue with working vacations has been analyzed by Aissani et al. [1] to derive the expressions for joint probability distribution of the server state and system size probabilities of the queue when the server is in steady-state by using the Laplace and z-transforms. Recently, Vijaya Laxmi and Rajesh [23] analyzed single sever queue with customer impatience and variant working vacation policy. They obtained the explicit expression for system size probabilities and some performance measures at steady state.

Gelenbe [9] introduced the notion of catastrophes and it has been gaining significant scholarly attention during the last few decades since their applications are widely used in service systems, computer systems, manufacturing systems and. Catastrophes occur at random time when server is to complete the servicing for all the customers at that time or the server inactivates until a new arrival. This situation can be considered as negative customer arrival in queueing system and they have a property to remove all the customers or some of them in the queueing system. It may be possible to happen either from another service station or from outside the system. A mail server with an infected virus can be considered as an example for a queueing system with catastrophes. Since this email transmit the virus during its transferring to the other processors, disasters may occur to clear the operation of all emails stored in the system. Krishna Kumar and Arivudainambi [14] analyzed the transient solution for an $M/M/1$ queue with catastrophes. Chao [6] has extended the research which has been done by Di Crescenzo et al. [7] for the $M/M/1$ queue with catastrophes to a network of queues. An $M/M/R/N$ queueing system with balking, reneging and server break-downs was analyzed by Wang and Chang [24].

Queueing system with repairable servers often arise in the field of computer and communication switching systems and web servicing systems where the processors have to handle failing and repairing of them [16]. Therefore, the studying of queues subjected to catastrophes and breakdowns and repairable servers has got more attention of the researchers. An $M/M/1$ queue which has N servers with server breakdowns and repairs has been analyzed by Neuts and Lucantoni [17]. A single server priority queue with server failures and queue flushing has been discussed by Towsley and Tripathi [22]. The transient solution for an $M/M/1$ queue subjected to catastrophes with server failure and non-zero repairable time has been derived by Krishna Kumar and Pavavi [15]. Giorno et al. [10] has obtained jump diffusion approximation for a double ended queue with catastrophes and repairs. Kalidass and et al. [13] derived the transient solution of an N-policy single server queueing system with catastrophes and repairable server. A single server queueing system with balking, catastrophes, server failures and repairs was analyzed by Tarabia [21] extending the model of Krishna Kumar and Pavai [15] with balking feature and he has obtained transient and steady state probabilities with the use of probability generating function technique and a direct approach.

Yechiali [27] has obtained the time independent probabilities of the system size of the queue with system break downs and customer impatience. Expanding this model, Sudesh [20] derived the transient solutions for the probabilities of number of customers in the system with the use of generating function methods and continued fractions. Considering an $M/M/1$ queue with working vacation and multiple types of server breakdowns, the distribution for number of jobs in the system was derived by Jain and Jain [12]. An $M/M/1$ queueing system with second optional service and unreliable server has been extensively researched. Using the matrix geometric technique, Jain and Chauhan [11] have analyzed a single server queue with unreliable server and second optional service. Dealing with a feedback retrial $M/G/1$ queue with multiple working vacations and vacation interruption, Rajadurai et al. [18] has obtained the time independent probabilities for the system size and some performance measures.

In existing literature, analyzing a repairable single server queue with working vacations and system disasters in transient state is less researched. Therefore, in this research, the transient solutions of an $M/M/1$ queue with working vacations and system disasters are obtained using Laplace transform, probability generating function technique and continued fractions. As the performance measures, mean and variance of the system size are explicitly expressed. The findings of this study is applicable in manufacturing systems, computer communication systems, network systems and inventory systems etc. Therefore, the results of this research may help people who use queueing theory to deal with congestion problems in the systems.

This paper has organized as follows. Section II includes the model for a repairable single server queue with working vacations and system disasters in transient state. The results of the explicit expressions for the time dependent system size probabilities are derived in section III. Section IV presents the time dependent expected values. Conclusions of this work is discussed under section V.

II. Model Description

A single server queueing model with system failure and working vacations is considered. The assumptions of the system are build up as follows:

1) Arrivals are allowed to join the system according to a Poisson process with rate λ and service takes place according to an exponential distribution with rate μ.
2) The server takes a working vacation when there are no customers in the system. Working vacation policy has an exponential distribution with mean $1/\gamma$ and the server serves the customers with service rate $\mu_v(< \mu)$ during the working vacation.
3) The system faces server breakdowns at a Poisson rate η. It means that life time of the system is exponentially distributed with mean $1/\eta$. When it suffers a server breakdown, all customers in the queue are flushed away and the server goes to the failure state.
4) The repairing process is started immediately, when server comes to the failure state and the repair time has an exponential distribution with mean $1/\nu$.
5) It is assumed that inter-arrival times, service times, repair times and vacation times are mutually independent and the service discipline is First-In, First-Out (FIFO).

Let $\{X(t), t \geq 0\}$ denotes the total number of customers in the system at time t and let $J(t)$ represents the state of the system at time t, which is defined as follows:

$$J(t) = \begin{cases} 0, & \text{if the server being in failure state} \\ & \text{at time } t \\ 1, & \text{if the server being in functional state} \\ & \text{at time } t \\ 2, & \text{if the server being in working vacation} \\ & \text{at time } t \end{cases}$$

Then $\{J(t), X(t), t \geq 0\}$ is a two-dimensional continuous time Markov process on the state space $S = \{(j,n); j = 0,1,2; n = 0,1,2,...\}$. Let $P_{j,n}(t)$ be the time dependent probabilities for the system to be in the state j with n customers at time t. Let

$$\begin{aligned} P_{0,n}(t) = &\quad Prob\{J(t)=0, X(t)=n\}, \quad n=0,1,2,.... \\ P_{1,n}(t) = &\quad Prob\{J(t)=1, X(t)=n\}, \quad n=1,2,3,.... \\ P_{2,n}(t) = &\quad Prob\{J(t)=2, X(t)=n\}, \quad n=0,1,2,.... \end{aligned}$$

Then, the set of forward Kolmogorov differential difference equations governing the process is given by

$$P_{0,0}^{'}(t)=-(\lambda+\nu)P_{0,0}(t)+\eta\sum_{n=1}^{\infty}\left(P_{1,n}(t)+P_{2,n}(t)\right) \quad (1)$$

$$P_{0,n}^{'}(t)=\lambda P_{0,n-1}(t)-(\lambda+\nu)P_{0,n}(t), n\geq 1 \quad (2)$$

$$P_{1,1}^{'}(t)=-(\lambda+\mu+\eta)P_{1,1}(t)+\mu P_{1,2}(t)+\nu P_{0,1}(t) +\gamma P_{2,1}(t) \quad (3)$$

$$P_{1,n}^{'}(t)=\lambda P_{1,n-1}(t)-(\lambda+\mu+\eta)P_{1,n}(t)+\mu P_{1,n+1}(t) +\nu P_{0,n}(t)+\gamma P_{2,n}(t); n\geq 2 \quad (4)$$

$$P_{2,0}^{'}(t)=-\lambda P_{2,0}(t)+\mu_v P_{2,1}(t)+\mu P_{1,1}(t)+\nu P_{0,0}(t) \quad (5)$$

$$P_{2,n}^{'}(t)=\lambda P_{2,n-1}(t)-(\lambda+\mu_v+\eta+\gamma)P_{2,n}(t) +\mu_v P_{2,n+1}(t); n\geq 1 \quad (6)$$

Initially, it is assumed that there are no customers in the queueing system and the server being in the working vacation state, i.e., $P_{0,0}(0)=0$ and $P_{2,0}(0)=1$ and $P_{j,n}(0)=0$ for $n\geq 1$ and $j=0,1,2$.

III. Transient Probabilities

In this section, the transient solution of the above described model is derived by employing generating functions, Laplace transform and continued fractions. Time dependent analysis is used to understand the behavior of a queueing system, when the parameters are perturbed.

A. Evaluation of $P_{1,n}(t)$

Define the generating function as follows, for $|z|\leq 1$

$$P(z,t)=\sum_{n=1}^{\infty}P_{1,n}(t)z^n,$$

with initial condition $P(z,0)=0$.

Multiplying the Equations (3) and (4) by appropriate powers of z and summing over $n\geq 1$, we can obtain

$$\frac{\partial P(z,t)}{\partial t}=\sum_{n=1}^{\infty}P_{1,n}^{'}(t)z^n$$

$$\frac{\partial P(z,t)}{\partial t}=-\left[\lambda(1-z)+\mu(1-z^{-1})+\eta\right]P(z,t) +\nu\sum_{n=1}^{\infty}P_{0,n}(t)z^n+\gamma\sum_{n=1}^{\infty}P_{2,n}(t)z^n -\mu P_{1,1}(t) \quad (7)$$

Since the Equation (7) is a first-order partial differential equation for $P(z,t)$, after solving the Equation (7) using the integrating factor $\exp\left\{\left[\lambda(1-z)+\mu(1-z^{-1})+\eta\right]t\right\}$, we will have

$$P(z,t)=\nu\int_0^t\left(\sum_{m=1}^{\infty}P_{0,m}(u)z^m\right) \times e^{-\left[\lambda(1-z)+\mu(1-z^{-1})+\eta\right](t-u)}du +\gamma\int_0^t\left(\sum_{m=1}^{\infty}P_{2,m}(u)z^m\right) \times e^{-\left[\lambda(1-z)+\mu(1-z^{-1})+\eta\right](t-u)}du -\mu\int_0^t P_{1,1}(u)e^{-\left[\lambda(1-z)+\mu(1-z^{-1})+\eta\right](t-u)}du \quad (8)$$

It is well known that if $\alpha=2\sqrt{\lambda\mu}$ and $\beta=\sqrt{\frac{\lambda}{\mu}}$, then

$$\exp\left[\left(\lambda z+\frac{\mu}{z}\right)t\right]=\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind.

Substituting this equation to the equation (8), we have

$$P(z,t)=\nu\int_0^t\left(\sum_{m=1}^{\infty}P_{1,m}(u)z^m\right)e^{-(\lambda+\mu+\eta)(t-u)} \times\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha(t-u))du +\gamma\int_0^t\left(\sum_{m=1}^{\infty}P_{2,m}(u)z^m\right)e^{-(\lambda+\mu+\eta)(t-u)} \times\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha(t-u))du -\mu\int_0^t P_{1,1}(u)e^{-(\lambda+\mu+\eta)(t-u)} \times\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha(t-u))du \quad (9)$$

Comparing the coefficients of z^n in the Equation (9) for $n=1,2,3,....$ leads to

$$P_{1,n}(t)=\nu\int_0^t\sum_{m=1}^{\infty}P_{0,m}(u)\beta^{n-m}I_{n-m}(\alpha(t-u)) \times e^{-(\lambda+\mu+\eta)(t-u)}du +\gamma\int_0^t\sum_{m=1}^{\infty}P_{2,m}(u)\beta^{n-m}I_{n-m}(\alpha(t-u)) \times e^{-(\lambda+\mu+\eta)(t-u)}du -\mu\int_0^t P_{1,1}(u)\beta^n I_n(\alpha(t-u)) \times e^{-(\lambda+\mu+\eta)(t-u)}du \quad (10)$$

Using the fact that $I_{-n}(\cdot)=I_n(\cdot)$ and comparing the coefficients of z^{-n} in the Equation (9) yields

$$0=\nu\int_0^t\sum_{m=1}^{\infty}P_{0,m}(u)\beta^{n-m}I_{n+m}\left(\alpha(t-u)\right)\times e^{-(\lambda+\mu+\eta)(t-u)}du+\gamma\int_0^t\sum_{m=1}^{\infty}P_{2,m}(u)\beta^{n-m}I_{n+m}\left(\alpha(t-u)\right)\times e^{-(\lambda+\mu+\eta)(t-u)}du-\mu\int_0^t P_{1,1}(u)\beta^n I_n\left(\alpha(t-u)\right)e^{-(\lambda+\mu+\eta)(t-u)}du \quad (11)$$

Subtracting the Equation (10) from the Equation (11) for $n=1,2,3,...$, we have

$$P_{1,n}(t)=\nu\int_0^t\sum_{m=1}^{\infty}P_{0,m}(u)\beta^{n-m}\left[I_{n-m}\left(\alpha(t-u)\right)-I_{n+m}\left(\alpha(t-u)\right)\right]e^{-(\lambda+\mu+\eta)(t-u)}du+\gamma\int_0^t\sum_{m=1}^{\infty}P_{2,m}(u)\beta^{n-m}\left[I_{n-m}\left(\alpha(t-u)\right)-I_{n+m}\left(\alpha(t-u)\right)\right]e^{-(\lambda+\mu+\eta)(t-u)}du \quad (12)$$

It is clear that $P_{1,n}(t)$ are expressed in terms of $P_{0,n}(t)$ and $P_{2,n}(t)$ and they are given by the Equations (15) and (19) respectively.

B. *Evaluation of* $P_{0,n}(t)$

$\hat{P}_{j,n}(s)$ represents Laplace transform of $P_{j,n}(t)$. Taking the Laplace transform of the Equation (2), we can obtain

$$s\hat{P}_{0,n}(s)-P_{0,n}(s)=-(\lambda+\nu)\hat{P}_{0,n}(s)+\lambda\hat{P}_{0,n-1}(s)$$

Substituting the initial value and after some algebra, we have

$$\hat{P}_{0,n}(s)=\left(\frac{\lambda}{s+\lambda+\nu}\right)^n\hat{P}_{0,0}(s)$$

We can obtain the following equation after taking the Laplace transform of the Equation (1) and applying the initial condition

$$(s+\lambda+\nu)\hat{P}_{0,0}(s)=\eta\sum_{n=1}^{\infty}\left(\hat{P}_{1,n}(s)+\hat{P}_{2,n}(s)\right) \quad (13)$$

Clearly for $t>0$,

$$\sum_{n=0}^{\infty}P_{0,n}(t)+\sum_{n=1}^{\infty}P_{1,n}(t)+\sum_{n=0}^{\infty}P_{2,n}(t)=1$$

The above equation can be expressed as follows after taking Laplace transform and some algebra

$$\sum_{n=1}^{\infty}\left(\hat{P}_{1,n}(s)+\hat{P}_{2,n}(s)\right)=\frac{1}{s}-\hat{P}_{2,0}(s)-\sum_{n=0}^{\infty}\hat{P}_{0,n}(s)$$

Substituting the above equation to the Equation (13) and after some mathematical calculations, we are able to derive

$$\hat{P}_{0,0}(s)=A(s)\left[\frac{1}{s}-\hat{P}_{2,0}(s)\right] \quad (14)$$

where

$$A(s)=\frac{\eta}{(s+\lambda+\nu)}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\eta}{s+\nu}\right)^k$$

We will have the following equation after taking inversion of the above equation,

$$P_{0,0}(t)=A(t)*[1-P_{2,0}(t)]$$

where

$$A(t)=e^{-(\lambda+\nu)t}\sum_{k=0}^{\infty}(-1)^k\eta^{k+1}e^{-\nu t}\frac{t^{k-1}}{(k-1)!}$$

Then

$$\hat{P}_{0,n}(s)=A(s)\left[\frac{1}{s}-\hat{P}_{2,0}(s)\right]\left(\frac{\lambda}{s+\lambda+\nu}\right)^n$$

After taking inverse Laplace transform transform of the above equation, we have

$$P_{0,n}(t)=\lambda^n A(t)*(1-P_{2,0}(t))*e^{-(\lambda+\nu)t}\frac{t^{n-1}}{(n-1)!} \quad (15)$$

where "*" denotes the convolution. The terms for $P_{0,0}(t)$ and $P_{0,n}(t)$ are expressed in terms of $P_{2,0}(t)$ which is given by the Equation (21).

C. *Evaluation of* $P_{2,n}(t)$

Laplace transform can be used to derive the following equation by the Equation (6),

$$s\hat{P}_{2,n}(s)-P_{2,n}(0)=\lambda\hat{P}_{2,n-1}(s)-(\lambda+\mu_v+\eta+\gamma)\hat{P}_{2,n}(s)+\mu_v\hat{P}_{2,n+1}(s)$$

Applying the initial condition to the above equation and after some algebra, we have

$$\frac{\hat{P}_{2,n}(s)}{\hat{P}_{2,n-1}(s)}=\frac{\lambda}{(s+\lambda+\mu_v+\eta+\gamma)-\mu_v\frac{\hat{P}_{2,n+1}(s)}{\hat{P}_{2,n}(s)}}$$

Iterating the above equation, we have

$$\frac{\hat{P}_{2,n}(s)}{\hat{P}_{2,n-1}(s)}=\frac{\lambda}{(s+\lambda+\mu_v+\eta+\gamma)-\frac{\lambda\mu_v}{(s+\lambda+\mu_v+\eta+\gamma)-\frac{\lambda\mu_v}{(s+\lambda+\mu_v+\eta+\gamma)-\ldots\ldots}}}$$

It can be rewritten as follows

$$\frac{\hat{P}_{2,n}(s)}{\hat{P}_{2,n-1}(s)}=\frac{\lambda}{(s+\lambda+\mu_v+\eta+\gamma)-\Phi(s)} \quad (16)$$

where

$$\Phi(s)=\frac{\lambda\mu_v}{(s+\lambda+\mu_v+\eta+\gamma)-\frac{\lambda\mu_v}{(s+\lambda+\mu_v+\eta+\gamma)-\ldots\ldots}} \quad (17)$$

Clearly $\Phi(s)$ satisfies the quadric equation $\Phi^2(s) - (s+\lambda+\mu_v+\eta+\gamma)\,\Phi(s) + \lambda\mu_v = 0$. This equation has two roots $\frac{P+\sqrt{P^2-4\lambda\mu_v}}{2}$ and $\frac{P-\sqrt{P^2-4\lambda\mu_v}}{2}$. Here, since $\frac{P-\sqrt{P^2-4\lambda\mu_v}}{2} < 1$, it is the real root of $\Phi(s)$. Where $P = s+\lambda+\mu_v+\eta+\gamma$.

Substituting $\Phi(s)$ to the Equation (17) and after some algebra, we will have

$$\hat{P}_{2,n}(s) = \left(\frac{2\lambda}{P+\sqrt{P^2-\theta^2}}\right)^n \hat{P}_{2,0}(s) \tag{18}$$

where $\theta = 2\sqrt{\lambda\mu_v}$.

Taking the Laplace transform of the above equation, we can obtain

$$P_{2,n}(t) = \left(\frac{2}{\theta}\right)^{n-1} \lambda^n \left[I_{n-1}(\theta t) - I_{n+1}(\theta t)\right] \times e^{-(\lambda+\mu_v+\eta+\gamma)t} * P_{2,0}(t) \tag{19}$$

It is clear that $P_{2,n}(t)$ are expressed in terms of $P_{2,0}(t)$ which is given by the Equation (21). Where "*" denotes the convolution.

D. Evaluation of $P_{2,0}(t)$

We have the following equation after taking the Laplace transform of the Equation (5)

$$s\hat{P}_{2,0}(s) - P_{2,0}(0) = -\lambda\hat{P}_{2,0}(s) + \mu_v\hat{P}_{2,1}(s) + \mu\hat{P}_{1,1}(s) + \nu\hat{P}_{0,0}(s)$$

Applying the initial condition and substituting the Equation (18) for $n=1$, we can derive

$$\hat{P}_{2,0}(s) = \sum_{j=0}^{\infty} \frac{(2\lambda\mu_v)^j}{(s+\lambda)^{j+1}\left(P+\sqrt{P^2-\theta^2}\right)^j} \times \left[1+\mu\hat{P}_{1,1}(s)+\nu\hat{P}_{0,0}(s)\right]$$

And again, substituting the Equation (14) to the above equation, we have

$$\hat{P}_{2,0}(s) = \left(1+\mu\hat{P}_{1,1}(s)\right) G_i(s)B(s) - \frac{G_{i+1}(s)}{s} \tag{20}$$

where

$$B(s) = \sum_{j=0}^{\infty} \frac{(2\lambda\mu_v)^j}{(s+\lambda)^{j+1}\left(P+\sqrt{P^2-\theta^2}\right)^j}$$

and

$$G_n(s) = \sum_{n=0}^{\infty} (-1)^n \nu^n \left[A(s)\right]^n \left[B(s)\right]^n$$

Inversion of the Equation (21) yields,

$$P_{2,0}(t) = (1+\mu P_{1,1}(t)) * G_i(t) * B(t) - \int_0^t G_{i+1}(u)du \tag{21}$$

where

$$B(t) = \sum_{j=0}^{\infty} \left(\frac{2}{\theta}\right)^{j-1} (\lambda\mu_v)^j e^{-\lambda t} \frac{t^{j-1}}{(j-1)!} * \left[I_{m-1}(\theta t) - I_{m+1}(\theta t)\right] e^{-(\lambda+\mu_v+\eta+\gamma)t}$$

and

$$G_n(t) = \sum_{n=0}^{\infty} (-1)^n \nu^n \left[A(s)\right]^{*n} * \left[B(s)\right]^{*n}$$

It is clear that $P_{2,0}(t)$ is expressed in terms of $P_{1,1}(t)$ and $P_{1,1}(t)$ is expressed by the Equation (24). Where "*" denotes the convolution, while "$*n$" represents the n-fold convolution.

E. Evaluation of $P_{1,1}(t)$

Substituting $n=1$ to the Equation (12) and using the fact that $I_{-n}(\cdot) = I_n(\cdot)$, we can obtain

$$\begin{aligned} P_{1,1}(t) = {} & \nu \int_0^t \sum_{m=1}^{\infty} P_{0,m}(u)\beta^{1-m} \left[I_{m-1}\left(\alpha(t-u)\right) - I_{m+1}\left(\alpha(t-u)\right)\right] e^{-(\lambda+\mu+\eta)(t-u)} du \\ & + \gamma \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u)\beta^{1-m} \left[I_{m-1}\left(\alpha(t-u)\right) - I_{m+1}\left(\alpha(t-u)\right)\right] e^{-(\lambda+\mu+\eta)(t-u)} du \end{aligned} \tag{22}$$

Using the following Bessel identity

$$I_{m-1}\left(\alpha(t-u)\right) - I_{m+1}\left(\alpha(t-u)\right) = 2m\frac{I_m\left(\alpha(t-u)\right)}{\alpha(t-u)}$$

The Equation (22) can be rewritten as follows,

$$\begin{aligned} P_{1,1}(t) = {} & \nu \int_0^t \sum_{m=1}^{\infty} P_{0,m}(u)\beta^{1-m} 2m \frac{I_m\left(\alpha(t-u)\right)}{\alpha(t-u)} \times e^{-(\lambda+\mu+\eta)(t-u)} du \\ & + \gamma \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u)\beta^{1-m} 2m \frac{I_m\left(\alpha(t-u)\right)}{\alpha(t-u)} \times e^{-(\lambda+\mu+\eta)(t-u)} du \end{aligned}$$

Taking the Laplace transform of the above equation and after some algebra, we have

$$\begin{aligned} \hat{P}_{1,1}(s) = {} & 2\nu \sum_{m=1}^{\infty} \hat{P}_{0,m}(s) \frac{\beta^{1-m}}{\alpha^{m+1}} \left(P_1 - \sqrt{P_1^2-\alpha^2}\right)^m \\ & + 2\gamma \sum_{m=1}^{\infty} \hat{P}_{2,m}(s) \frac{\beta^{1-m}}{\alpha^{m+1}} \left(P_1 - \sqrt{P_1^2-\alpha^2}\right)^m \end{aligned}$$

where $P_1 = \lambda + \mu + \eta$.

Again, substituting the Equation (15) to above equation, we can obtain

$$\hat{P}_{1,1}(s)=\frac{A(s)H(s)}{s} + \left[K(s) - A(s)H(s)\right]\hat{P}_{2,0}(s)$$

Finally, we can derive the following expression for $\hat{P}_{1,1}(t)$ substituting the Equation (20) to the above equation and doing some mathematical calculations,

$$\begin{aligned}\hat{P}_{1,1}(s)=&\left\{\frac{A(s)H(s)}{s} + \left[K(s) - A(s)H(s)\right]\right.\\ &\left.\left[G_i(s)B(s) - \frac{G_{i+1}(s)}{s}\right]\right\}\\ &\times\left\{\sum_{r=1}^{\infty}\mu^r\left[G_i(s)\right]^r\left[B(s)\right]^r\right.\\ &\left.\times\sum_{m=0}^{\infty}(-1)^m\binom{r}{m}\left[K(s)\right]^m\left[A(s)H(s)\right]^{r-m}\right\} \quad (23)\end{aligned}$$

where

$$\begin{aligned}H(s)=&2\nu\sum_{m=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\nu}\right)^m\frac{\beta^{1-m}}{\alpha^{m+1}}\\ &\times\left(P_1 - \sqrt{P_1^2 - \alpha^2}\right)^m\end{aligned}$$

and

$$\begin{aligned}K(s)=&2\gamma\sum_{m=1}^{\infty}\left(\frac{2\lambda}{P + \sqrt{P^2 - \theta^2}}\right)^m\frac{\beta^{1-m}}{\alpha^{m+1}}\\ &\times\left(P_1 - \sqrt{P_1^2 - \alpha^2}\right)^m\end{aligned}$$

Inversion of the Equation (23) provides the following results

$$\begin{aligned}P_{1,1}(t)=&\left\{\int_0^t A(u) * H(u)du + \left[K(t) - A(t) * H(t)\right]\right.\\ &\left. * \left[G_i(t) * B(t) - \int_0^t G_{i+1}(u)du\right]\right\}\\ & * \left\{\sum_{r=1}^{\infty}\mu^r\left[G_i(t)\right]^{*r} * \left[B(t)\right]^{*r} * \sum_{m=0}^{\infty}(-1)^m\binom{r}{m}\right.\\ &\left.\times\left[K(t)\right]^{*m} * \left[A(t) * H(t)\right]^{*(r-m)}\right\} \quad (24)\end{aligned}$$

where

$$\begin{aligned}H(t)=&\nu\sum_{m=1}^{\infty}\lambda^m\beta^{1-m}e^{-(\lambda+\nu)t}\frac{t^{m-1}}{(m-1)!} * \left[I_{m-1}(\alpha t)\right.\\ &\left.-I_{m+1}(\alpha t)\right]e^{-(\lambda+\mu+\eta)t}\end{aligned}$$

and

$$\begin{aligned}K(t)=&\gamma\sum_{m=1}^{\infty}\left(\frac{2}{\theta}\right)^{m-1}\lambda^m\beta^{1-m}\left[I_{m-1}(\theta t)\right.\\ &\left.-I_{m+1}(\theta t)\right]e^{-(\lambda+\mu_v+\eta+\gamma)t}\\ & * \left[I_{m-1}(\alpha t) - I_{m+1}(\alpha t)\right]e^{-(\lambda+\mu+\eta)t}\end{aligned}$$

where '*' denotes convolution while '$*r$', '$*m$' and '$*(r-m)$', represent r-fold convolution, m-fold convolution and '$(r-m)$' convolution respectively.

All the time-dependent probabilities are explicitly derived in terms of modified Bessel function of the first kind by making use of Laplace transform, probability generating function techniques and continued fractions.

IV. Time dependent mean and variance

In this section, time dependent expected value and variance of the system size distribution are derived.

A. Mean

Let $X(t)$ denotes the number of jobs in the system at time t. The average number of jobs in the system at time t is given by

$$m(t) = E(X(t)) = \sum_{n=1}^{\infty} n\left(P_{0,n}(t) + P_{1,n}(t) + P_{2,n}(t)\right)$$

$$m(0) = \sum_{n=1}^{\infty} n\left(P_{0,n}(0) + P_{1,n}(0) + P_{2,0}(t)\right) = 0$$

$$m'(t) = \sum_{n=1}^{\infty} n\left(P'_{0,n}(t) + P'_{1,n}(t) + P'_{2,n}(t)\right)$$

By the Equations (2), (3), (4) and (6) and after some algebra, we have the following equation

$$\begin{aligned}m'(t)=&\lambda - \eta\left[\sum_{n=1}^{\infty} n\left(P_{1,n}(t) + P_{2,n}(t)\right)\right]\\ &-\mu\sum_{n=1}^{\infty}P_{1,n}(t) - \mu_v\sum_{n=1}^{\infty}P_{2,n}(t)\end{aligned}$$

By using the initial condition $m(0) = 0$ and integrating it by t, the solution of the above equation can be obtained as follows;

$$\begin{aligned}m(t)=&\lambda t - \eta\sum_{n=1}^{\infty} n\left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right]\\ &-\mu\sum_{n=1}^{\infty}\int_0^t P_{1,n}(u)du\\ &-\mu_v\sum_{n=1}^{\infty}\int_0^t P_{2,n}(u)du \quad (25)\end{aligned}$$

where $P_{1,n}(t)$ and $P_{2,n}(t)$ are given by the Equations (12) and (19) respectively.

B. Variance

Let $X(t)$ denotes the number of jobs in the system at time t. The variance for number of jobs in the system at time t is given by

$$\begin{aligned}Var(X(t))=&E(X^2(t)) - \left[E(X(t))\right]^2,\\ Var(X(t))=&k(t) - \left[m(t)\right]^2 \quad (26)\end{aligned}$$

where

$$k(t) = E(X^2(t)) = \sum_{n=1}^{\infty} n^2 \left(P_{0,n}(t) + P_{1,n}(t) + P_{2,n}(t)\right)$$

Also,

$$k(0) = \sum_{n=1}^{\infty} n^2 \left(P_{0,n}(0) + P_{1,n}(0) + P_{2,n}(0)\right) = 0$$

and

$$k^{'}(t)=\sum_{n=1}^{\infty} n^2 \left(P^{'}_{0,n}(t) + P^{'}_{1,n}(t) + P^{'}_{2,n}(t)\right)$$

By Equations (2), (3), (4) and (6) and after some algebra, we have the following equation

$$\begin{aligned} k^{'}(t)=&2\lambda m(t) + \lambda - \eta \sum_{n=1}^{\infty} n^2 \left[P_{1,n}(t) + P_{2,n}(t)\right] \\ &-2\mu \sum_{n=1}^{\infty} nP_{1,n}(t) - 2\mu_v \sum_{n=1}^{\infty} nP_{2,n}(t) \\ &+\mu \sum_{n=1}^{\infty} P_{1,n}(t) + \mu_v \sum_{n=1}^{\infty} P_{2,n}(t) \end{aligned}$$

By using the initial condition $k(0) = 0$ and integrating it by t, the solution of the above equation can be obtained as follows;

$$\begin{aligned} k(t)=&2\lambda \int_0^t m(u)du + \lambda t \\ &-\eta \sum_{n=1}^{\infty} n^2 \left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right] \\ &-2\mu \sum_{n=1}^{\infty} n \int_0^t P_{1,n}(u)du - 2\mu_v \sum_{n=1}^{\infty} n \int_0^t P_{2,n}(u)du \\ &+\mu \sum_{n=1}^{\infty} \int_0^t P_{1,n}(u)du + \mu_v \sum_{n=1}^{\infty} \int_0^t P_{2,n}(u)du \end{aligned}$$

Substituting above equation into the Equation (26), we will have

$$\begin{aligned} Var(X(t))=&2\lambda \int_0^t m(u)du + \lambda t \\ &-\eta \sum_{n=1}^{\infty} n^2 \left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right] \\ &-2\mu \sum_{n=1}^{\infty} n \int_0^t P_{1,n}(u)du \\ &-2\mu_v \sum_{n=1}^{\infty} n \int_0^t P_{2,n}(u)du \\ &+\mu \sum_{n=1}^{\infty} \int_0^t P_{1,n}(u)du + \mu_v \sum_{n=1}^{\infty} \int_0^t P_{2,n}(u)du \\ &- \left[m(t)\right]^2 \end{aligned}$$

where $P_{1,n}(t)$, $P_{2,n}(t)$ and $m(t)$ are given by the Equations (12), (19) and (25) respectively.

V. Conclusions

A repairable single server queue with working vacations and system disasters is considered in transient regime and the explicit expression for system size probabilities of the queueing system are derived in terms of the modified Bessel function of first kind. Probability generating function method, Laplace transform and continued fractions are used to derive the transient solution. Additionally, the mean and variance for number of jobs in the system at time t are derived as the performance measures.

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