On a linearization method for the dynamical systems associated to excitable media models

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Abstract. In the present paper there are presented recent results concerning the behavior of the mixing flow dynamical system. From analytical standpoint, the feedback linearization of this dynamical system issue special interpretations. This technique contains two fundamental nonlinear controller design techniques: input-output linearization and state-space linearization. The approach is usually referred as *input-output linearization* or *feedback linearization* and is based on concepts from nonlinear systems theory. The resulting controller includes the inverse of the dynamic model of the process, providing that such an inverse exists. The results will be used for further analysis of 3D mixing flow dynamical system

Keywords—Mixing flow model, feedback linearization, controllable system, phase portrait, random event

I. INTRODUCTION. MIXING FLOW MATHEMATICAL CONTEXT

The mixing flow theory appears in an area with far from complete solving problems: the flow kinematics. Its methods and techniques developed the *significant relation between turbulence and chaos*. The turbulence is an important feature of dynamic systems with few freedom degrees, the so-called "far from equilibrium systems". These are widespread between the models of excitable media, and a recent goal is to find a consistent and coherent theory to stand up that a mixing model in excitable media leads to a far from equilibrium model.

After a hundred years of stability study, the problems of flow kinematics are far from complete solving. Since the beginnings, considering the stability of laminar flows with infinitesimal turbulences was a fruitful investigation method. This context becomes more difficult if the non-linearity is in the sense of increasing of the growing rate of linear unstable modes. In fact, we are talking about *strong turbulence problems*, an area which still needs a lot of analysis.

Generally, the statistical idea of a flow is represented by a map:

$$x = \Phi_t(X), X = \Phi_{t=0}(X) \quad (1)$$

We say that **X** is *mapped in* ***x*** after a time t. In the continuum mechanics the relation (1) is named *flow*, and it is a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation

$$J = \det\left(D\left(\Phi_t(X)\right)\right) = \det\left(\frac{\partial x_i}{\partial X_j}\right) \quad (2)$$

where the derivation is with respect to the reference configuration, in this case **X**. The relation (2) implies two particles, X_1 and X_2, which occupy the same position **x** at a moment.

With respect to **X** there is defined the basic measure of deformation, the *deformation gradient*, **F,** namely:

$$\mathbf{F} = \left(\nabla_X \Phi_t(\mathbf{X})\right)^T,\ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right) \quad (3)$$

where ∇_X denotes differentiation with respect to **X**. According to (3), **F** is non singular. The basic measure for the deformation with respect to **x** is the *velocity gradient.*

After defining the basic deformation of a material filament and the corresponding relation for the area of an infinitesimal material surface, we can define the basic deformation measures: the *length deformation* λ and *surface deformation* η, with the relations [7]:

$$\lambda = (\mathbf{C} : \mathbf{MM})^{1/2},\ \eta = (\det F)\cdot\left(\mathbf{C}^{-1} : \mathbf{NN}\right)^{1/2} \quad (4)$$

whith **C** (=$\mathbf{F}^T\cdot\mathbf{F}$) the *Cauchy-Green deformation tensor,* and the vectors **M**,**N** - the orientation versors in length and surface respectively, defined by

$$\mathbf{M} = \frac{d\mathbf{X}}{|d\mathbf{X}|},\ \mathbf{N} = \frac{d\mathbf{A}}{|d\mathbf{A}|} \quad (5)$$

In the above context, we say that the flow $\mathbf{x}=\Phi_t(\mathbf{X})$ has *a good mixing* if the mean values $D(\ln\lambda)/Dt$ and $D(\ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations **M** and **N**.

From both analytical and computational standpoint, the following relations are basic in the flow kinematics: there is

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defined the *deformation efficiency in length*, $e_\lambda = e_\lambda$ *(**X**,**M**,t)* of the material element d**X**, as:

$$e_\lambda = \frac{D(\ln \lambda)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \le 1 \tag{6}$$

and similarly, the *deformation efficiency in surface*, $e_\eta = e_\eta$ *(**X**,**N**,t)* of the area element d**A**: in the case of an isochoric flow (the jacobian equal 1), we have:

$$e_\eta = \frac{D(\ln \eta)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \le 1 \tag{7}$$

where **D** is the deformation tensor [7].

The deformation tensor **F** and the associated tensors **C**, $\mathbf{C}^{-1}$, form the fundamental quantities for the analysis of deformation of infinitesimal elements. In most cases, the flow $x = \Phi_t(X)$ is unknown and has to be obtained by integration from the Eulerian velocity field. If this can be done analytically, then **F** can be obtained by differentiation of the flow with respect to the material coordinates **X.**

The flows of interest belong to two classes: i) flows with a special form of ∇v and ii) flows with a special form of **F**. The second class is of very large interest, as it contains the so-called Constant Stretch History Motion – CSHM flows.

II. Recent Results and Methods

Studying a *mixing* for a flow implies the analysis of successive *stretching* and *folding* phenomena for its particles, together with the influence of parameters and initial conditions. It can concern simple mixing phenomena, or the phenomena of a mixing of a biological material in a host fluid.

In the previous works, the mixing phenomenon produced when a biological material is vortexed in a host fluid was studied. A first aim was to study the deformation efficiency in length and surface for the mixing flow model. The mathematical model used as start point in the analysis is basically the widespread isochoric two-dimensional flow, namely [7]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1, \ \ -1 \langle K \langle 1, G \rangle 0 \end{cases} \tag{8}$$

In the 3d case, the associated mathematical model was constructed according to the experiments realized for a vortex phenomenon. This was realized by simply adding the vortex velocity as third component [3]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 \ \ , -1 \langle K \langle 1, c = const. \\ \dot{x}_3 = c \end{cases} \tag{9}$$

The study of the 3D non-periodic models exhibited a quite complicated behaviour. In agreement with experiments, there were involved some significant events - the so-called "*rare events*". The variation of parameters had a great influence on the length and surface deformations. The experiments were realized with a special vortex installation, it was used a well-known aquatic algae as biologic material, and the water as basic fluid [3].

There were obtained quite complex relations for e_λ and e_η. The analysis of the mathematical model contained an analytical and a computational / simulation stage. There were used procedures of MAPLE soft for discrete time. The events studied were very few, about 60 both for 2d and 3d case.

This stage of the analysis produced a panel of *random events* for the mixing flow mathematical model. Although it has a not very complicated mathematical form, it is going to turbulence for certain values of the parameters. A very important point is that when adding similar terms to the model, in 2d case, the model turns its behaviour into a far from equilibrium one.

The analysis recently has been continued with more computational simulations, for 2D model, both in periodic and non-periodic case, and for 3D model, too. A lot of comparisons between periodic and non-periodic case, 2D and 3D case were realized [4]. In the same time, the computational appliances were varied. If initially, the model was studied from the standpoint of mixing efficiency, in the works that come after, new appliances of the MAPLE11 soft were tested [1], in order to collect more statistical data for the turbulent mixing model behavior. Also, some interesting versions of the mixing dynamical system were analyzed, perturbing the model with a logistic-type term [4]. In the analysis, the same set of parameters values there were taken into account, for a better accuracy of the comparative analysis. The phase-portrait analysis offered new features concerning the influence on parameters on the model behaviour.

III. Feedback Linearization for 2D Mixing Flow

The aim of the paper is to analyze the mixing model dynamical system from a new analytical standpoint, the *feedback linearization method*. This approach is based on concepts from nonlinear systems theory and contains two fundamental nonlinear controller design techniques: input-output linearization and *state-space linearization.* [2,5]. The resulting controller includes the *inverse of the dynamic model* of the process, providing that such an inverse exists. Thus, the method is applicable to broad classes of nonlinear control problems.

The feedback method is applied generally to differential systems of the form:

$$\begin{aligned} &\dot{x} = f(x) + g(x) \cdot u, \\ &f, g : D \subset R^n \to R^n, \ u \in R \end{aligned} \tag{10}$$

We search for a diffeomorphism $T: D \subset \mathbb{R}^n \to \mathbb{R}^n$, which defines a coordinate transformation

$$\mathbf{z} = \mathbf{T}(\mathbf{x}) \tag{11}$$

in order to find for the system (10), a state - space realization of the form

$$\mathbf{Z} = A\mathbf{z} + Bv \tag{12}$$

The method is presented in detail in [5]. In this approach, u has the role of control, and the relation with v (the new control) is given by the following relation:

$$u = \phi(\mathbf{x}) + \omega^{-1}(\mathbf{x}) \cdot v \tag{13}$$

where Φ and ω are scalar functions obtained by imposed relations on the vector functions **f**, **g** and the transformation **T**. The transformation **T** has to be obtained in special conditions for the partial derivatives of f and g [5]. Also, A and B are in the *controllable form* (*n x n*, *n x 1* respectively):

$$A_C = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_C = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{14}$$

Synthesizing, we can say that, given the nonlinear system (10), the problem of feedback linearization consists in finding, if possible, a coordinate transformation of the form (11), and a static feedback control law of the form

$$u = \alpha(\mathbf{x}) + \beta(\mathbf{x}) \cdot v \tag{15}$$

where v is the new control and β(x) is assumed to be non zero for all **x**, such that the composed dynamics of the new system

$$\dot{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \alpha(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \beta(\mathbf{x}) \cdot v, \ \mathbf{x} \in R^n \tag{16}$$

expressed in the new coordinates z, is the controllable system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\dots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned} \tag{17}$$

Taking into account that we are analyzing a far from equilibrium model, we shall apply the feedback linearization method first for a mixing flow model perturbed with a logistic-type term.. Thus, let us consider the following perturbed version for the dynamical system (8), analyzed also in [4]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 + G \cdot (x_2 - x_1), \end{cases} \quad -1 \langle K \langle 1, \ G \rangle 0 \tag{18}$$

We are in the case n=2, and we search for a transformation T, $\mathbf{T}(\mathbf{x}) = \begin{pmatrix} T_1(\mathbf{x}) \\ T_2(\mathbf{x}) \end{pmatrix}$, in order to transform this system. According to the above statements, let us put the system (18) in the vector form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} Gx_2 \\ KGx_1 \end{pmatrix} + \begin{pmatrix} 0 \\ Gx_2 - Gx_1 \end{pmatrix} \cdot u \tag{19}$$

In this form we consider the vectors **f** and **g** by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} Gx_2 \\ KGx_1 \end{pmatrix}, \ \mathbf{g}(\mathbf{x}) = \begin{pmatrix} 0 \\ Gx_2 - Gx_1 \end{pmatrix} \tag{20}$$

The transformation $\mathbf{T} = (T_1 \quad T_2)^T$ is found, after calculus, as

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} x_1^2 + KG \\ 2Gx_1x_2 \end{pmatrix}, \quad \mathbf{x} = (x_1 \quad x_2) \tag{21}$$

Further there are found the functions ω and Φ as follows:

$$\omega(\mathbf{x}) = 2G^2 x_1 \cdot (x_2 - x_1), \quad \phi(\mathbf{x}) = -\frac{x_2^2 + Kx_1^2}{x_1 \cdot (x_2 - x_1)} \tag{22}$$

Thus, taking the controllers A=A_C and B=B_C as

$$A_C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the inverse system becomes, in the coordinate $\mathbf{z} = (z_1 \quad z_2)$,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \omega \cdot (u - \phi) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

That means we get the following system:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = 2G^2 \cdot \left(z_2^2 + Kz_1^2\right) + 2G^2 z_1 \cdot (z_2 - z_1) \cdot u \end{cases} \quad -1 \langle K \langle 1, G \rangle 0 \tag{23}$$

Thus, the dynamical system associated to the 2d mixing flow model, in a perturbed form like (18), admits an inverse system in a controllable form. The system (23) proves that the form (17) of the inverse mode can be reached. The transformation T does exist if certain conditions are fulfilled for the functions f and **g** of the system. In fact, the form (17) is the form that is reached by the feedback linearization method, provided that T exists. In [6] there was approached an optimal control for a feedback linearized form of the dynamical system associated to the 2d mixing flow model in the basic form.

IV. Graphical Comparative Analysis for the Initial and Linearized Mixing Model

In this section, the comparative analysis for the systems' trajectories is realized. The focus is on the influence of the parameters on the trajectory trend, both for the initial and linearized system.

A. Let us take into account first the system (18) of the 2d mixing flow, perturbed with a logistic type term.

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 + G \cdot (x_2 - x_1), \end{cases}$$
$$-1 \langle K \langle 1,\ G \rangle 0$$

For this system, the phase-portrait was realized with MAPLE software, in some specific conditions for the parameters. For the continuity of the analysis and comparisons, there were used the same value sets for the parameters as in previous works [4], namely:

$$\begin{aligned} &i)\ G = 0.25,\ KG = -0.035; \\ &ii)\ G = 0.755,\ KG = -0.65; \qquad (24) \\ &iii)\ G = 0.85,\ KG = -0.25 \end{aligned}$$

These parameter cases are chosen in order to get a better influence on the trajectory, like is the negative form taken in the values' sets. The system's trajectories are realized based on numerical methods; the default method of integration is the "Forward Runge-Kutta" method, of order 4-5, abbreviated by rkf45 [1].

Also, the interactive style of MAPLE procedures allow us to change the time units as desired. For the present graphical analysis, there were sufficient 50 time units in order to get a definite trend of the trajectory.

Below there are presented the picture for the cases i- iii of simulation. Each case is labeled on the figure.

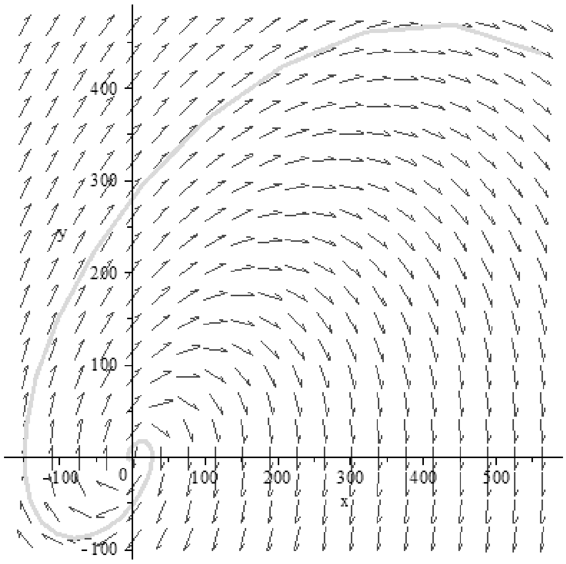

Fig.1. The case i) of simulation for the system (18)

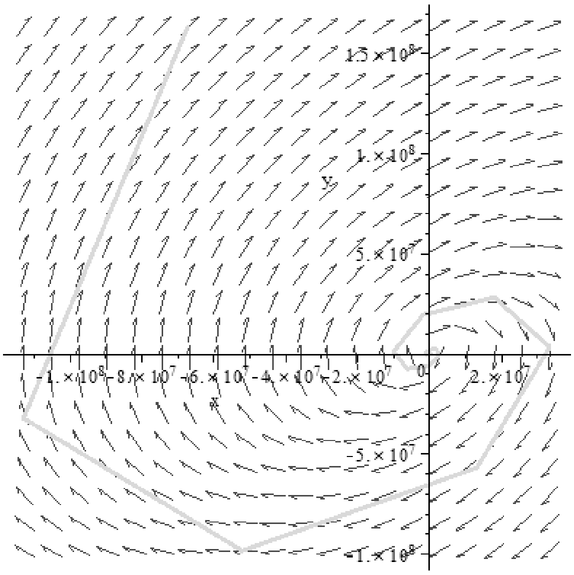

Fig.2. The case ii) of simulation for the system (18)

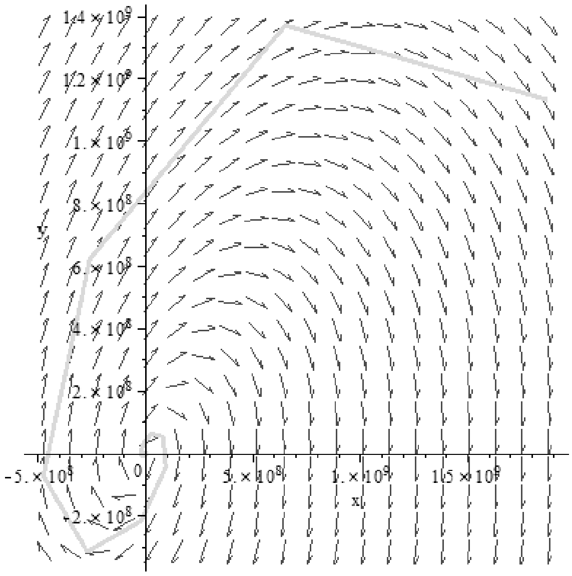

Fig.3. The case iii) of simulation for the system (18)

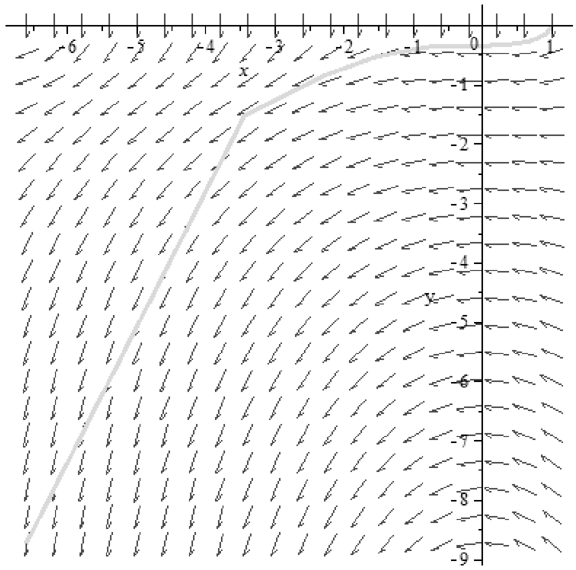

Fig.4. The case i) of simulation for the system (23)

B. We take now into account the linearized system associated to (18). We search for a comparison between the behavior of the mixing system perturbed with a logistic type term, (18), and that of its inverse system, namely the system (23), in the same simulation cases. The initial conditions, associated to the Cauchy problem for simulation, are also the same as for the system (18):

$$x(0)=1,\ y(0)=0$$

Since the parameter distribution is very different in the inverse system (23), it is expected to see a difference in the trajectory form, too. After re-noting

$$z_1 = x,\ z_2 = y$$

in order to preserve the pictures legend, the pictures for the cases (24) are as follows.

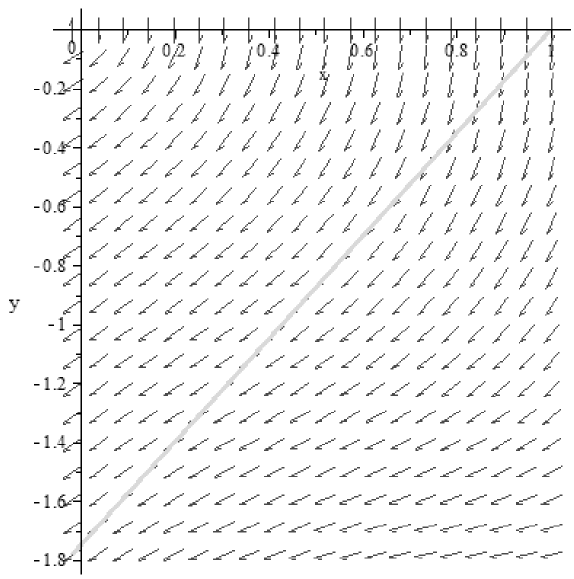

Fig.5. The case ii) of simulation for the system (23)

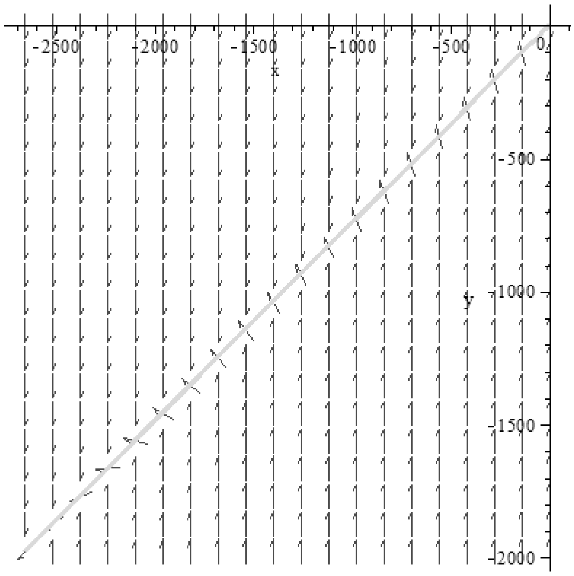

Fig.6. The case iii) of simulation for the system (23)

V. REMARKS.

The feedback linearization method presented above has a significant importance in the qualitative analysis of dynamical systems. First of all, we have to mention the special form of the linearized system, associated to the initial system. The parameters have a very different repartition in the inverse model. As the form (17) shows, in the inverse system, the last equation contains all the non-linearity of the model. This fact is confirmed by the inverse system (23) associated to the 2d mixing model perturbed with a logistic term. We can state that this form is similar with that of the second order non-linear oscillator with *polynomial nonlinearities*.

It is easy to observe in fact, that the general controllable form (17), and the form (23) for the 2d case, is more convenient for some analytical standpoints. For example the stability properties, like the topological degree, can be easier evaluated in the case of an inverse model, and this is a next target.

The graphical approach produces very important remarks in the above analysis. First, both for the system (18) and (23), we see that increasing the simulation time provides an important influence on the trajectory. We mention that in MAPLE the time units are dimensionless, so we can adapt the simulations, for the user needs.

For the system (18) the origin is a centre, as it can be easily seen, and the trajectory is changing the positivity. There can be observed quite large values for x(t) and y(t). By contrast, for the system (23), although the origin is also a solution, it is no more a centre. It is very important to mention that the software detected a "possible singularity", and there were recorded medium values for the trajectory.

Thus, although the trajectory becomes linear, as it can be easily seen from the pictures 4,5,6, we have to take into account the singular points. So the figures 4-6 have a double interpretation: on one hand, they show a linear behaviour, expected from a linearized system (23), and on the other hand, concern with possible singular points of the inverse system (23). Therefore, another important conclusion is that the inverse system obtained by feedback linearization could have singular points, and a future target is to find them.

We have also to notice that for the system (23) it was taken into account for the present aim, only the value u=1 for the scalar control. Some other values of u would produce other trends for the inverse system's trajectory.

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Basic domains of interest: Differential Equations and Dynamical Systems; Computational Fluid Dynamics; Qualitative Analysis and Optimization Techniques for industrial and real life models; Feedback linearization and LQG Controllers for dynamical systems; Nonlinear Control Systems.

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