2D Filters with Elliptical and Circular Frequency Response Based on Digital Prototypes

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Abstract—In this work an analytical design procedure is proposed for a class of two-dimensional recursive filters, having a frequency response of elliptical or circular shape. The design method relies on efficient digital filter prototypes, to which a specific complex frequency mapping is applied. This allows obtaining directly the transfer function of the desired 2D filter, in a factored form. In the proposed design some accurate approximations are used, like Chebyshev-Padé method, but no global optimization algorithm. Finally the 2D filter matrices are derived, as a convolution of smaller size matrices, an advantage in implementation. The filter results adjustable, its coefficients depending explicitly on the specified orientation and bandwidth. As the design examples show, the obtained 2D filters have an accurate shape, with very low distortions even near the margins of frequency plane and are efficient, of good selectivity and relatively low complexity.

Keywords—2D filters; elliptical, circular frequency response; approximations; frequency transformations

I. Introduction

Two-dimensional filters, both in FIR and IIR version, and their design methods have been an essential field of research since the advent of digital signal processing era, for their important applications in the image processing domain [1]. Apart from the classical numerical optimization methods, many analytical design techniques have been elaborated, which use 1D prototype filters and spectral transformations to derive 2D filters with a desired frequency response [2]. A convenient and largely used tool for 2D FIR filter design is the well-known McClellan transform [3], [4]. Anisotropic filters were studied extensively and used in interesting applications, like remote sensing for directional smoothing applied to weather images, texture segmentation and pattern recognition [5], [6]. In particular, filters with elliptically-shaped frequency response are useful in image processing and various design methods were developed in early papers like [7]-[9]. A fast space-variant filtering using Gaussian elliptic window is proposed in [10]. Elliptical filters also found applications in biometrics, like pose robust human detection [11], palm print identification [12], iris recognition [13], fingerprint enhancement [14]. Relevant papers proposing various design methods for circular filters are [15]-[17]. Other analytical design methods for directional filters, in particular elliptically shaped, were proposed by the author in [18]-[21]. Stability of 2D filters and stabilization methods are important and rather difficult issues, studied in papers like [22], [23].

We approach here the design of a class of 2D filters, having an elliptically-shaped support of frequency response in the frequency plane. The design method is mainly analytical and uses approximations, but not any numerical optimization algorithms. It is based on 1D low-pass prototype filters and frequency mappings. Design examples using proposed method are given, both for elliptically and circularly shaped filters.

II. Low-Pass Filter Prototypes

As a starting point in the design of an elliptically-shaped 2D filter we will use an efficient recursive filter prototype. The most efficient filter for a specified selectivity and steepness is the elliptic type filter (both in digital and analog versions). It is known to result of a lower order than other common approximations, like Butterworth or Chebyshev.

Next, consider a digital elliptic filter with specifications: order $N = 6$, peak-to-peak ripple $R_P = 0.15\,\text{dB}$, minimum stop-band attenuation $R_S = 39\,\text{dB}$ and normalized pass-band edge frequency $\omega_P = 0.5$ (the value 1 corresponding to half the sample rate). Using MATLAB, these specifications yield the following transfer function in complex frequency z:

$$H_P(z) = \frac{\begin{pmatrix} 0.08908 \cdot z^6 + 0.24421 \cdot z^5 + 0.44757 \cdot z^4 \\ +0.24421 \cdot z^3 + 0.44757 \cdot z^2 + 0.24421 \cdot z + 0.08908 \end{pmatrix}}{\begin{pmatrix} z^6 - 0.52714 \cdot z^5 + 1.7397 \cdot z^4 - 0.78876 \cdot z^3 \\ +0.87416 \cdot z^2 - 0.27316 \cdot z + 0.09866 \end{pmatrix}} \quad (1)$$

which has the factored expression (where $k = 0.089079$):

$$H_P(z) = 0.089079 \cdot \frac{\left(z^2 + 1.687693 \cdot z + 1\right)}{\left(z^2 + 0.049849 \cdot z + 0.911314\right)} \cdot \frac{\left(z^2 + 0.705134 \cdot z + 1\right)}{\left(z^2 - 0.124832 \cdot z + 0.628466\right)} \cdot \frac{\left(z^2 + 0.348702 \cdot z + 1\right)}{\left(z^2 - 0.452157 \cdot z + 0.172254\right)} \quad (2)$$
$$= k \cdot H_{B1}(z) \cdot H_{B2}(z) \cdot H_{B3}(z)$$

Therefore, the 1D elliptic digital filter transfer function in z is

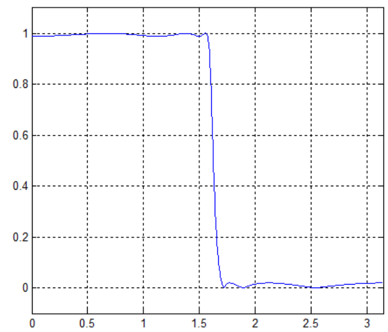


Fig. 1. Magnitude of the elliptic filter frequency response

factored into three so-called bi-quad functions $H_{B1}(z)$, $H_{B2}(z)$ and $H_{B3}(z)$. However, the factors of the nominator and denominator in (2) can be coupled in pairs in several ways. In Fig.1, the magnitude of the transfer function (1) is shown for $\omega \in [0,\pi]$; as can be noticed, it has a steep transition and very small ripple in the pass-band and stop-band as well.

III. Design Method for Low-pass 2D Filters with Elliptically-Shaped Frequency Response

In this section we derive the frequency transformation which leads from the chosen low-pass prototype to the desired elliptically-shaped filter. The filter transfer function in matrix form is derived, then some design examples are presented.

A. Frequency Transformation for Elliptically-Shaped Filters

In this paragraph a 2D LP filter with elliptical symmetry is obtained, starting from an usual digital prototype with a transfer function in variable *z*. This 2D filter will be specified by imposing the values of the ellipse semi-axes, and the orientation is given by the angle of the large axis with respect to ω_2 – axis. Starting from the frequency response of a 1D filter given by (2), we derive a 2D elliptically-shaped filter using the frequency mapping $\omega^2 \to E_\varphi(\omega_1,\omega_2)$, where:

$$E_\varphi(\omega_1,\omega_2) = \omega_1^2\left(\frac{\cos^2\varphi}{E^2}+\frac{\sin^2\varphi}{F^2}\right)+\omega_2^2\left(\frac{\sin^2\varphi}{E^2}+\frac{\cos^2\varphi}{F^2}\right) + \omega_1\omega_2\sin(2\varphi)\left(\frac{1}{F^2}-\frac{1}{E^2}\right) = a\cdot\omega_1^2+b\cdot\omega_2^2+c\cdot\omega_1\omega_2 \quad (3)$$

The following linear transformation of the spatial frequencies describes how an elliptically shaped filter can be obtained from a circular filter:

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}=\begin{bmatrix}E & 0\\ 0 & F\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}\omega_1'\\ \omega_2'\end{bmatrix} \quad (4)$$

where usually we take $E > F$; in (4) , (ω_1,ω_2) are the current variables and (ω_1',ω_2') are the former (rotated) variables. Thus, the unit circle is stretched along the axes ω_1 and ω_2 with the factors *E* and *F* respectively, then rotated counter-clockwise with angle φ, thus becoming an oriented ellipse.

Therefore, starting from a 1D prototype filter, a corresponding 2D filter with elliptical support results, specified by parameters *E*, *F* and φ which impose the orientation and shape, using the mapping $\omega \to \sqrt{E_\varphi(\omega_1,\omega_2)}$, also written:

$$\omega^2 \to E_\varphi(\omega_1,\omega_2) = a_0\cdot\omega_1^2+b_0\cdot\omega_2^2+c_0\cdot\omega_1\omega_2 \quad (5)$$

Using in (5) the identity $\omega_1\omega_2 = 0.5\cdot\left((\omega_1+\omega_1)^2-\omega_1^2-\omega_2^2\right)$, we find another expression for $E_\varphi(\omega_1,\omega_2)$:

$$\omega^2 \to E_\varphi(\omega_1,\omega_2) = a\cdot\omega_1^2+b\cdot\omega_2^2+c\cdot(\omega_1+\omega_2)^2 \quad (6)$$

where

$$\begin{aligned} a &= a_0-0.5c_0 = p+q\cdot\cos(2\varphi)+q\cdot\sin(2\varphi)\\ b &= b_0-0.5c_0 = p-q\cdot\cos(2\varphi)+q\cdot\sin(2\varphi)\\ c &= 0.5c_0 = -q\cdot\sin(2\varphi)\end{aligned} \quad (7)$$

Here we have used the notations:

$$p = 1/E^2+1/F^2\ ,\ q=1/E^2-1/F^2 \quad (8)$$

The next step in the design of the 2D elliptically-shaped filter is applying the frequency mapping (6) to the digital prototype $H_P(z)$ from (2). Thus we substitute $z=\exp(j\omega)$ by $\exp\left(j\cdot\sqrt{E_\varphi(\omega_1,\omega_2)}\right)=\exp\left(\sqrt{-E_\varphi(\omega_1,\omega_2)}\right)$. In order to derive more efficient 2D filters, we use the following first-order rational approximation for $\exp\sqrt{-\omega}$ on frequency range $\omega\in[0,\pi]$:

$$\exp\left(\sqrt{-\omega}\right)=\exp\left(j\sqrt{\omega}\right)\cong F(\omega)= =\frac{(0.664635-0.2127933\cdot\omega)+j\cdot(0.896434+0.251929\cdot\omega)}{(1+0.209288\cdot\omega)+j\cdot(0.515203-0.268335\cdot\omega)} \quad (9)$$

This approximation is accurate enough for our purpose, and practically reduces twice the filter order, while maintaining the correct frequency response shape, with very low distortions. The real and imaginary part and their approximations are plotted comparatively in Fig.2. The approximation (9) can be made scalable of frequency axis, i.e. substituting the current variable ω by $k\cdot\omega$ ($k>0$), it remains valid for a certain range of the scaling parameter *k* (which means stretching for $k<1$ or shrinking for $k>1$). Therefore, in order to obtain a parametric filter, the above approximation is written as [21]:

$$\exp\left(j\sqrt{k\cdot\omega}\right)=\exp\left(\sqrt{-k\cdot\omega}\right) \cong\frac{(0.664635-0.212793\cdot k\cdot\omega)+j\cdot(0.896434+0.25193\cdot k\cdot\omega)}{(1+0.209288\cdot k\cdot\omega)+j\cdot(0.515203-0.268335\cdot k\cdot\omega)} \quad (10)$$

A generic bi-quad function $H_{Bi}(z)$ in variable *z* has the form:

$$H_{Bi}(z)=\frac{\left(z^2+v_1\cdot z+v_0\right)}{\left(z^2+u_1\cdot z+u_0\right)} \quad (11)$$

To obtain a 2D filter with elliptical symmetry, we simply make the substitution (frequency mapping) $\omega^2\to E_\varphi(\omega_1,\omega_2)$ and the following frequency transformation results:

$$z \to \exp\left(j\sqrt{k \cdot E_\varphi(\omega_1,\omega_2)} \right) \cong \frac{\left(0.664635 - 0.212793 \cdot k \cdot E_\varphi(\omega_1,\omega_2)\right) + j\cdot\left(0.896434 + 0.25193 \cdot k \cdot E_\varphi(\omega_1,\omega_2)\right)}{\left(1 + 0.209288 \cdot k \cdot E_\varphi(\omega_1,\omega_2)\right) + j\cdot\left(0.515203 - 0.268335 \cdot k \cdot E_\varphi(\omega_1,\omega_2)\right)} \quad (12)$$

where $E_\varphi(\omega_1,\omega_2)$ is substituted by its expression (6). Using Chebyshev-Padé method, the following rational trigonometric approximation is obtained for the square function ω^2:

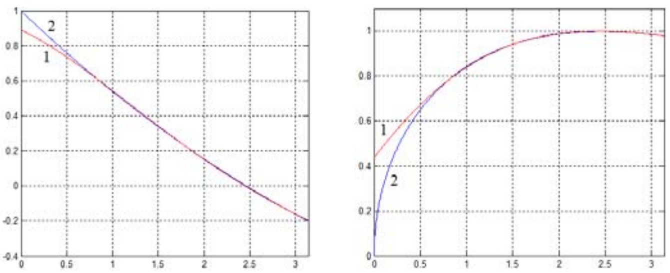
$$\omega^2 \cong 2.35753 \cdot (1 - 0.946216 \cdot \cos\omega) / (1 + 0.46301 \cdot \cos\omega) \quad (13)$$

displayed in Fig. 3. This is obviously a very efficient and accurate approximation on the frequency range $\omega \in [-\pi, \pi]$, having a small distortion only at the margins of the specified interval.

Using the approximation (13) for the variables ω_1 and ω_2 respectively, expressions of ω_1^2 and ω_2^2 are then substituted into the mapping (12). Next, a matrix form of this mapping results, for the parameter value $k = 1$.

Using identities $\cos\omega_1 = 0.5 \cdot (z_1 + z_1^{-1})$, $\cos\omega_2 = 0.5 \cdot (z_2 + z_2^{-1})$ expressed in complex variables $z_1 = e^{j\omega_1}$, $z_2 = e^{j\omega_2}$, the transformation (12) may be finally written in the matrix form:

$$z \to H_C(z_1, z_2) = \frac{B(z_1,z_2)}{A(z_1,z_2)} = \frac{\mathbf{z_1} \times \mathbf{B} \times \mathbf{z}_2^T}{\mathbf{z_1} \times \mathbf{A} \times \mathbf{z}_2^T} \quad (14)$$



(a) (b)

Fig. 2. (a) Plot of the function $\cos\sqrt{\omega}$ and the real part of the approximation $\mathrm{Re}(F(\omega))$; (b) Plot of the function $\sin\sqrt{\omega}$ and the imaginary part of the approximation, $\mathrm{Im}(F(\omega))$.

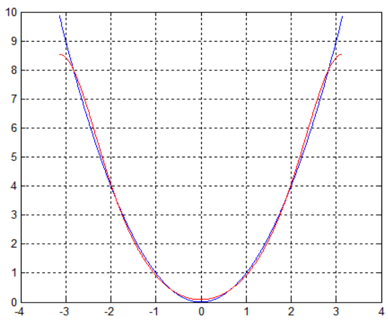


Fig. 3. The parabolic function (in blue) and its first-order rational trigonometric approximation (in red)

where $\times$ is inner product and where the vectors $\mathbf{z_1}$ and $\mathbf{z_2}$ are:

$$\mathbf{z_1} = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 & z_1^4 \end{bmatrix} ; \mathbf{z_2} = \begin{bmatrix} 1 & z_2 & z_2^2 & z_2^3 & z_2^4 \end{bmatrix} \quad (15)$$

The matrices **A** and **B** with complex elements corresponding to the numerator and denominator are expressed as:

$$\mathbf{A} = \mathbf{A}_R + j \cdot \mathbf{A}_I \qquad \mathbf{B} = \mathbf{B}_R + j \cdot \mathbf{B}_I \quad (16)$$

where the component matrices are given by:

$$\begin{aligned} \mathbf{A}_R &= 0.664635 \cdot \mathbf{B_1} - 0.212793 \cdot k \cdot (a \cdot \mathbf{A_1} + b \cdot \mathbf{A_2} + c \cdot \mathbf{A_3}) \\ \mathbf{A}_I &= 0.896434 \cdot \mathbf{B_1} + 0.25193 \cdot k \cdot (a \cdot \mathbf{A_1} + b \cdot \mathbf{A_2} + c \cdot \mathbf{A_3}) \\ \mathbf{B}_R &= \mathbf{B_1} + 0.209288 \cdot k \cdot (a \cdot \mathbf{A_1} + b \cdot \mathbf{A_2} + c \cdot \mathbf{A_3}) \\ \mathbf{B}_I &= 0.515203 \cdot \mathbf{B_1} - 0.268335 \cdot k \cdot (a \cdot \mathbf{A_1} + b \cdot \mathbf{A_2} + c \cdot \mathbf{A_3}) \end{aligned} \quad (17)$$

The matrices $\mathbf{A}_R$, $\mathbf{A}_I$, $\mathbf{B}_R$, $\mathbf{B}_I$ from (17) are linear combinations of 5×5 matrices $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ and $\mathbf{B_1}$, having the generic form:

$$\mathbf{A_1} = \begin{bmatrix} 0 & 0 & \lambda & \delta & \lambda \\ 0 & \delta & \gamma & \beta & \mu \\ \lambda & \beta & \alpha & \beta & \lambda \\ \mu & \beta & \gamma & \delta & 0 \\ \lambda & \delta & \lambda & 0 & 0 \end{bmatrix} ; \quad \mathbf{A_2} = \begin{bmatrix} 0 & 0 & \lambda & \mu & \lambda \\ 0 & \delta & \beta & \beta & \delta \\ \lambda & \gamma & \alpha & \gamma & \lambda \\ \delta & \beta & \beta & \delta & 0 \\ \lambda & \mu & \lambda & 0 & 0 \end{bmatrix}$$
$$\mathbf{A_3} = \begin{bmatrix} 0 & 0 & \lambda & \delta & \lambda \\ 0 & \mu & \beta & \gamma & \delta \\ \lambda & \beta & \alpha & \beta & \lambda \\ \delta & \gamma & \beta & \mu & 0 \\ \lambda & \delta & \lambda & 0 & 0 \end{bmatrix} \quad (18)$$

The 5×5 matrices $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are centrally-symmetric and their elements have the values $\alpha = 2.237977$, $\beta = 0.287569$, $\gamma = -0.989016$, $\delta = -0.258214$, $\lambda = -0.059778$, $\mu = 0.126352$. Matrix $\mathbf{B_1}$ of size 5×5 has the diagonally symmetric form:

$$\mathbf{B_1} = \begin{bmatrix} 0 & 0 & 0.0248 & 0.1072 & 0.0248 \\ 0 & 0.1072 & 0.5702 & 0.5702 & 0.1072 \\ 0.0248 & 0.5702 & 1.0248 & 0.5702 & 0.0248 \\ 0.1072 & 0.5702 & 0.5702 & 0.1072 & 0 \\ 0.0248 & 0.1072 & 0.0248 & 0 & 0 \end{bmatrix} \quad (19)$$

The numerator $B(z_1, z_2)$ and denominator $A(z_1, z_2)$ in (14) are in fact the Discrete Space Fourier Transforms of the corresponding matrices **B** and **A** with complex elements. Next, substituting z in (11) by the 1D to 2D mapping (14), we find the factor corresponding to the bi-quad function $H_{Bi}(z)$:

$$H_{Bi}(z) \to H_{Bi}(z_1,z_2) = \frac{P(z_1,z_2)}{Q(z_1,z_2)} = \frac{P_R(z_1,z_2) + j \cdot P_I(z_1,z_2)}{Q_R(z_1,z_2) + j \cdot Q_I(z_1,z_2)} = \frac{\mathbf{z}_1 \times \mathbf{P} \times \mathbf{z}_2^T}{\mathbf{z}_1 \times \mathbf{Q} \times \mathbf{z}_2^T} \quad (20)$$

Applying this mapping to all three bi-quad factors of prototype (2), we finally derive the factored transfer function in z_1 and z_2 for the overall 2D elliptically-shaped filter.

B. Design Examples of Elliptically-Shaped Filters

Some design examples are presented for elliptically-shaped filters specified by the values of scale parameter *k*, the semi-axes *E* and *F*, and orientation angle φ. The frequency response magnitudes and contour plots given in Fig.4 show a relatively accurate elliptical shape in the frequency plane, a maximally flat top and a small ripple in the stop band. Since applications of elliptically-shaped filters in image processing are relatively well known, simulation results were not included here. The aim of this paper was limited to presenting this analytic design method and to highlight its advantages over a completely numerical optimization method.

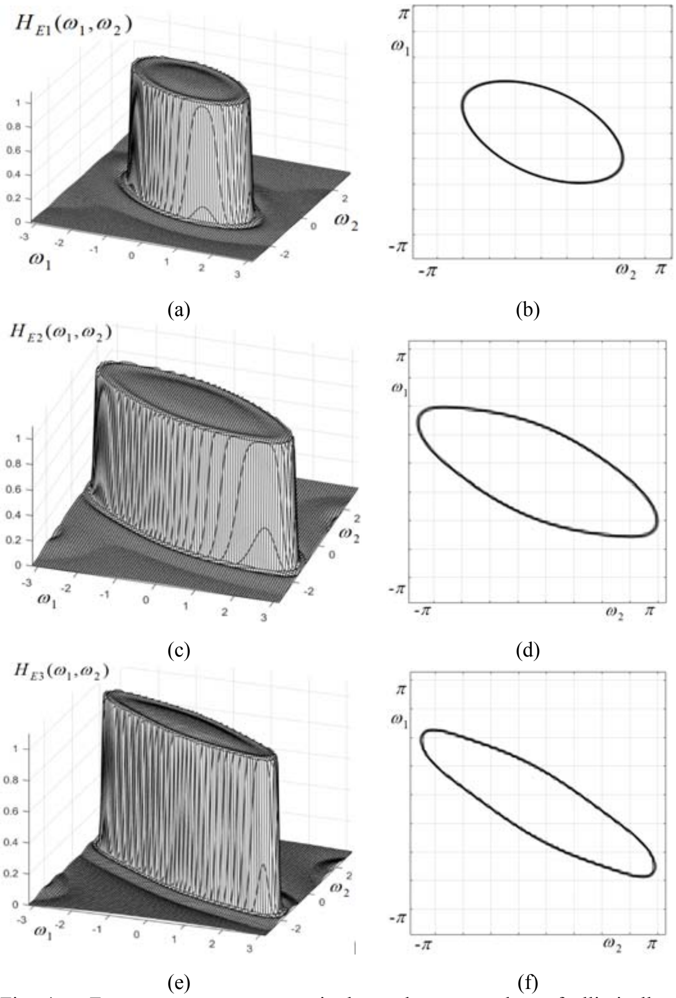


Fig. 4. Frequency response magnitudes and contour plots of elliptically-shaped filter for parameters: (a), (b) $k=0.1$, $\varphi=\pi/8$, $E=0.4$, $F=0.2$; (c), (d) $k=0.1$, $\varphi=\pi/8$, $E=0.6$, $F=0.2$; (e), (f) $k=0.1$, $\varphi=\pi/6$, $E=0.6$, $F=0.14$

C. Design of Circular Filters as a Particular Case

Using the proposed method, we can also obtain 2D filters with circular symmetry as a particular case. Indeed, if in (3) we set equal semi-axes $E=F=1$, the mapping $\omega^2 \rightarrow E_\varphi(\omega_1,\omega_2)$ takes the simpler form corresponding to a circular filter:

$$\omega^2 \rightarrow \omega_1^2+\omega_2^2 \tag{21}$$

In this particular case, the mapping (12) written for the general case of an elliptically-shaped filter, takes the simpler form:

$$z \rightarrow \exp\left(j\sqrt{k\cdot\left(\omega_1^2+\omega_2^2\right)}\right) \cong \frac{\left(0.664635-0.212793\cdot k\cdot\left(\omega_1^2+\omega_2^2\right)\right) +j\cdot\left(0.896434+0.25193\cdot k\cdot\left(\omega_1^2+\omega_2^2\right)\right)}{\left(1+0.209288\cdot k\cdot\left(\omega_1^2+\omega_2^2\right)\right) +j\cdot\left(0.515203-0.268335\cdot k\cdot\left(\omega_1^2+\omega_2^2\right)\right)} \tag{22}$$

Using again in mapping (22) the approximation given by (13),

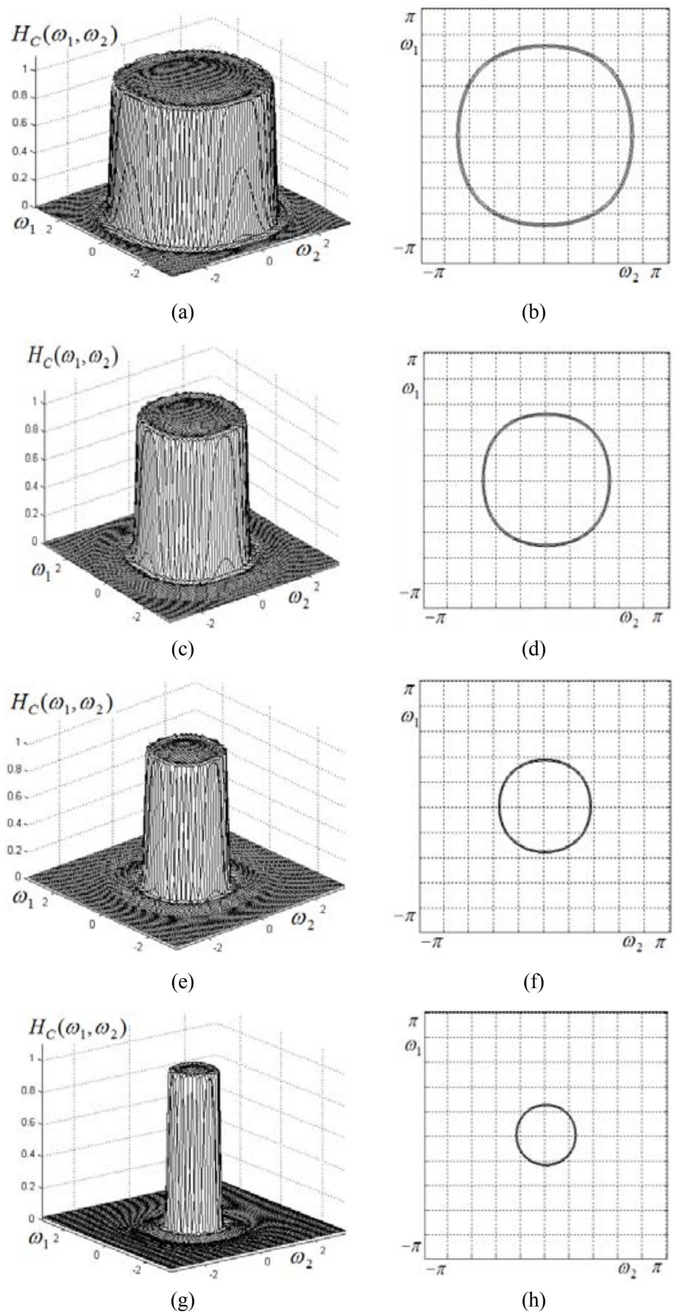


Fig. 5. Frequency response magnitude and contour plot of the circular filter for parameter values: (a), (b) $k=0.5$; (c), (d) $k=1$; (e), (f) $k=2$; (g), (h) $k=4$.

we obtain a frequency transformation in matrix form similar to (14), but somewhat simpler, in which the component matrices are of size 3×3 instead of 5×5. Thus, the order of the designed circular filter (or the size of filter matrices) results half compared to the order of the elliptically-shaped filter.

As in the previous case of an elliptically shaped filter, some design examples are shown to illustrate the design procedure. In Fig.5 the frequency response magnitudes and corresponding contour plots are displayed for given values of the scale parameter *k;* we notice that they generally have an accurate circular shape in the frequency plane, a maximally flat top and a small ripple in the stop band.

The stability of the designed 2D filters was not investigated in this paper, but it will be studied in detail in further work on this subject. As is well known, to analyze and ensure the stability of 2D systems is generally much more complicated than in the case of 1D systems. If the prototype filter is stable, and if the frequency transformations used in design preserve stability, the derived 2D filters should also result stable. There exist various criteria to test and ensure stability [22] and also stabilization procedures which can be applied [23]. Moreover, it is known that some unstable filters, with poles both inside and outside the unit circle, have transfer functions that can be separated into one stable part and one unstable part. In specific cases, the latter can be implemented using so-called backward filtering. The input sequence is first filtered in the forward direction by the stable part, then the inverted sequence is filtered backwards by the unstable part. The applicability of such technique for the designed filters will be studied in further work.

IV. Conclusion

An efficient analytical design technique was proposed for 2D recursive filters with elliptically or circularly shaped frequency response, having adjustable orientation and bandwidth. This method starts from a low-pass prototype filter, to which a specific frequency transformation is applied, thereby obtaining directly the desired 2D transfer function in a factored form. The derived filters are parametric, since their frequency response depends explicitly on the specified parameters giving the orientation angle and selectivity. The filters also have an accurate elliptical or circular shape, with low shape distortions. The main advantage of the introduced method is that the 2D filter is designed in a closed form and is adjustable. Changing the specifications, the 2D filter matrices result directly, without the need to resume the entire design procedure from the start. Further research envisages an efficient implementation of these filters and testing them on various real-life images.

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