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Riesz spaces using modulus function

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***Abstract*—The aim of this paper is to introduce the new techniques of modulus function involving the sequences of Riesz nature. Some of its basic properties has been established.**

***Index Terms*—Modulus function, paranormed sequence, infinite matrices.**

I. Introduction

We denote the set of all sequences with complex terms by ω. It is a routine verification that Ω is a linear space with respect to the coordinate wise addition and scalar multiplication of sequences which are defined, as usual, by

$$\zeta+\eta=(\zeta_k)+(\eta_k)=(\zeta_k+\eta_k)$$

and

$$\beta\zeta=\beta(\zeta_k)=(\beta\zeta_k),$$

respectively; with $\zeta=(\zeta_k)$, $\eta=(\eta_k)\in\Omega$ and $\beta\in\mathbf{C}$. By a $sequence\ \ space$ we define a linear subspace of ω i.e., the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper N and C denotes the set of non-negative integers and the set of complex numbers, respectively. Let ℓ_∞, c and c_0, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by ℓ_1, $\ell(p)$, cs and bs we denote the spaces of all absolutely, p-absolutely convergent, convergent and bounded series, respectively.

We call a sequence space is as a function space whose elements are functions from natural numbers $\mathbb{N}$ to the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ the complex numbers. The set of every sequences (real or complex) will be abbreviated by Ψ. The bounded sequences, p-absolutely sequence, convergent sequences and null sequences will be abbreviated by ℓ_∞, ℓ_p, c and c_0 respectively as defined in [8], [24].

For an infinite matrix $T=(t_{i,j})$ and $\nu=(\nu_k)\in\Psi$, the T-transform of ν is $T\nu=\{(T\nu)_i\}$ provided it exists $\forall\ i\in\mathbb{N}$, where $(T\nu)_i=\sum_{j=0}^{\infty}t_{i,j}\nu_j$.

For an infinite matrix $T=(t_{i,j})$, the set G_T, where

$$G_T=\{\nu=(\nu_j)\in\Psi\ :\ T\nu\in G\},\qquad(1)$$

is known as the matrix domain of T in G [13].

A infinite matrix $G=(\varrho_{nk})$ is said to be regular if and only if the following conditions hold:

$(i)\quad \lim_{n\to\infty}\sum_{k=0}^{\infty}\varrho_{nk}=1,$

$(ii)\quad \lim_{n\to\infty}\varrho_{nk}=0,\quad (k=0,1,2,...),$

$(iii)\quad \sum_{k=0}^{\infty}|\varrho_{nk}|<M,\quad (M>0,\ n=0,1,2,...).$

Let (q_k) be a sequence of positive numbers and let us write, $Q_n=\sum_{k=0}^{n}q_k$ for $n\in N$. Then the matrix $R^q=(r^q_{nk})$ of the Riesz mean (R,q_n) is given by

$$r^q_{nk}=\begin{cases}\frac{q_k}{Q_n}, & \text{if } 0\le k\le n,\\ 0 & \text{if } k>n\end{cases}$$

The Riesz mean (R,q_n) is regular if and only if $Q_n\to\infty$ as $n\to\infty$ (see, [18], [22]).

Quite recently, in (see, [21]) the author has introduced the following:

$$r^q(u,p)=\left\{\zeta=(\zeta_k)\in\omega:\sum_k\left|\frac{1}{Q_k}\sum_{j=0}^{k}u_jq_j\zeta_j\right|^{p_k}<\infty\right\}$$

where, $0<p_k\le H<\infty$.

In [10], the author had given the difference sequence spaces $W(\triangle)$ as follows

$$W(\triangle)=\{\zeta=(\zeta_k)\in\omega:(\triangle\zeta_k)\in W\}$$

where, $W\in\{\ell_\infty,c,c_0\}$ and $\triangle\zeta_k=\zeta_k-\zeta_{k+1}$.

In [2], the author has studied the sequence space as

$$bv_p=\left\{\zeta=(\zeta_k)\in\omega:\sum_k|\Delta x_k|^p<\infty\right\},$$

where $1\le p<\infty$. With the notation of (1), the space bv_p can be redefined as

$$bv_p=(l_p)_\triangle,1\le p<\infty$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \leq k \leq n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

In [14], the author introduced the concept of modulus function. We call a function $\mathcal{F} : [0,\infty) \to [0,\infty)$ to be modulus function if

(i) $\mathcal{F}(\zeta) = 0$ if and only if $\zeta = 0$,

(ii) $\mathcal{F}(\zeta+\eta) \leq \{(\zeta) + \{(\eta)\ \forall\ \zeta \geq 0, \eta \geq 0$

(iii) $\mathcal{F}$ is increasing, and

(iv) $\mathcal{F}$ is continuous from the right at 0.

One can easily see that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are modulus functions then so is $\mathcal{F}_1 + \mathcal{F}_2$; and the function $\mathcal{F}^j$ $(j \in \mathbf{N})$, the composition of a modulus function $\mathcal{F}$ with itself j times is also modulus function.

Recently, in [19] the new space was introduced by using notion of modulus function as follows:

$$L(\mathcal{F}) = \left\{ \zeta = (\zeta_r) : \sum_r |\mathcal{F}(\zeta_r|)| < \infty \right\}$$

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors. $(\ell_\infty)_{N_q}$ and c_{N_q}(see, [23]), $(\ell_p)_{C_1} = X_p$ and $(\ell_\infty)_{C_1} = X_\infty$(see, [17]), $(\ell_\infty)_{R^t} = r^t_\infty$, $(c)_{R^t} = r^t_c$ and $(c_o)_{R^t} = r^t_0$(see, [9]), $(\ell_p)_{R^t} = r^t_p$ (see, [1]), $(\ell_p)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$ (see, [3]), $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$(see, [4]), $[c_0(u,p)]_{A^r} = a^r_0(u,p)$ and $[c(u,p)]_{A^r} = a^r_c(u,p)$ (see, [5], $r^q(u,p) = \{l(p)\}_{R^q_u}$ (see, [21])and etc.

II. The sequence space $r^q_{\mathcal{F}}(\triangle^p_g)$ of non-absolute type

In this section, we define the Riesz sequence space $r^q_{\mathcal{F}}(\triangle^p_g)$, and prove that the space $r^q_{\mathcal{F}}(\triangle^p_g)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $\ell(p)$.

Let Λ be a real or complex linear space, define the function $\tau : \Lambda \to \mathbb{R}$ with $\mathbb{R}$ as set of real numbers. Then, the paranormed space is a pair $(\Lambda; \tau)$ and τ is a paranorm for Λ, if the following axioms are satisfied for all $\zeta,\ \eta \in \Lambda$ and for all scalars β:

(i) $\tau(\theta) = 0$,

(ii) $\tau(-\zeta) = \tau(\zeta)$,

(iii) $\tau(\zeta+\eta) \leq \tau(\zeta) + \tau(\eta), and$

(iv) scalar multiplication is continuous, that is,

$|\beta_n - \beta| \to 0$ and $h(\zeta_n - \zeta) \to 0$ imply $\tau(\beta_n\zeta_n - \beta\zeta) \to 0$ for all $\beta's$ in $\mathbb{R}$ and $\zeta's$ in Λ , where θ is a zero vector in the linear space Λ. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$and $M = max\{1, H\}$. Then, the linear space $\ell_\infty(p)$ was defined by Maddox [13] as follows :

$$\ell_\infty(p) = \{\zeta = (\zeta_k) : \sup_k |\zeta_k|^{p_k} < \infty\}$$

which is complete space paranormed by

$$\tau_1(\zeta) = \left[\sup_k |\zeta_k|^{p_k}\right]^{1/M}.$$

We shall assume throughout that $p_k^{-1} + \{p_k'\}^{-1}$ provided $1 < inf p_k \leq H < \infty$, and we denote the collection of all finite subsets of N by F, where N={0, 1, 2, ... }.

Following Altay (see, [1]- [3]), Başarir and Öztürk (see, [6]), Choudhary and Mishra (see, [7]), Ganie and Neyaz (see, [8]), Mursaleen (see, [15]), Ganie [21], Ruckle [19], Sengönül [20], we define the difference sequence space $r^q_{\mathcal{F}}(\triangle^p_g)$ as follows:

$$r^q_{\mathcal{F}}(\triangle^p_g) = \left\{ \zeta = (\zeta_k) \in \omega : \sum_k \left| \mathcal{F}\left(\frac{1}{Q_k} \sum_{j=0}^k g_k q_j \triangle\zeta_j \right) \right|^{p_k} < \infty \right\}$$

where, $0 < p_k \leq H < \infty$.

By (1), it can be redefined as

$$r^q_{\mathcal{F}}(\triangle^p_g) = \{l(p)\}_{R^q_{\mathcal{F}}(\triangle_g)}.$$

Define the sequence $\xi = (\xi_k)$, which will be used, by the $R^q_{\mathcal{F}}\triangle_g$-transform of a sequence $\zeta = (\zeta_k)$, i.e.,

$$\xi_k = f \frac{1}{Q_k} \sum_{j=0}^k g_j q_j \triangle\zeta_j. \tag{2}$$

Now, we begin with the following theorem which is essential in the text.

Theorem $r^q_{\mathcal{F}}(\triangle^p_g)$ is a complete linear metric space paranormed by $h_\triangle$, defined as

$$h_\triangle(\zeta) = \left[\sum_k \left| \mathcal{F}\left[\frac{1}{Q_k} \sum_{j=0}^{k-1} (g_j q_j - g_{j+1} q_{j+1})\zeta_j + \frac{q_k g_k}{Q_k}\zeta_k \right] \right|^{p_k} \right]^{\frac{1}{M}}$$

with $0 < p_k \leq H < \infty$.

Proof: The linearity of $r^q_{\mathcal{F}}(\triangle^p_g)$ with respect to the coordinatewise addition and scalar multiplication follows from from the inequalities which are satisfied for $\zeta, \xi \in r^q_{\mathcal{F}}(\triangle^p_g)$(see [14], p.30])

$$\left[\sum_k \left|\mathcal{F}\left[\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j - g_{j+1}q_{j+1})(\zeta_j+\eta_j) + \frac{q_kg_k}{Q_k}(\zeta_k+\eta_k)\right]\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq \left[\sum_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j - g_{j+1}q_{j+1})\zeta_j + \frac{q_kg_k}{Q_k}\zeta_k\right)\right|^{p_k}\right]^{\frac{1}{M}}$$

$$+\left[\sum_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j - g_{j+1}q_{j+1})\eta_j + \frac{q_kg_k}{Q_k}\eta_k\right)\right|^{p_k}\right]^{\frac{1}{M}} \quad (3)$$

and for any $\alpha \in \mathbf{R}$ (see, [13])

$$|\alpha|^{p_k} \leq max(1, |\alpha|^M). \quad (4)$$

It is clear that, $h_\triangle(\theta)$=0 and $h_\triangle(\zeta) = h_\triangle(-\zeta)$ for all $\zeta \in r^q_{\mathcal{F}}(\triangle^p_g)$. Again the inequality (3) and (4), yield the subadditivity of $h_\triangle$ and

$$h_\triangle(\alpha\zeta) \leq max(1, |\alpha|)h_\triangle(\zeta).$$

Let $\{\zeta^n\}$ be any sequence of points of the space $r^q_{\mathcal{F}}(\triangle^p_g)$ such that $h_\triangle(\zeta^n - \zeta) \to 0$ and (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$h_\triangle(x^n) \leq h_\triangle(x) + h_\triangle(x^n - x)$$

holds by subadditivity of $h_\triangle$, $\{h_\triangle(\zeta^n)\}$ is bounded and we thus have

$$h_\triangle(\alpha_n\zeta^n - \alpha\zeta)$$

$$= \left[\sum_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k}(g_jq_j - g_{j+1}q_{j+1})(\alpha_n\zeta^n_j - \alpha\zeta_j)\right)\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq |\alpha_n - \alpha|^{\frac{1}{M}}\, h_\triangle(\zeta^n) + |\alpha|^{\frac{1}{M}}\, h_\triangle(\zeta^n - \zeta)$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, $h_\triangle$ is paranorm on the space $r^q_{\mathcal{F}}(\triangle^p_g)$.

It remains to prove the completeness of the space $r^q_{\mathcal{F}}(\triangle^p_g)$. Let $\{\zeta^j\}$ be any Cauchy sequence in the space $r^q_{\mathcal{F}}(\triangle^p_g)$, where $\zeta^i = \{\zeta^i_0, \zeta^i_1, ...\}$. Then , for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$h_\triangle(\zeta^i - \zeta^j) < \epsilon \quad (5)$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $h_\triangle$ and for each fixed $k \in \mathbf{N}$ that

$$\left|(R^q_{\mathcal{F}}\triangle_g\zeta^i)_k - (R^q_g\triangle\zeta^j)_k\right| \leq \left[\sum_k \left|(R^q_{\mathcal{F}}\triangle_g\zeta^i)_k - (R^q_{\mathcal{F}}\triangle_g\zeta^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R^q_{\mathcal{F}}\triangle_g\zeta^0)_k, (R^q_{\mathcal{F}}\triangle_g\zeta^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbf{N}$. Since R is complete, it converges,say, $(R^q_{\mathcal{F}}\triangle_g\zeta^i)_k \to ((R^q_{\mathcal{F}}\triangle_g\zeta)_k$ as $i \to \infty$. Using these infinitely many limits $(R^q_{\mathcal{F}}\triangle_g\zeta)_0, (R^q_g\triangle\zeta)_1, \ldots$, we define the sequence $\{(R^q_{\mathcal{F}}\triangle_g\zeta)_0, (R^q_{\mathcal{F}}\triangle_g\zeta)_1, \ldots\}$. From (5) for each $m \in \mathbf{N}$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{m}\left|(R^q_{\mathcal{F}}\triangle_g\zeta^i)_k - (R^q_{\mathcal{F}}\triangle_g\zeta^j)_k\right|^{p_k} \leq h_\triangle(\zeta^i - \zeta^j)^M < \epsilon^M. \quad (6)$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (6) and then $m \to \infty$, we obtain

$$h_\triangle(\zeta^i - \zeta) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (6) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in N$ that

$$\left[\sum_{k=0}^{m}|(R^q_{\mathcal{F}}\triangle_g\zeta)_k|^{p_k}\right]^{\frac{1}{M}} \leq h_\triangle(\zeta^i - \zeta) + h_\triangle(\zeta^i) \leq 1 + h_\triangle(\zeta^i)$$

which implies that $\zeta \in r^q_{\mathcal{F}}(\triangle^p_g)$. Since $h_\triangle(\zeta - \zeta^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $\zeta^i \to \zeta$ as $i \to \infty$, hence we have shown that $r^q_{\mathcal{F}}(\triangle^p_g)$ is complete, hence the proof .

Note that one can easily see the absolute property does not hold on the spaces $r^q_{\mathcal{F}}(\triangle^p_g)$, that is $h_\triangle(\zeta) \neq h_\triangle(|\zeta|)$ for atleast one sequence in the space $r^q_{\mathcal{F}}(\triangle^p_g)$ and this says that $r^q_{\mathcal{F}}(\triangle^p_g)$ is a sequence space of non-absolute type.

III. Inclusion relations

In this section, we investigate some of its inclusions properties .

Theorem If p_k and t_k are bounded sequences of positive real numbers with $0 < p_k \leq t_k < \infty$ for each $k \in \mathbf{N}$, then for any modulus function $\mathcal{F}$, $r^q_{\mathcal{F}}(\triangle^p_g) \subseteq r^q_{\mathcal{F}}(\triangle^t_g)$

Proof: For $\zeta \in r^q_{\mathcal{F}}(\triangle^p_g)$ it is obvious that

$$\sum_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j - g_{j+1}q_{j+1})\zeta_j + \frac{q_kg_k}{Q_k}\zeta_k\right)\right|^{p_k} < \infty.$$

Consequently, for sufficiently large values of k say $k \geq k_0$ for some fixed $k_0 \in \mathbf{N}$.

$$\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j - g_{j+1}q_{j+1})\zeta_j + \frac{q_kg_k}{Q_k}\zeta_k\right)\right| < \infty$$

But $\mathcal{F}$ being increasing and $p_k \leq t_k$, we have

$$\sum_{k\geq k_0}\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j-g_{j+1}q_{j+1})\zeta_j+\frac{q_kg_k}{Q_k}\zeta_k\right)\right|^{t_k}$$
$$\leq\sum_{k\geq k_0}\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k-1}(g_jq_j-g_{j+1}q_{j+1})\zeta_j+\frac{q_kg_k}{Q_k}\zeta_k\right)\right|^{p_k}<\infty.$$

From this, it is clear that $\zeta\in r^q_{\mathcal{F}}(\triangle^t_g)$ and the result follows.$\diamond$

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