

Optimal Identifying Codes in Oriented Paths and Circuits

Ahmed Semri

LaROMaD, Faculty of Mathematics, USTHB,
 BP 32 Bab Ezzouar, El-Alia 16111,
 Algiers, Algeria.

Hillal Touati

LaROMaD, Faculty of Mathematics, USTHB,
 BP 32 Bab Ezzouar, El-Alia 16111,
 Algiers, Algeria.

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Abstract—Identifying codes in graphs are related to the classical notion of dominating sets [1]. Since their first introduction in 1998 [2], they have been widely studied and extended to several applications, such as: detection of faulty processor in multiprocessor systems, locating danger or threats in sensor networks.

Let $G=(V,E)$ an unoriented connected graph. The minimum identifying code in graphs is the smallest subset of vertices C , such that every vertex in V have a unique set of neighbors in C . In our work, we focus on finding minimum cardinality of an identifying code in oriented paths and circuits.

Keywords: Identifying code, Oriented paths, Circuits.

I. INTRODUCTION

After their introduction for modelize the problem of detecting a failures in multiprocessors system[2], the notion of identifying code have been also used in numerous applications such as locating and detecting danger or threats in indoor environments [3] and wireless Network Monitoring [4]. Actually, the theories and the applications of identifying code attracted the attention of many researchers. This led to many results that have been obtained in hypercubes [5], [6], grids [7], [8], paths and cycles [9], [10], [11], [12]. An updated bibliography on the subject can be found in [13].

Let $G = (V, E)$ a simple, connected and undirected graph, where V is the set of vertices and E the set of edges. We call a *code* any noempty subset of vertices and its elements a *codewords*. We define $B_r(v)$, a *ball* of center v and radius r by $B_r(v) = \{u \in V | d(u, v) \leq r\}$, where $d(x, y)$ denotes the length (number of edges) of the shortest path between the vertices x and y .

Thus, an r -identifying code is any subset $C \subseteq V$ such that:

- 1) $\forall v \in V, B_r(v) \cap C \neq \emptyset$,
- 2) $B_r(u) \cap C \neq B_r(v) \cap C$, for all $u, v \in V, u \neq v$.

Therefore, the first condition ensures that every vertice of the graph is covered by at least one codeword, and the second one ensures that every pair of different vertices is separated. In other words, each vertex of the graph G is covered by a unique set of codewords. The set $B_r(v) \cap C$, denoted also by $I_r(v)$, is called the *r -identifying set* of v (simply identifying set when $r = 1$).

For an oriented graph $G = (V, A)$, we just replace $B_r(v) \cap C$ by $\Gamma_r^-(v) \cap C = I_r^-(v)$, where the set $\Gamma_r^-[x] = \{y \in V |$

$d(y, x) \leq r\}$ contains all the predecessors at distance at most r from x (x within).

The problem with identifying code is finding one with the fewest elements. This problem is known to be an NP-complete problem [14].

Our work studies this problem in oriented graphs, particularly in oriented paths and circuits. Thus, some partial results were obtained.

II. IDENTIFYING CODE IN ORIENTED PATHS

As mentioned before, we are interested in finding an optimal identifying code in oriented paths and circuits. First, we give some notations that will be used in the next paragraphs.

We denote by \mathcal{P}_n an oriented path of length n , ie it contains exactly $n + 1$ vertices, and \mathcal{C}_n a circuit of length n . Let $M_r^-(G)$ denotes the minimum cardinality of an r -identifying code in graph G .

First, we investigate the 1-identifying code (or simply identifying code, if there's no ambiguity) then the 2-identifying code.

A. 1-Identifying Code

Lemma 1: A subset $C \subseteq V$ is an identifying code in \mathcal{P}_n if and only if: 1.

- 1) The two vertices x_0 and x_1 belong to the code C ,
- 2) For every pair of consecutive vertices x_i and x_{i+1} , $i \in \{2, 3, \dots, n-1\}$, x_i or x_{i+1} is a codeword.
- 3) For every triplet of consecutive vertices x_i, x_{i+1} and x_{i+2} , $i \in \{2, 3, \dots, n-2\}$, x_i or x_{i+2} is a codeword.

Proof: For (1), x_0 is covered by itself, then x_0 must be a codeword. In addition, x_1 must belong to code to separate the pairs of vertices (x_0, x_1) .

For the second condition, suppose that $x_i \notin C$ and $x_{i+1} \notin C$. Then $I^-(x_{i+1}) = \emptyset$ (x_{i+1} isn't covered). Then either x_i or x_{i+1} must belong to the code.

For (3), suppose that neither x_i nor x_{i+2} belong to the code. Then we have two cases:

Case 1 If $x_{i+1} \in C$, then $I^-(x_{i+1}) = I^-(x_{i+2}) = \{x_{i+1}\}$, ie the two vertices x_{i+1} and x_{i+2} aren't separated.

Case 2: If $x_{i+1} \notin C$, necessarily the two vertices x_{i+1} and x_{i+2} will not be covered because $\mathcal{I}^-(x_{i+1}) = \mathcal{I}^-(x_{i+2}) = \emptyset$.

Thus, in the two cases either x_i or x_{i+2} must be a codeword. One can see the necessity and the sufficiency of the three conditions to cover all the vertices of \mathcal{P}_n , this comes from the fact that every semi-ball contains exactly two consecutive vertices.

Now, let's show the sufficiency of the three conditions for the separation.

Let x_i and x_j be two vertices, then we have two cases:

Case 1 The vertices are neighbours. Without loss of generality, we put $j = i + 1$. Above, we have shown that Condition (1) separates the vertices x_0 and x_1 . Therefore, by Condition (3), we know that $x_{i-1} \in C$ or $x_{i+1} \in C$, then we have $\mathcal{I}^-(x_i) \neq \mathcal{I}^-(x_j)$. Thus, x_i and x_j were separated.

Case 2 x_i et x_j are not neighbours, ie the distance $d(x_i, x_j) \geq 2$.

Suppose, without loss of generality, that $j = i + 2$. Then, we have $\Gamma_1^-[x_i] = \{x_{i-1}, x_i\}$ and $\Gamma_1^-[x_j] = \{x_{i+1}, x_{i+2}\}$, but by Condition (2), we have $\mathcal{I}^-(x_i) \neq \mathcal{I}^-(x_j)$. Then x_i and x_j are separated.

This completes the proof of the lemma. ■

By the following theorem we give a minimum cardinality of an identifying code in oriented paths.

Theorem 1: For an oriented path \mathcal{P}_n , we have:

$$M_1^-(\mathcal{P}_n) = \begin{cases} 2p & \text{if } n=3p, \\ 2p + 1 & \text{if } n=3p+1, \\ 2p + 2 & \text{if } n=3p+2. \end{cases}$$

Proof: Let V the set vertices of \mathcal{P}_n . If we denote by L the set of vertices identified by one codeword (or covered by one codeword). Then, the other vertices ($|V| - |L|$) are covered by at least two codewords. In other words, C double covers these vertices. Thus, using the fact that $|L| \leq |C|$ (at most $|C|$ vertices are covered by one codeword), therefore we have the following inequality

$$2(|V| - |L|) + |L| \leq \sum_{x_i \in C} |\Gamma_1^-[x_i]| \leq 2|C|$$

so

$$\begin{aligned} 2|V| - |L| &\leq 2|C| \Leftrightarrow 2|V| - |C| \leq 2|C| \\ &\Leftrightarrow \frac{2}{3}|V| \leq |C| \end{aligned}$$

which leads to

$$|C| \geq \left\lceil \frac{2n}{3} \right\rceil$$

Let $n = 3p + q$, with $q \in \{0, 1, 2\}$. Thus we obtain:

$$\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{2(3p+q)}{3} \right\rceil = 2p + \left\lceil \frac{2q}{3} \right\rceil$$

Therefore: If $q = 0$, then $\lceil \frac{2q}{3} \rceil = 0$. If $q = 1$, then $\lceil \frac{2q}{3} \rceil = 1$. And finally, if $q = 2$, then $\lceil \frac{2q}{3} \rceil = 2$.

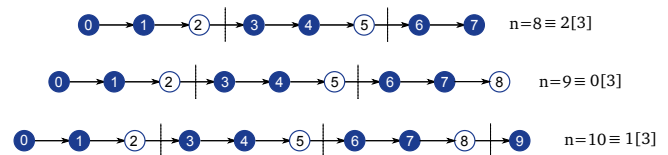


Fig. 1 – An Example of identifying code in oriented paths of length 7,8 and 9

To conclude, we exhibit an identifying code which reaches the bound for each case. Thus, we can choose $C = \{x_i | i \equiv 0[3] \text{ and } i \equiv 1[3]\}$ for all cases (see figure 1). ■

B. 2-Identifying Code

Before proceeding to the proof of our results we need the following result:

Let $\mathcal{P}_n = \{x_0, x_1, \dots, x_n\}$ an oriented path of length n , and C a code in \mathcal{P}_n .

Lemma 2: A subset C is a 2-identifying code in \mathcal{P}_n if and only if the following three conditions are satisfied: 1.

- 1) The vertices x_0, x_1 and x_2 must belong to C ,
- 2) For every group of three consecutive vertices, x_i, x_{i+1}, x_{i+2} , $i \in \{3, 4, \dots, n-2\}$, at least one belong to the code C ,
- 3) For every group of four consecutive vertices, $x_i, x_{i+1}, x_{i+2}, x_{i+3}$, $i \in \{3, 4, \dots, n-3\}$, we can't have $x_i \notin C$ and $x_{i+3} \notin C$.

Proof: For the condition (1), if $x_0 \notin C$, then we have $\Gamma_2^-[x_0] = \emptyset$ (the vertex is not covered). Thus x_0 must be a codeword.

For (2), we can see that if any of the three vertices x_i, x_{i+1}, x_{i+2} , for all $i \in \{3, 4, \dots, n-2\}$, is not a codeword then $\mathcal{I}^-(x_{i+2}) = \emptyset$ which contradicts the fact that C is a covering code.

Finally for the condition (3), suppose that neither x_i nor x_{i+3} is in C then the two vertices x_{i+2} and x_{i+3} will not be separated because $\mathcal{I}_2^-(x_{i+2}) = \mathcal{I}_2^-(x_{i+3}) = \{x_{i+1}, x_{i+2}\}$. Thus $x_i \in C$ or $x_{i+3} \in C$, for all $i \in \{3, 4, \dots, n-2\}$.

We remark that the condition (1) and (2) are necessary and sufficient for the condition that $\Gamma_2^-[x_i] \cap C \neq \emptyset$ for all $i \in \{0, 1, \dots, n\}$.

We need to show that the three conditions of the previous lemma are sufficient for the separation. Let x_i and x_j be two distinct vertices. Thus, two cases appear:

Case 1 The two vertices are neighbours, $j = i + 1$. In this case, by (1) we have the pairs (x_0, x_1) , (x_1, x_2) , (x_2, x_3) separated, and by the condition (3), we have $x_{i-3} \in C$ and $x_i \in C$ for all pairs (x_i, x_{i+1}) , where $i \in \{3, 4, \dots, n-1\}$. We can observe that $\mathcal{I}_2^-(x_i) \neq \mathcal{I}_2^-(x_{i+1})$ for all pairs of consecutive vertices. Thus, the vertices x_i and x_j are separated by the code C .

Case 2 The two vertices x_i and x_j are at distance at least 2, ie $d(x_i, x_j) \geq 2$. In this case, if $d(x_i, x_j) > 2$ ($j > i + 2$), then by (2) we have $\mathcal{I}_2^-(x_i) \neq$

$\mathcal{I}_2^-(x_j)$, and if $d(x_i, x_j) = 2$, then by the conditions (2) and (3) we have also $\mathcal{I}_2^-(x_i) \neq \mathcal{I}_2^-(x_j)$. Thus, in this case, also x_i and x_j are separated by the code C .



For more clearness, we will denote each vertex x_i by its subscript i .

We know that each 2-identifying code is 2-separator in \mathcal{P}_n . Also, we know that for each vertex $i \in \mathcal{P}_n$ ($i \geq 2$) we have $\Gamma_2^-[i] \Delta \Gamma_2^-[i+1] = \{i-2, i+1\}^1$, then $\forall i \in \mathcal{P}_n$ one of the vertices $i-2$ and $i+1$ must belong to C (condition (3) of lemma 2). Thus, we have $i-2 \in C$ or $i+1 \in C$ for each vertex $i \in \{2, n-1\}$ from \mathcal{P}_n (see figure 2). Such disjunction will be called (ie $i-2 \in C$ or $i+1 \in C$) *Elementary Constraint* (EC), so it is abbreviated as $i-2 \vee i+1$.

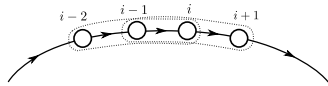


Fig. 2 – One of the two vertices i or $i+3$ belong to a code to separate $i+2$ and $i+3$

Next, we introduce an example which clear up some notations that we will use in the rest of this paper.

Example 1: Let $\mathcal{P}_{10} = x_0, x_1, \dots, x_9$, an oriented path of length 9. To obtain a 2-identifying code we have to separate nine pairs of consecutive vertices. Thus, we have nine ECs to satisfy those we enumerate as follows : $0 \vee 3, 3 \vee 6, 6 \vee 9, 1 \vee 4, 4 \vee 7, 2 \vee 5, 5 \vee 8$. Note that we omit the two ECs which separate the pairs (0,1) et (1,2), because they are separated (condition (1) of lemma 2). The set of these constraints will be called *General Constraint* (GC). This set of elementary constraints (or GC) can be partitioned in three subsets of constraints, called *Partial Constraints* (PC), such as :

$$\begin{aligned} &0 \vee 3, 3 \vee 6, 6 \vee 9 \\ &1 \vee 4, 4 \vee 7 \\ &2 \vee 5, 5 \vee 8 \end{aligned}$$

However, in order to get a general formulation, we give some adaptation for this notation. Thus, we can write the above PCs as follows:

$$\begin{aligned} &0 \vee 0 + 1 \times 3, 0 + 1 \times 3 \vee 0 + 2 \times 3, 0 + 2 \times 3 \vee 0 + 3 \times 3 \quad (C_0) \\ &1 \vee 1 + 1 \times 3, 1 + 1 \times 3 \vee 1 + 2 \times 3 \quad (C_1) \\ &2 \vee 2 + 1 \times 3, 2 + 1 \times 3 \vee 2 + 2 \times 3 \quad (C_2) \end{aligned}$$

we call $C_i, i = 0, 1, 2$, the partial constraint i .

In general, if the number of vertices is n , then we suppose that $n = 3p + q$, with $p \in \{1, 2, \dots, \lfloor \frac{n}{3} \rfloor\}$ and $q \in \{0, 1, 2\}$. Thus the PC i has the following form:

$$i \vee i + 1 \times 3, i + 1 \times 3 \vee i + 2 \times 3, \dots, i + (s_i - 1) \times 3 \vee i + s_i \times 3$$

¹ $A \Delta B = A \cup B \setminus A \cap B$, called symmetric difference

where s_i is the greatest integer for which the following inequality checked:

$$i + s_i \times 3 \leq n.$$

In the above example, we have $s_0 = 3$ for the PC C_0 .

Let V_i be the set of vertices in the PC i . We remark that $V_i \cap V_j = \emptyset$, for all $i \neq j | i, j \in \{0, 1, 2\}$. In other word, all the PCs have disjoint sets of vertices.

Thus, satisfying the GC, to obtain an 2-identifying code, amounts to satisfy all PCs.

Using this notation, we get the following result:

Theorem 2: Given an oriented path \mathcal{P}_n of length n , where $n = 3p + q$ and $q \in \{0, 1, 2\}$. If C is an 2-identifying code in \mathcal{P}_n . Then:

- (1) If $p = 0$, then $M_2^-(\mathcal{P}_n) = q + 1$
- (2) If $q = 0, p \geq 1, M_2^-(\mathcal{P}_n) = \begin{cases} \frac{3p}{2} + 1 & \text{if } p \text{ is even} \\ \frac{3(p+1)}{2} & \text{if } p \text{ is odd} \end{cases}$
- (3) If $q = 1, p \geq 1, M_2^-(\mathcal{P}_n) = \begin{cases} \frac{3p}{2} + 2 & \text{if } p \text{ is even} \\ \frac{3(p+1)}{2} & \text{otherwise} \end{cases}$
- (4) If $q = 2, p \geq 1, M_2^-(\mathcal{P}_n) = \begin{cases} \frac{3p}{2} + 2 & \text{if } p \text{ is even} \\ \frac{3(p+1)}{2} + 1 & \text{otherwise} \end{cases}$

Proof: If $p = 0$ ($n \leq 2$) the minimum cardinality of a 2-identifying code in \mathcal{P}_n is deduced from the first condition of lemma 2.

For the second case, ie $p \geq 1$ and $q = 0$, we know, by the condition (1) of lemma 2, that the vertices 0, 1 and 2 belong to the code, which satisfies the first EC of the PCs 0,1 and 2. On the other hand, we have, by condition (2) of lemma 2, necessary at least one codeword between 3, 4 and 5, thus we have one EC between the PCs 0,1 and 2 for which two vertices are a codeword. Without loss of generality, let 3 this vertex, then this satisfies two ECs in the PC 0. In this case, we need to satisfy $(p-3)$ ECs, then at least $\lceil \frac{p-3}{2} \rceil$ codeword are needed to satisfy the rest of ECs in PC 0.

In addition, for each of the partial constraints 1 and 2 we have one elementary constraint satisfied (since $1, 2 \in C$), then $(p-2)$ ECs aren't satisfied for each one. Thus, at least $\lceil \frac{p-2}{2} \rceil$ codewords are needed to satisfy the rest of ECs for PCs 1 and 2.

We conclude that we need, totally, at least:

$$4 + 2 \left\lceil \frac{p-2}{2} \right\rceil + \left\lceil \frac{p-3}{2} \right\rceil$$

codewords to satisfy the general constraint.

If p is even, then $M_2^-(\mathcal{P}_n) \geq \frac{3p}{2} + 1$. Else, $M_2^-(\mathcal{P}_n) \geq \frac{3(p+1)}{2}$.

Finally, we construct a 2-identifying code that reaches the bound to conclude. Indeed, we use the following construction: We take all vertices $i \in V$, where i is even and adding vertices 0 and 1.

The proof of the case $p \geq 1$ and $q = 1$ is similar. Indeed, concerning the first EC of CP 0, we have $(p-1)$ ECs to satisfy,



Fig. 3 – 2-identifying code ($C = \{2, 4, 6, 8\} \cup \{0, 1\}$) for an oriented path of length 8 ($n = 3 \times 3 + 0$ vertices)

since $0 \in C$. Thus, we need at least $\lceil \frac{p-1}{2} \rceil$ codewords. For the PCs 1 and 2 we need, respectively, at least $\lceil \frac{p-3}{2} \rceil$ and $\lceil \frac{p-2}{2} \rceil$ codewords to satisfy the rest of elementary constraints. Thus, we need at least :

$$4 + \left\lceil \frac{p-3}{2} \right\rceil + \left\lceil \frac{p-2}{2} \right\rceil + \left\lceil \frac{p-1}{2} \right\rceil$$

codeword.

Then, if p is even, then $M_2^-(\mathcal{P}_n) \geq \frac{3p}{2} + 2$. Else, $M_2^-(\mathcal{P}_n) \geq \frac{3(p+1)}{2}$.



Fig. 4 – A 2-identifying in an oriented path having $n = 3 \times 3 + 1$ vertices ($p = 3, q = 1$)

To conclude, we exhibit a code reaching these bound. Indeed, we remark that for the code $C = \{i | i \text{ is even}\} \cup \{0, 1\}$ the bound is attained (see figure 4).

Finally, the proof for the last case ($p \geq 1$ and $q = 2$) is also similar, we have $(p-1)$ ECs to satisfy for the PCs 0 and 1, and we need respectively at least $\lceil \frac{p-1}{2} \rceil$ and $\lceil \frac{p-2}{2} \rceil$ codewords. For the PC 2 we have $(p-1)$ EC, since $2 \in C$ then we have $(p-2)$ ECs to satisfy, then at least $\lceil \frac{p-2}{2} \rceil$ codewords are needed. Thus, we need at least :

$$4 + 2 \left\lceil \frac{p-2}{2} \right\rceil + \left\lceil \frac{p-1}{2} \right\rceil$$

codewords to satisfy the GC.

If p is even, then $M_2^-(\mathcal{P}_n) \geq \frac{3p}{2} + 2$. If p is odd, then $M_2^-(\mathcal{P}_n) \geq \frac{3(p+1)}{2}$. To conclude, we just consider the same construction as the previous cases to exhibit a 2-identifying code reaching the bound. ■

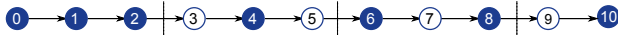


Fig. 5 – An Example of 2-identifying code in an oriented path with $3 \times 3 + 2 = 11$ vertices ($p = 3, q = 2$)

III. IDENTIFYING CODE IN CIRCUITS

In circuits we give an optimal 2-identifying code.

A. 2-Identifying Code

In the case of circuit, the two conditions of the lemma 2 are still valid. Then:

Lemma 3: Let $C_n = \{1, 2, \dots, n, 1\}$ a circuit of length n . C is a 2-identifying code for C_n if and only if the following conditions are satisfied:

- 1) For all group of three consecutive vertices x_i, x_{i+1} and x_{i+2} at least one of them is a codeword,
- 2) For all group of four consecutive vertices x_i, \dots, x_{i+3} we could not have $x_i \notin C$ and $x_{i+3} \notin C$.

Proof: The proof is similar to lemma 2. Except adding the condition that the distance between i and j isn't greater than that between j and i to show the sufficiency and the necessity of the conditions (1) and (2). ■

In the case of circuit, Although the reasoning is similar as in oriented path, there is, however, some differences. Thus, we define a partial constraint i ($i \in \{1, 2, 3\}$) as follow:

$$i \vee i+1 \times 3, i+1 \times 3 \vee i+2 \times 3, \dots, i+(s_i-1) \times 3 \vee i+s_i \times 3, i+s_i \times 3 \vee h_i$$

where s_i is the greatest integer such:

$$i + s_i \times 3 \leq n$$

and h_i is such that $i + (s_i + 1) \times 3 \equiv h_i \pmod{n}$, (ie $h_i \in \{1, 2, 3\}$).

Example 2: Let $C_n = \{1, 2, \dots, n, 1\}$ be a circuit of length n . Suppose that $n = 10$, thus $p = 3$ and $q = 1$. Then the PCs, 1, 2 et 3, can be written as follows:

$$\begin{aligned} 1 \vee 4, 4 \vee 7, 7 \vee 10, 10 \vee 3, & \quad (i = 1) \\ 2 \vee 5, 5 \vee 8, 8 \vee 1 & \quad (i = 2) \\ 3 \vee 6, 6 \vee 9, 9 \vee 2 & \quad (i = 3) \end{aligned}$$

denoting by $i|j$ the elementary constraint $i \vee j$. Then the GC is written:

$$1|4|7|10|3|6|9|2|5|8|1$$

If, for example, $n = 12$, then the GC will be:

$$1|4|7|10|1, 2|5|8|11|2, 3|6|9|12|3$$

Thus, we have determined the optimal 2-identifying code. The result is given by the following theorem:

Theorem 3: Let C_n be a circuit of length n . Then:

$$M_2^-(C_n) = \begin{cases} \emptyset & \text{if } n \leq 3, & (1) \\ 3 & \text{if } n = 4, & (2) \\ k & \text{if } n = 2k, k \geq 3, & (3) \\ k+1 & \text{if } n = 2k+1, k \geq 2, & (4) \end{cases}$$

Proof: For (1), it is clear that, if $n \leq 3$, then C_n can't admit a 2-identifying code because there are twin vertices².

²We call twin vertices every two vertices u, v such that $\Gamma_r^-(u) = \Gamma_r^-(v)$

For (2), we show that there is no 2-identifying code of cardinality 2 in a circuit of length 4. Indeed, suppose that there are only two vertices as codeword. Without loss of generality, let 1 and 3 be these vertices, then $I_2^-(1) = I_2^-(3) = \{1, 3\}$. Therefore at least three vertices must belong to a code. Finally, it suffices to exhibit a code with cardinality 3 to conclude (see figure 6).

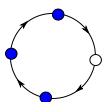


Fig. 6 – Exemple de code 2-identifiant dans C_4

Concerning (3), ie the case where the length of the circuit is even ($n = 2k$), we know that there is $\frac{n}{2} = k$ ECs to satisfy, therefore we need at least k codewords. It suffices to exhibit a 2-identifying code of cardinality k to conclude. Thus, we can take as a code the set $C = \{i | i \text{ even}, 1 \leq i \leq n\}$ (see the figure 7) hence the result.

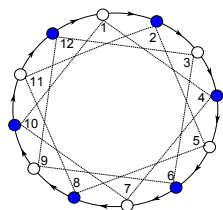


Fig. 7 – An example of 2-identifying code in circuit of length 12. Six codewords are needed to cover and separate all the vertices of the circuit.

Similarly to the previous case, when $n = 2k + 1$ (n is odd) we need at least $k + 1$ codewords to satisfy all the ECs. Thus, there is at least one EC which has its vertices belong to the code. We want to show that $|C| > k + 1$. To do it, we suppose that we can find a 2-identifying code C of cardinality $k + 1$ in a circuit of length n , and we get to a contradiction. Since we have $k + 1$ vertices as codewords, then necessarily two codewords are adjacent. Without loss of generality, let 1 and 2 these two vertices, or one of the two vertices n and 3 must be a codeword by the condition (2) of lemma 3. Thus, for every 2-identifying code at least three consecutive vertices are codewords.

Now, there are two cases:

Case 1: Suppose that the length of the circuit is equal to $n = 4p + 3$ ($k = 2p + 1$). Since at least three consecutive vertices are codewords (as mentioned previously), then we need to cover and separate $4p$ vertices. But by conditions (1) and (2) of lemma 3, we know that for every four consecutive vertices, at least two of them are codewords. Thus, we need at least $2p$ vertices as codeword, therefore $2p + 3 = k + 2$ vertices belong to a code (see the example of the figure 8). Hence the contradiction.

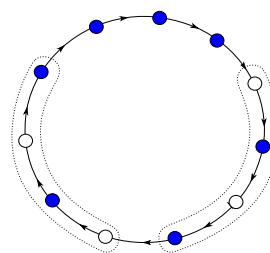


Fig. 8 – An example of optimal 2-identifying code of cardinality $2 \times 2 + 3$ in circuit of length $4 \times 2 + 3$

Case 2: In this case, we have $n = 4p + 1$ ($k = 2p$). Observing this case, we see that it's similar to the first one. We have $n = 4p + 1 = [4(p - 1) + 3] + 2$ ($k = 2p$). Thus, $2(p - 1) + 3 = k + 1$ vertices are codewords among the $4(p - 1) + 3$ vertices that a circuit contains (by condition (2) of lemma 3) adding the three consecutive vertices belong to the code.

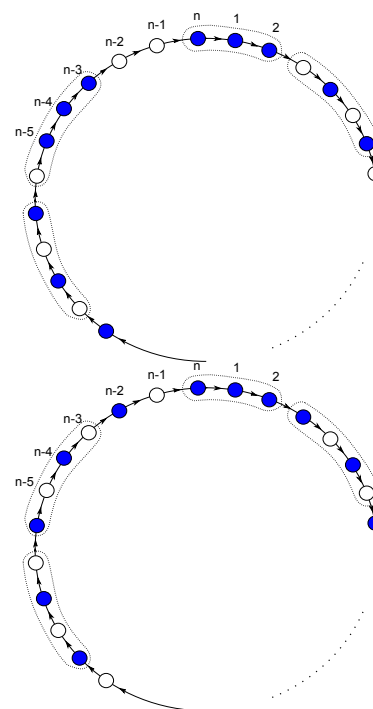


Fig. 9 – An example of two optimal 2-identifying code

In the case where the remaining two vertices don't belong to the code, saying, without loss of generality, $n - 1$ and $n - 2$, then necessarily the three vertices $n - 3, n - 4$ and $n - 5$ belong to the code (see the left representation in figure 9). Thereby, we have $4(p - 2)$ vertices that are covered and separated by $2(p - 2)$ codewords plus $n - 3, n - 4$ et $n - 5$ and the three consecutive vertices. In total there is $2(p - 2) + 6 = 2p + 2 = k + 2$ codewords. If, either $n - 1$ or $n - 2$ belong to the code then, we will need $2(p - 1)$ codewords for covering and separating the $4(p - 1)$ vertices (conditions of lemma 3) adding the three consecutive vertices

(see the left representation in figure 9). Thus, in total, we will have $1+2(p-1)+3 = 2p+2 = k+2$ codewords.

Therefore, in the two cases, we will have at least $k + 2$ codewords.

We conclude by exhibiting a 2-identifying code reaching this bound. The latter, constituted of the set of vertices $C = \{2\} \cup \{i \equiv 1[2], 1 \leq i \leq n\}$. ■

IV. CONCLUSION

In this work we gave some results about identifying code in oriented paths and circuits. It remains to determine the minimum cardinality for the case of a 1-identifying code in circuit. In addition, the question of the general case, ie r -identifying code is also an open problem.

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