An Approximate Analytical Approach for Systems of Fredholm Integro-Differential Equations of Fractional Order

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Abstract- A new technique for solving a system of fractional Fredholm integro-differential equations (IDEs) is introduced in this manuscript. Furthermore, we present a review for the derivation of the residual power series method (RPSM) to solve fractional Fredholm IDEs in the paper done by Syam, as well as, corrections to the examples mentioned in that paper. The numerical results demonstrated the new technique's applicability, efficacy, and high accuracy in dealing with these systems. On the other hand, a comparison has been done between the two schemes using the two corrected examples in addition to a problem that had been solved in many previous studies, and the results of these studies were compared with the new technique and RPSM. The comparison demonstrated clear superiority of our method over the RPSM for solving this class of equations. Moreover, they dispel the misconception that the RPSM works effectively on fractional Fredholm IDEs as mentioned in the paper done by Syam, whereas two problems solved by the RPSM produced an unaccepted error. Also, the comparison with the previous studies indicates the importance of the new method in dealing with the fractional Fredholm IDEs despite its simplicity, ease of use, and negligible computational time.

Keywords- Caputo derivative, residual power series method, fractional power series, fractional Fredholm integro-differentional equations.

I INTRODUCTION

During the last century, the fractional calculus was used to model many natural phenomena in different fields including natural sciences, engineering, and numerical analysis such as electric-circuit analysis [1], earthquake [2], fluid-dynamic traffic [3], measurement of viscoelastic material properties [4], control theory [5], solid mechanics models [6] and others.

The fractional integro-differential equations (FIDEs) have a fundamental role in modeling many phenomena in some of the above-mentioned disciplines such as presenting the control process and the dynamics of different systems, Electric circuit analysis, the activity of synaptically coupled networks of excitatory and inhibitory neurons, and viscoelastic material dynamics [7, 8, 9]. Many techniques were applied to find approximate solutions for FIDEs since the majority of them do not have exact solution, or it is difficult to calculate.

A lot of methods have been developed to find approximate solutions for single fractional Fredholm integro-differential equations (IDEs). In the lines that follow, some of these approaches will be discussed. For instance, Saeedi, Moghadam, Mollahasani and Chuev [10] utilized a new approach to solve nonlinear fractional Fredholm IDEs that relied on the Sine and Cosine wavelets. In [11] Samimi and Saeedi introduced a new technique, based on the Homotopy Perturbation method, to approximate the solution for a class of nonlinear Fredholm IDEs of fractional order. Saeedi used Block Pulse functions and Haar wavelets to solve a nonlinear fractional Fredholm IDEs in [12]. Amit Setia, Yucheng Liu and Vatsala [13] find approximate solutions for linear Fredholm FIDEs by using the Chebyshev wavelets method. Darweesh, Alquran and Aghzawi [14] developed a new algorithm, based on Haar wavelets, to find approximate solutions for a class of two-dimensional fractional Fredholm IDEs, and they introduced a modification to that method by employing the Laplace transform.

In the last fifteen years, a lot of attention has been paid to develop new techniques that solve systems of fractional IDEs due to their repeated appearance in many disciplines such as engineering and chemistry [15, 16]. Some of these techniques will be listed in the lines that follow. For instance, Qaralleh and Momani [17] used the Adomian decomposition approach to approximate the solutions for linear and nonlinear coupled systems of fractional IDEs. Sweilam and Khader [18] utilized a new approach, based on the Chebyshev pseudo-spectral method, to find approximate solutions for a coupled system of linear and nonlinear fractional Volterra IDEs in one-dimensional space. Khan and Khalil [19] proposed operational matrices based on the shifted Legendre polynomials to generate approximate solutions for a coupled system of one-dimensional linear fractional Fredholm IDEs. Wang, Xu, Wei and Xie [20] used a new technique that relies on Bernoulli wavelets to approximate the solution of coupled systems of nonlinear fractional IDEs of Volterra type. Mahdy [21] introduced a new technique with the aid of the least squares method and Hermite method to produce approximate solutions for a coupled system of one-dimensional linear Fredholm FIDEs. Mohammed and Malik [22] utilized a modified computational algorithm to solve a coupled system of one-dimensional linear fractional Volterra IDEs. Xie, Wang, Ren, Zhang and Quan[23] employed the Haar wavelets method to find approximate solutions for a coupled system of one-dimensional fractional IDEs of Volterra type.

In our framework, we are concerned with studying a system of fractional Fredholm IDEs of the following form:

$$\begin{aligned}
D^{\alpha_1}u_1(x) &= f_1(x)+\int_0^1 (u_1(t)K_{1,1}(x,t)+\\
&\qquad u_2(t)K_{1,2}(x,t)+\ldots+u_n(t)K_{1,n}(x,t))dt\\
D^{\alpha_2}u_2(x) &= f_2(x)+\int_0^1 (u_1(t)K_{2,1}(x,t)+\\
&\qquad u_2(t)K_{2,2}(x,t)+\ldots+u_n(t)K_{2,n}(x,t))dt \qquad (1)\\
&\vdots\\
D^{\alpha_n}u_n(x) &= f_n(x)+\int_0^1 (u_1(t)K_{n,1}(x,t)+\\
&\qquad u_2(t)K_{n,2}(x,t)+\ldots+u_n(t)K_{n,n}(x,t))dt,
\end{aligned}$$

with

$$\alpha_i \in (0,1] \ \ \forall\, i=1,2,3,..,n, \text{ and } x,t\in[0,a],$$

subject to the initial conditions

$$u_i(0)=c_{i,0}\ \forall i=1,2,3,\cdots,n,$$

where the fractional derivatives are in the Caputo sense, $K_{i,j}(x,t)$ are arbitrary continuous kernels over $[0,a]\ \forall\, i,j = 1,2,3,..,n$, $u_i(x)$ are analytic unknown functions to be calculated with $a>0\ \forall\, i=1,2,3,..,n$, a is the convergent radius, and $f_i(x)$ are smooth functions $\forall\, i=1,2,3,...,n$.

Our primary objectives of this work are to first find an analytic-numeric solution for system (1) using a modified scheme that is similar to the residual power series method (RPSM). Second, we review paper [24], which employed the RPSM to solve the single form of system 1, and correct the findings presented in that paper. Finally, we compare the proposed technique with the RPSM.

The residual power series method was introduced and utilized by Abu Arqub in [25] for solving fuzzy differential equations. The RPSM was applied to solve first-order initial value problems in [26] by Al-Smadi. Also, some papers solved several kinds of partial differential equations with fractional order depending on the use of RPSM [27, 28, 29]. Komashynska, Al-Smadi, Ateiwi, and Al-Obaidy [30] used the RPSM to find approximate solutions for a system of Fredholm integral equations. Moreover, the RPSM was utilized to solve several types of fractional IDEs, we mention some of them in the lines that follow. Alshammari, Al-Smadi, Hashim, and Alias [31] solved Volterra IDEs with fractional order using the RPSM, and they solved fractional mixed IDEs using the same method in [32].

The rest of this article is composed as follows; In section II, two background topics are revised concerning fractional derivative and power series generalization to fractional power series (FPS). In section III, we present the modified scheme that deals with the system (1). The performance of the scheme is illustrated and investigated in section IV, with two numerical examples to prove its efficiency. In section V, we correct the work done, in employing the RPSM to find the recursive formula for the coefficients of the unknown function, in paper [24]. Moreover, we correct the first two examples written in that paper and solve them by the two methods with a brief comparison between the two methods in solving the single equation form of the system (1). In section VI, we comment and discuss the most salient points that resulted from subsection V.B. Moreover, we mention the features of the proposed scheme in section III and its limitations. Finally, we conclude this article with brief observations and conclusions in section VII.

II Preliminaries and Basic Definitions

II.A Fractional Derivative

Throughout this section, we review the Caputo fractional derivative definition and some basic results about it. In this framework, we use the fractional derivatives described in Caputo's definition from a variety of fractional derivative definitions.

Definition 2.1.1 [33]$\Big($Caputo fractional derivative$\Big)$. Suppose that $\alpha>0, x>0,\ \alpha,x\in\mathbb{R}$. The operator of fractional calculus:

$$D_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x f^{(n)}(\xi)(x-\xi)^{n-\alpha-1}d\xi, if\\ \qquad\qquad\qquad\qquad n-1<\alpha<n\in\mathbb{N},\\ \dfrac{d^n}{dx^n}f(x), \qquad\qquad \alpha=n\in\mathbb{N}.\end{cases}$$

is called the Caputo fractional derivative or Caputo differential operator of fractional calculus of order α.

Theorem 2.1.2 [33]. Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$, and $\lambda,z\in\mathbb{C}$. Let $f(x)$ and $g(x)$ be such that both $D_x^\alpha f(x)$ and $D_x^\alpha g(x)$ exist. Then

1) Linearity: $D_x^\alpha(\lambda f(x)+g(x))=\lambda D_x^\alpha f(x)+D_x^\alpha g(x)$.

2) $D_x^\alpha c=0$ for any constant $c\in\mathbb{R}$.

3)

$$D_x^\alpha x^p=\begin{cases}\dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}, if\\ \qquad\qquad n-1<\alpha<n,\ p>n-1,\ p\in\mathbb{R},\\ 0, if \qquad n-1<\alpha<n,\ p\le n-1,\ p\in\mathbb{N}_0.\end{cases}$$

4) The Caputo fractional derivative of the exponential function has the following form:

$$D_x^\alpha e^{\lambda x}=\sum_{k=0}^{\infty}\frac{\lambda^{k+n}x^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)}=\lambda^n x^{n-\alpha}E_{1,n-\alpha+1}(\lambda x).$$

where

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$$

where $\alpha,\ \beta>0,\ \alpha,\ \beta\in\mathbb{R},\ z\in\mathbb{C}.$ is generalized Mittag-Leffler function.

II.B Fractional Power Series

In this section, we recall some fundamental definitions and theorems concerning power series generalization to fractional power series (FPS) [34].

Definition 2.2.1 [34]. A power series representation of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^{n\alpha} = c_0 + c_1(x-a)^{\alpha} + c_2(x-a)^{2\alpha} + \cdots \quad (2)$$

where $0 \leq m-1 < \alpha \leq m$ and $x \geq a$ is called a fractional power series about a, where x is a variable and c_n's are the constants called the coefficients of the series.

Theorem 2.2.2 [34]. Suppose that f has a FPS representation at a of the form:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^{n\alpha},\ 0 \leq m-1 < \alpha \leq m,\ a \leq x < a+R \quad (3)$$

If $f(x) \in C[a,a+R)$ and $D^{n\alpha} f(x) \in C(a,a+R)$ for $n = 0,1,2,\cdots$, then the coefficients c_n in equation (3) will take the form $c_n = \frac{D^{n\alpha} f(a)}{\Gamma(n\alpha+1)}$, where $D^{n\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha} \quad (n-times)$.

III The methodology of the proposed method

The purpose of this section is to obtain an analytic-numeric solution based on the fractional power series expansion for a system of Fredholm fractional IDEs subject to certain initial conditions in the form of (1).
We expand the solution to the system (1) as fractional power series about $x=0$

$$\begin{aligned}
u_1(x) &= \sum_{j=0}^{\infty} c_{1,j} \frac{x^{j\alpha_1}}{\Gamma(j\alpha_1+1)} = u_1(x,k) + R_1(x,k+1)\\
u_2(x) &= \sum_{j=0}^{\infty} c_{2,j} \frac{x^{j\alpha_2}}{\Gamma(j\alpha_2+1)} = u_2(x,k) + R_2(x,k+1)\\
&\vdots\\
u_n(x) &= \sum_{j=0}^{\infty} c_{n,j} \frac{x^{j\alpha_n}}{\Gamma(j\alpha_n+1)} = u_n(x,k) + R_n(x,k+1).
\end{aligned} \quad (4)$$

Where $u_i(x,k) = \sum_{j=0}^{k} c_{i,j} \frac{x^{j\alpha_i}}{\Gamma(j\alpha_i+1)}$ is the k-th truncated series of $u_i(x)$, and $R_i(x,m) = \sum_{j=m}^{\infty} c_{i,j} \frac{x^{j\alpha_i}}{\Gamma(j\alpha_i+1)}$. We require that $R_i(x,m)$ as small as negligible for $m \geq k+1$ on the interval $x \in (0,1)$ for all $i = 1,2,\cdots,n$. Accordingly, we replace each $u_i(x)$ by the $k-th$ truncated series $u_i(x,k)$ for all $i = 1,2,\cdots,n$ in (1). Now, we aim to find the $k-th$ approximation labeled as $u_i(x,k)$ for all $i = 1,2,\cdots,n$ to the system (1), which can be summarized in the following steps:

Step 1: Consider system (1) after replacing each $u_i(x)$ by the $k-th$ truncated series $u_i(x,k)$ for all $i = 1,2,\cdots,n$, we obtain:

$$\begin{aligned}
D^{\alpha_1} u_1(x,k) &= f_1(x) + \int_0^1 (u_1(t,k) K_{1,1}(x,t)\\
&\quad + u_2(t,k) K_{1,2}(x,t) + \ldots + u_n(t,k) K_{1,n}(x,t))dt\\
D^{\alpha_2} u_2(x,k) &= f_2(x) + \int_0^1 (u_1(t,k) K_{2,1}(x,t)\\
&\quad + u_2(t,k) K_{2,2}(x,t) + \ldots + u_n(t,k) K_{2,n}(x,t))dt\\
&\vdots\\
D^{\alpha_n} u_n(x,k) &= f_n(x) + \int_0^1 (u_1(t,k) K_{n,1}(x,t)\\
&\quad + u_2(t,k) K_{n,2}(x,t) + \ldots + u_n(t,k) K_{n,n}(x,t))dt,
\end{aligned} \quad (5)$$

Step 2: For all $m = 1,2,\cdots,k$, we apply $D^{(m-1)\alpha_i}|_{x=0}$ for all $i = 1,2,\cdots,n$ on both sides of the equations of the system (5) as follows:

$$\begin{aligned}
D^{m\alpha_1} u_1(x,k)|_{x=0} &= D^{(m-1)\alpha_1} f_1(0) +\\
&\int_0^1 u_1(t,k) D^{(m-1)\alpha_1} K_{1,1}(0,t)dt\\
&+ \int_0^1 u_2(t,k) D^{(m-1)\alpha_1} K_{1,2}(0,t)dt +\\
&\ldots + \int_0^1 u_n(t,k) D^{(m-1)\alpha_1} K_{1,n}(0,t)dt\\
D^{m\alpha_2} u_2(x,k)|_{x=0} &= D^{(m-1)\alpha_2} f_2(0) +\\
&\int_0^1 u_1(t,k) D^{(m-1)\alpha_2} K_{2,1}(0,t)dt\\
&+ \int_0^1 u_2(t,k) D^{(m-1)\alpha_2} K_{2,2}(0,t)dt +\\
&\ldots + \int_0^1 u_n(t,k) D^{(m-1)\alpha_2} K_{2,n}(0,t)dt\\
&\vdots\\
D^{m\alpha_n} u_n(x,k)|_{x=0} &= D^{(m-1)\alpha_n} f_n(0) +\\
&\int_0^1 u_1(t,k) D^{(m-1)\alpha_n} K_{n,1}(0,t)dt +\\
&\int_0^1 u_2(t,k) D^{(m-1)\alpha_n} K_{n,2}(0,t)dt +\\
&\ldots + \int_0^1 u_n(t,k) D^{(m-1)\alpha_n} K_{n,n}(0,t)dt,
\end{aligned}$$

$for\ all \quad m = 1,2,\cdots,k.$

where $D^{m\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha} \quad (m-times)$.

Step 3: The previous step yields $k \times n$ equations. Moreover, we use the initial conditions given in (1) to find the values of $c_{i,0},\ \forall\ i = 1,2,\cdots,n$ by substituting 0 in each equation of the system (4) . Finally, we solve the previous $k \times n$ equations for $c_{i,j}$ for all $i = 1,2,\cdots,n,\ \ j = 1,2,\cdots,k$ to obtain our approximate solution to the system (1):
$u(x) \approx \big(u_1(x,k), u_2(x,k), \cdots, u_n(x,k)\big)$.

IV Numerical Examples

Two numerical examples are provided in this section, to demonstrate the efficiency of the proposed scheme.

Example 1: Consider the following system of linear fractional Fredholm IDEs:

$$
\begin{aligned}
D^{\frac{1}{2}} u_1(x) &= f_1(x) + \int_0^1 x u_2(t) dt \\
D^{\frac{1}{2}} u_2(x) &= f_2(x) + \int_0^1 (t u_1(t) + \sqrt{x} u_2(t)) dt,
\end{aligned} \tag{6}
$$

subject to the initial condition $u_1(0) = 0$ and $u_2(0) = 1$, where

$$
\begin{aligned}
f_1(x) &= \sqrt{x} \sum_{j=0}^{\infty} \frac{(2ix)^j (1 + (-1)^j)}{\Gamma(j + \frac{3}{2})} - \frac{1}{4} x \sin(4) \\
f_2(x) &= 2i\sqrt{x} \sum_{j=0}^{\infty} \frac{(4ix)^j (1 - (-1)^j)}{\Gamma(j + \frac{3}{2})} \\
&\quad + \frac{1}{4}(2\cos(2) - \sin(2) - \sqrt{x} \sin(4)),
\end{aligned}
$$

the exact solution is $u_1(x) = \sin(2x)$, $u_2(x) = \cos(4x)$, and $i = \sqrt{-1}$.
We expand the solution to the system (6) as fractional power series representation about $x = 0$ of the form

$$
\begin{aligned}
u_1(x) &= \sum_{j=0}^{\infty} c_j \frac{x^{\frac{j}{2}}}{\Gamma((\frac{j}{2}) + 1)} \\
u_2(x) &= \sum_{j=0}^{\infty} b_j \frac{x^{\frac{j}{2}}}{\Gamma((\frac{j}{2}) + 1)}.
\end{aligned} \tag{7}
$$

Now, we apply the steps mentioned in section III by first replacing each $u_l(x)$ by the $k-th$ truncated series $u_l(x,k)$ in system (6) for all $l = 1, 2$. Accordingly, we have

$$
\begin{aligned}
D^{\frac{1}{2}} u_1(x,k) &= f_1(x) + \int_0^1 x u_2(t,k) dt \\
D^{\frac{1}{2}} u_2(x,k) &= f_2(x) + \int_0^1 (t u_1(t,k) + \sqrt{x} u_2(t,k)) dt.
\end{aligned}
$$

Here, we write down some calculations needed for step 2:

1)

$$
D^{\frac{1}{2}(n)} f_1(x) = \sum_{j=\beta_n}^{\infty} \frac{(2i)^j (1 + (-1)^j) x^{j - \frac{n-1}{2}}}{\Gamma(j - \frac{n-3}{2})}, \forall n \geq 3.
$$

$$
Where
$$

$$
\beta_n = \begin{cases} \frac{n}{2} & \text{if } n = 2m, m = 2,3,4\cdots \\ \frac{n-1}{2} & \text{if } n = 2m-1, m = 2,3,4\cdots \end{cases}
$$

2) From note number one, we conclude that:

$$
D^{\frac{1}{2}(n)} f_1(0) = \begin{cases} 0 & \text{if } n = 2m, m = 2,3,4\cdots \\ 0 & \text{if } n = 2m-1, m = 2,4,6\cdots \\ 2^{\frac{n+1}{2}} \times i^{\frac{n-1}{2}} & \text{if } n = 2m-1, m = 3,5,7\cdots \end{cases}
$$

3)

$$
D^{\frac{1}{2}(n)} f_2(x) = \sum_{j=\beta_n}^{\infty} \frac{(2)^{2j+1} i^{j+1} (1 - (-1)^j) x^{j - \frac{n-1}{2}}}{\Gamma(j - \frac{n-3}{2})},
$$

$$
\forall n \geq 2. \; Where
$$

$$
\beta_n = \begin{cases} \frac{n}{2} & \text{if } n = 2m, m = 1,2,3,4\cdots \\ \frac{n-1}{2} & \text{if } n = 2m-1, m = 2,3,4\cdots \end{cases}
$$

4) From note number three, we conclude that:

$$
D^{\frac{1}{2}(n)} f_2(0) = \begin{cases} 0 & \text{if } n = 2m, m = 1,2,3,4\cdots \\ 0 & \text{if } n = 2m-1, m = 3,5,7\cdots \\ 2^{n+1} \times i^{\frac{n+1}{2}} & \text{if } n = 2m-1, m = 2,4,6\cdots \end{cases}
$$

5)

$$
D^{\frac{1}{2}(n)} \left(\int_0^1 x u_2(t,k) dt \right) |_{x=0} = 0, \;\; \forall n \geq 3.
$$

6)

$$
D^{\frac{1}{2}(n)} \left(\int_0^1 (t u_1(t,k) + \sqrt{x} u_2(t,k)) dt \right) |_{x=0} = 0, \forall n \geq 2.
$$

Using the initial conditions given in this example by substituting 0 in (7), we find that $c_0 = 0$ and $b_0 = 1$. Now we write the $k \times 2$ equations as mentioned in step 2 of section 3 as follows:
For $m = 1$, we have:

$$
\begin{aligned}
c_1 &= 0 \\
b_1 - \frac{1}{4}(2\cos(2) - \sin(2)) - \sum_{j=0}^{k} c_j \frac{2+j}{2\Gamma(\frac{j}{2}+3)} &= 0.
\end{aligned} \tag{8}
$$

For $m = 2$, we have:

$$
\begin{aligned}
c_2 &= 2 \\
b_2 + \frac{\sqrt{\pi}\sin(4)}{8} - \sum_{j=0}^{k} \frac{\sqrt{\pi} b_j}{2\Gamma(\frac{j}{2}+2)} &= 0.
\end{aligned} \tag{9}
$$

For $m = 3$, we have:

$$
\begin{aligned}
c_3 + \frac{\sin(4)}{4} - \sum_{j=0}^{k} \frac{b_j}{\Gamma(\frac{j}{2}+2)} &= 0 \\
b_3 &= 0.
\end{aligned} \tag{10}
$$

Continuing in this process and using the calculations mentioned from 1 to 6 above we conclude the following:

$$
\begin{aligned}
c_{n+1} &= \begin{cases} 0 & \text{if } n = 2m, m = 2,3,4\cdots \\ 0 & \text{if } n = 2m-1, m = 2,4,6\cdots \\ 2^{\frac{n+1}{2}} \times i^{\frac{n-1}{2}} & \text{if } n = 2m-1, m = 3,5,7\cdots \end{cases} \\
b_{n+1} &= \begin{cases} 0 & \text{if } n = 2m, m = 1,2,3,4\cdots \\ 0 & \text{if } n = 2m-1, m = 3,5,7\cdots \\ 2^{n+1} \times i^{\frac{n+1}{2}} & \text{if } n = 2m-1, m = 2,4,6\cdots \end{cases}
\end{aligned} \tag{11}
$$

Now, we sum up the work done above. The unknown coefficients are b_1, b_2, and c_3. We can easily determine the value of these unknown coefficients, by solving the above equations for them, once we choose the level of approximation (the value of k) we want. As k gets larger, the approximation gets better.

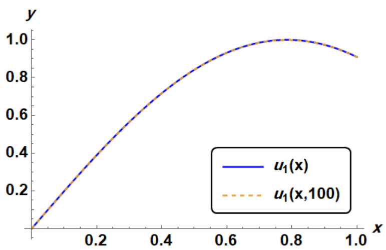
If we choose $k = 100$. Then we find the values of b_1, b_2, and c_3 by solving (8,9,10,11) for them. We conclude that $c_3 \approx -1.19409 \times 10^{-38}$, $b_1 \approx -2.56646 \times 10^{-39}$, and $b_2 \approx 1.05824 \times 10^{-38}$.

TABLE I. Comparison between the exact solution $u_1(x)$ and the approximate solution $u_1(x,100)$ together with the absolute errors at some points in $[0,1]$ for Example 1.

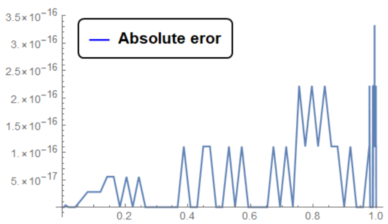
x	$u_1(x)$ $[Exact]$	$u_1(x,100)$ $[Approximate]$	$\lvert u_1(x)-u_1(x,100)\rvert$
0.0	0.0	0.0	0.0
0.1	1.986693308×10^{-1}	1.986693308×10^{-1}	2.77556×10^{-17}
0.2	3.894183423×10^{-1}	3.894183423×10^{-1}	5.55112×10^{-17}
0.3	5.646424734×10^{-1}	5.646424734×10^{-1}	0.0
0.4	7.173560909×10^{-1}	7.173560909×10^{-1}	1.11022×10^{-16}
0.5	8.414709848×10^{-1}	8.414709848×10^{-1}	0.0
0.6	9.32039086×10^{-1}	9.32039086×10^{-1}	1.11022×10^{-16}
0.7	9.8544973×10^{-1}	9.8544973×10^{-1}	0.0
0.8	9.99573603×10^{-1}	9.99573603×10^{-1}	0.0
0.9	9.738476309×10^{-1}	9.738476309×10^{-1}	0.0
1.0	9.092974268×10^{-1}	9.092974268×10^{-1}	0.0

TABLE II. Comparison between the exact solution $u_2(x)$ and the approximate solution $u_2(x,100)$ together with the absolute errors at some points in $[0,1]$ for Example 1.

x	$u_2(x)$ $[Exact]$	$u_2(x,100)$ $[Approximate]$	$\lvert u_2(x)-u_2(x,100)\rvert$
0.0	1.0	1	0.0
0.1	9.21060994×10^{-1}	9.21060994×10^{-1}	1.11022×10^{-16}
0.2	6.967067093×10^{-1}	6.967067093×10^{-1}	0.0
0.3	3.623577545×10^{-1}	3.623577545×10^{-1}	5.55112×10^{-17}
0.4	-2.91995223×10^{-2}	-2.91995223×10^{-2}	4.51028×10^{-17}
0.5	$-4.161468365\times10^{-1}$	$-4.161468365\times10^{-1}$	5.55112×10^{-17}
0.6	$-7.373937155\times10^{-1}$	$-7.373937155\times10^{-1}$	2.22045×10^{-16}
0.7	$-9.422223407\times10^{-1}$	$-9.422223407\times10^{-1}$	1.11022×10^{-16}
0.8	$-9.982947758\times10^{-1}$	$-9.982947758\times10^{-1}$	7.77156×10^{-16}
0.9	$-8.967584163\times10^{-1}$	$-8.967584163\times10^{-1}$	2.22045×10^{-16}
1.0	$-6.536436209\times10^{-1}$	$-6.536436209\times10^{-1}$	5.55112×10^{-16}

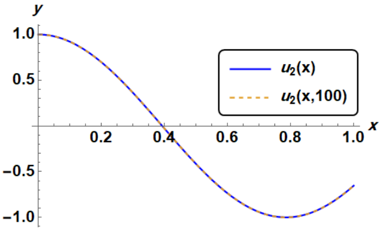


((a)) The graph of $u_1(x,100)$ with the exact solution $u_1(x)$.

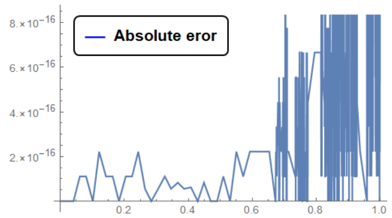


((b)) Graph of the Absolute error $|u_1(x,100)-u_1(x)|$.

Fig. 1. Behavior of the approximate solution $u_1(x,100)$ together with exact solution $u_1(x)$ and the Absolute error for Example 1.



((a)) The graph of $u_2(x,100)$ with the exact solution $u_2(x)$.



((b)) Graph of the Absolute error $|u_2(x,100)-u_2(x)|$.

Fig. 2. Behavior of the approximate solution $u_2(x,100)$ together with exact solution $u_2(x)$ and the Absolute error for Example 1.

The previous tables show that the approximate solution was in excellent agreement with the exact solution at some selected points. Moreover, figures 1 and 2 demonstrate that the accuracy is not limited to these points, but is consistent over the interval $[0,1]$.

Example 2: Consider the following system of linear fractional Fredholm IDEs:

$$\begin{aligned} D^{\frac{1}{2}}u_1(x) &= f_1(x)+\int_0^1 xu_2(t)dt \\ D^{\frac{1}{2}}u_2(x) &= f_2(x)+\int_0^1 (txu_1(t)+x^2u_3(t))dt \\ D^{\frac{1}{2}}u_2(x) &= f_2(x)+\int_0^1 (u_1(t)+xu_3(t))dt, \end{aligned} \tag{12}$$

subject to the initial condition $u_1(0)=1$, $u_2(0)=0$, and $u_3(0)=1$, where

$$f_1(x)=\sqrt{x}E_{1,\frac{3}{2}}(x)-\frac{4x}{3}.$$

$$f_2(x)=-x-\frac{1}{3}(e^3-1)x^2+\frac{3\pi+32x^{\frac{3}{2}}}{6\sqrt{\pi}}$$

$$f_3(x)=3\sqrt{x}E_{1,\frac{3}{2}}(3x)+1-e-\frac{1}{3}(e^3-1)x,$$

the exact solution is $u_1(x)=e^x$, $u_2(x)=2x^2+\sqrt{x}$, and $u_3(x)=e^{3x}$.
We expand the solution to the system (12) as fractional power series representation about $x=0$ of the form

$$\begin{aligned} u_1(x) &= \sum_{j=0}^{\infty} c_j\frac{x^{\frac{j}{2}}}{\Gamma((\frac{j}{2})+1)} \\ u_2(x) &= \sum_{j=0}^{\infty} b_j\frac{x^{\frac{j}{2}}}{\Gamma((\frac{j}{2})+1)} \\ u_3(x) &= \sum_{j=0}^{\infty} p_j\frac{x^{\frac{j}{2}}}{\Gamma((\frac{j}{2})+1)}. \end{aligned} \tag{13}$$

Now, we apply the steps mentioned in section III by first replacing each $u_l(x)$ by the $k-th$ truncated series $u_l(x,k)$ in system (12) for all $l=1,2,3$. Accordingly, we have

$$D^{\frac{1}{2}}u_1(x,k)=f_1(x)+\int_0^1 xu_2(t,k)dt$$
$$D^{\frac{1}{2}}u_2(x,k)=f_2(x)+\int_0^1 (txu_1(t,k)+x^2u_3(t,k))dt$$
$$D^{\frac{1}{2}}u_2(x,k)=f_2(x)+\int_0^1 (u_1(t,k)+xu_3(t,k))dt.$$

Here, we write down some calculations needed for step 2:

1)
$$D^{\frac{1}{2}(n)}f_1(x)=\sum_{j=\beta_n}^{\infty}\frac{x^{j-\frac{n-1}{2}}}{\Gamma(j-\frac{n-3}{2})},\quad \forall n\geq 3.\ Where$$
$$\beta_n=\begin{cases}\frac{n}{2} & \text{if } n=2m, m=2,3,4\cdots\\ \frac{n-1}{2} & \text{if } n=2m-1, m=2,3,4\cdots\end{cases}$$

2) From note number one, we conclude that:
$$D^{\frac{1}{2}(n)}f_1(0)=\begin{cases}0 & \text{if } n=2m, m=2,3,4\cdots\\ 1 & \text{if } n=2m-1, m=2,3,4\cdots\end{cases}$$

3)
$$D^{\frac{1}{2}(n)}f_2(0)=0,\quad \forall n\geq 5$$

4)
$$D^{\frac{1}{2}(n)}f_3(x)=\sum_{j=\beta_n}^{\infty}\frac{(3)^{j+1}x^{j-\frac{n-1}{2}}}{\Gamma(j-\frac{n-3}{2})}, \forall n\geq 3.\ Where$$
$$\beta_n=\begin{cases}\frac{n}{2} & \text{if } n=2m, m=2,3,4\cdots\\ \frac{n-1}{2} & \text{if } n=2m-1, m=2,3,4\cdots\end{cases}$$

5) From note number three, we conclude that:
$$D^{\frac{1}{2}(n)}f_3(0)=\begin{cases}0 & \text{if } n=2m, m=2,3,4\cdots\\ 3^{\frac{n+1}{2}} & \text{if } n=2m-1, m=2,3,4\cdots\end{cases}$$

6)
$$D^{\frac{1}{2}(n)}\left(\int_0^1 xu_2(t,k)dt\right)|_{x=0}=0,\quad \forall n\geq 3.$$

7)
$$D^{\frac{1}{2}(n)}\left(\int_0^1 (xtu_1(t,k)+x^2u_3(t,k))dt\right)|_{x=0}=0, \forall n\geq 5.$$

8)
$$D^{\frac{1}{2}(n)}\left(\int_0^1 (u_1(t,k)+xu_3(t,k))dt\right)|_{x=0}=0,\quad \forall n\geq 3.$$

Using the initial conditions given in this example by substituting 0 in (13), we find that $c_0=1$, $b_0=0$, and $p_0=1$. Now we write the $k\times 3$ equations as mentioned in step 2 of section III as follows:

For $m=1$, we have:
$$c_1=0,\ b_1=\frac{\sqrt{\pi}}{2},$$
$$p_1-1+e-\sum_{j=0}^{k}\frac{c_j}{\Gamma(\frac{j}{2}+2)}=0. \tag{14}$$

For $m=2$, we have:
$$c_2=1,\ b_2=0,\ p_2=3. \tag{15}$$

For $m=3$, we have:
$$c_3+\frac{4}{3}-\sum_{j=0}^{k}\frac{b_j}{\Gamma(\frac{j}{2}+2)}=0$$
$$b_3+1-\sum_{j=0}^{k}\frac{(2+j)c_j}{2\Gamma(3+\frac{j}{2})}=0 \tag{16}$$
$$p_3+\frac{1}{3}(e^3-1)-\sum_{j=0}^{k}\frac{p_m}{\Gamma(2+\frac{j}{2})}=0.$$

For $m=4$, we have:
$$c_4=1,\ b_4=4,\ p_4=9. \tag{17}$$

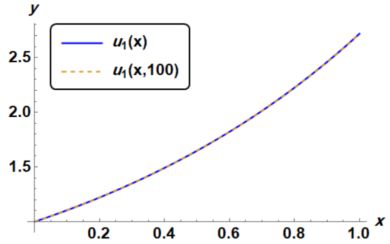
For $m=5$, we have:
$$c_5=0,\ p_5=0,$$
$$b_5+\frac{2}{3}(e^3-1)-\sum_{j=0}^{k}\frac{2p_j}{\Gamma(\frac{j}{2}+2)}=0. \tag{18}$$

Continuing in this process and using the calculations mentioned from 1 to 8 above we conclude the following:

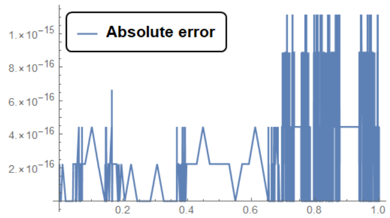
$$c_{n+1}=\begin{cases}0 & \text{if } n=2m, m=2,3,4\cdots\\ 1 & \text{if } n=2m-1, m=2,4,6\cdots\end{cases}$$
$$b_n=0,\quad \forall n\geq 6 \tag{19}$$
$$p_{n+1}=\begin{cases}3^{\frac{n+1}{2}} & \text{if } n=2m-1, m=2,3,4\cdots\\ 0 & \text{if } n=2m, m=2,3,4\cdots\end{cases}$$

Now, we sum up the work done above. The unknown coefficients are p_1, c_3, b_3, p_3, and b_5. We can easily determine the value of these unknown coefficients, by solving the above equations for them, once we chose the level of approximation (the value of k) we want. As k gets larger, the approximation gets better.

If we choose $k=100$. Then we find the values of p_1, c_3, b_3, p_3, and b_5 by solving (14,15,16,17,18,19) for them. We conclude that $c_3=-7.91311\times 10^{-45}$, $b_3=-1.70076\times 10^{-45}$, $b_5=-8.60904\times 10^{-44}$, $p_1=-2.38106\times 10^{-45}$, and $p_3=-4.30452\times 10^{-44}$.

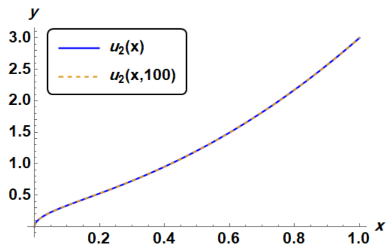


((a)) The graph of $u_1(x,100)$ with the exact solution $u_1(x)$.

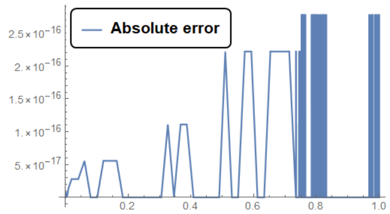


((b)) Graph of the Absolute error $|u_1(x,100) - u_1(x)|$.

Fig. 3. Behavior of the approximate solution $u_1(x,100)$ together with exact solution $u_1(x)$ and the Absolute error for Example 2.

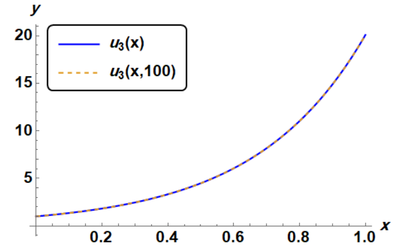


((a)) The graph of $u_2(x,100)$ with the exact solution $u_2(x)$.

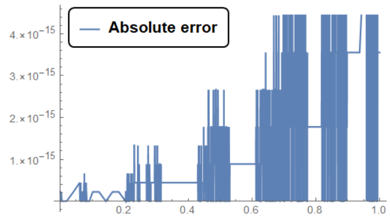


((b)) Graph of the Absolute error $|u_2(x,100) - u_2(x)|$.

Fig. 4. Behavior of the approximate solution $u_2(x,100)$ together with exact solution $u_2(x)$ and the Absolute error for Example 2.



((a)) The graph of $u_3(x,100)$ with the exact solution $u_3(x)$.



((b)) Graph of the Absolute error $|u_3(x,100) - u_3(x)|$.

Fig. 5. Behavior of the approximate solution $u_3(x,100)$ together with exact solution $u_3(x)$ and the Absolute error for Example 2.

V RESIDUAL POWER SERIES METHOD(RPSM) TO SOLVE FRACTIONAL FREDHOLM IDES

We divide this section into two parts. In the first part, we review and correct the main ideas mentioned in paper [24] which employed the residual power series method to solve a class of fractional Fredholm IDEs, which is the single equation form of the system (1). In the second part, we compare the presented scheme in section III with the RPSM and we correct the examples in paper [24] which put the reader under misapprehension that the RPSM works effectively on fractional Fredholm IDEs which is not the case as we are going to clarify that. Finally, we solve a fractional Fredholm IDE, that has been investigated in a number of previous studies, and compare our results with the previous studies ones.

V.A Review on applying the RPSM in the paper [24]

In the paper [24], Syam applied the RPSM to solve the following class of fractional Fredholm IDEs:

$$D^{\alpha}u(x) = f(x) + \lambda \int_a^b u(t)K(x,t)dt, \qquad 0 < \alpha \le 1, x \in \mathbb{R}, a \le t \le b \tag{20}$$

subject to the initial condition $u(a) = a_0$. where the fractional derivative is in the Caputo sense, a and b are constants, λ is a parameter, $K(x,t)$ is arbitrary continuous kernel over $[a,b]^2$, $u(x)$ is analytic unknown function to be calculated, and $f(x)$ is a smooth function. The residual power series method [1] expands the solution to (1) as fractional power series about $x = a$ of the following form:

$$u(x) = \sum_{j=0}^{\infty} c_j \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)} \tag{21}$$

Then, we approximate $u(x)$ by the k-th truncated series $u(x,k)$ of the form:

$$u(x,k) = \sum_{j=0}^{k} c_j \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)} \tag{22}$$

using the initial condition $u(a) = a_0 = c_0$, we rewrite $u(x,k)$ as:

$$u(x,k) = a_0 + \sum_{j=1}^{k} c_j \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}$$

Now, the residual functions to (20) can be defined as:

$$\mathrm{Res}_u(x) = D^\alpha u(x) - f(x) - \lambda \int_a^b u(t)K(x,t)dt$$

and the the k-th residual function as:

$$\mathrm{Res}_{u,k}(x) = D^\alpha u(x,k) - f(x) - \lambda \int_a^b u(t,k)K(x,t)dt$$

The fundamental properties of RPSM related to the residual functions as in [31, 35, 36] are

1) $\lim_{k\to\infty} \mathrm{Res}_{u,k}(x) = \mathrm{Res}_u(x) = 0$, for each $x > 0$.
2) $D_x^{(m-1)\alpha}\mathrm{Res}_{u,m}(0) = D_x^{(m-1)\alpha}\mathrm{Res}_u(0) = 0$,

   for $m = 1,2,3,\cdots$.

To find the coefficients c_m for $m = 1,2,3,\cdots,k$, we solve the following algebraic fractional differential equations for $m = 1,2,3,\cdots,k$,

$$D^{(m-1)\alpha}\mathrm{Res}_{u,m}(a) = 0, \tag{23}$$

where $D^{m\alpha} = D^\alpha \cdot D^\alpha \cdots D^\alpha \ \ (m-times)$.
The author in paper [24] made some mistakes in obtaining the analytic-numeric solution $u_k(x)$ for (20) using the RPSM. Specifically equations (31,32) in his paper [24] are not correct. Thus, here we derive the right recursive formula for generating the coefficients c_n resulted from applying the RPSM as follows.
The process of finding the coefficient c_m requires finding the coefficients $c_1, c_2, \cdots, c_{m-1}$ recursively. Thus, for any $n = 1,2,3,\cdots,k$, we first find $c_1, c_2, \cdots, c_{n-1}$ through solving algebraic fractional differential equation (23) for $m = 1,2,3,\cdots,n-1$. Then, we solve (23) for $m = n$ as follows:

$$\begin{aligned} D^{(n-1)}\mathrm{Res}_{u,n}(a) &= D^{n\alpha}u(a,n) - D^{(n-1)\alpha}f(a) \\ &\quad - \lambda \int_a^b u(t,n)D^{(n-1)\alpha}k(a,t)dt \\ &= c_n - c_n\lambda \int_a^b \frac{(t-a)^{n\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+n\alpha)}dt \\ &\quad - D^{(n-1)\alpha}f(a) \\ &\quad - \lambda \sum_{j=0}^{n-1} c_j \int_a^b \frac{(t-a)^{j\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+j\alpha)} \\ &= 0 \end{aligned}$$

Therefore, for any $n = 1,2,3,\cdots,k$, we have

$$c_n = \frac{D^{(n-1)\alpha}f(a) + \lambda \sum\limits_{j=0}^{n-1} c_j \int_a^b \frac{(t-a)^{j\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+j\alpha)}dt}{1 - \lambda \int_a^b \frac{(t-a)^{n\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+n\alpha)}dt}, \tag{24}$$

accordingly, the k-th RPS approximation is

$$\begin{aligned} u(x,k) = a_0 + \sum_{n=1}^{k} \Bigg(\Big(& D^{(n-1)\alpha}f(a) \\ & + \lambda \sum_{j=0}^{n-1} c_j \int_a^b \frac{(t-a)^{j\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+j\alpha)}dt \Big) / \\ \Big(1 - \lambda \int_a^b & \frac{(t-a)^{n\alpha}D^{(n-1)\alpha}k(a,t)}{\Gamma(1+n\alpha)}dt \Big) \Bigg) * \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned}$$

V.B comparison between the RPSM and the proposed scheme

The numerical examples presented in paper [24], to show the efficiency of the RPSM in solving (20), are not correct. The right-hand side of the three fractional Fredholm IDEs (33,38,45) does not equal the left-hand side of each one. In this section, we correct two of the three examples written in the paper [24] by fixing $u(x)$, $k(x,t)$, and changing $f(x)$ for each equation that satisfies it, in addition to solving and comparing a problem addressed by many previous studies.

Example 3:[17]. Consider the following Fredholm FIDE:

$$\begin{aligned} D^{\frac{1}{2}}u(x) = \frac{32}{3\sqrt{\pi}}x^{1.5} + \frac{16}{\sqrt{\pi}}x^{2.5} - 2x \\ + \int_0^1 xtu(t)dt, \end{aligned} \tag{25}$$

Subject to the initial condition $u(0) = 0$, and exact solution $u(x) = 4x^2 + 5x^3$.
We expand the solution to the equation (25) as a fractional power series representation about $x = 0$ in the form

$$u(x) = \sum_{n=0}^{\infty} c_n \frac{x^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{26}$$

Using the initial conditions given in this example by substituting 0 in (26), we find that $c_0 = 0$. Then we apply the recursive formula in (24) to find that: $c_1 = c_2 = c_5 = 0$,

$c_3 = \frac{-42\sqrt{\pi}}{-8+21\sqrt{\pi}}$, $c_4 = 8$, $c_6 = 30$,

since $D_x^{(n-1)\alpha}f(0) = D_x^{(n-1)\alpha}k(0,t) = 0$,

$\forall n \geq 7$. Thus, $c_n = 0$, $\forall n \geq 7$. Therefor, we find the residual power series approximation

$$u_R(x,k) = \frac{56}{8-21\sqrt{\pi}}x^{1.5} + 4x^2 + 5x^3, \ \forall k \geq 6$$

Now, we solve this example using the proposed scheme in section III by following the steps outlined there.
We first replace each $u(x)$ by the $k-th$ truncated series $u(x,k)$ in (25). Accordingly, we have:

$$D^{\frac{1}{2}}u(x,k) = f(x) + \int_0^1 xtu(t,k)dt,$$

where $f(x) = \frac{32x^{1.5}}{3\sqrt{\pi}} + \frac{16}{\sqrt{\pi}}x^{2.5} - 2x$. Here, we write down some calculations needed for step 2:

1)

$$D^{\frac{1}{2}(n)} f(x) = 0, \ \forall n \geq 6$$

2)

$$D^{\frac{1}{2}(n)} \int_0^1 xtu(t,k)dt = 0, \quad \forall n \geq 3.$$

Now, as stated in step 2 of section III, we write the $k \times 1$ equations as follows:
For $m = 1$, we have:

$$c_1 = 0. \tag{27}$$

For $m = 2$, we have:

$$c_2 = 0. \tag{28}$$

For $m = 3$, we have:

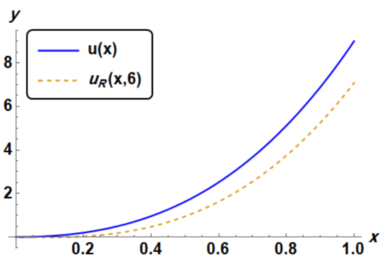
$$c_3 + 2 - \sum_{j=0}^{k} \frac{2c_j}{(4+j)\Gamma(\frac{j}{2}+1)} = 0 \tag{29}$$

Continuing in this process and using the calculations mentioned from 1 to 2 above, we conclude the following:

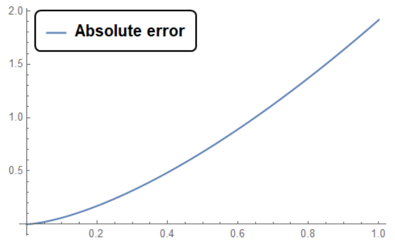
$$c_n = D^{\frac{1}{2}(n-1)} f(0) + D^{\frac{1}{2}(n-1)} \left(\int_0^1 xtu(t,k)dt \right) |_{x=0} = 0, \quad \forall n \geq 7. \tag{30}$$

Now, we sum up the work done above. The unknown coefficient is c_3 which can be easily determined by solving the above equations for it. The value of k in this example does not really matter, because $c_n = 0, \ \forall n \geq 7$, as long as $k \geq 6$.

If we choose $k \geq 6$. Then we find the value of c_3 by solving (27,28,29,30) for them. We conclude that $c_3 = 0$.



((a)) The graph of $u_R(x,6)$ with the exact solution $u(x)$.



((b)) Graph of the Absolute error $|u_R(x,6) - u(x)|$.

Fig. 6. Behavior of the RPSM approximate solution $u_R(x,6)$ together with exact solution $u(x)$ and the Absolute error for Example 3.

Using the proposed method in section III, we were able to obtain the exact solution for this example. whereas, the approximate solution in the RPSM is not suitable, according to the Absolute error graph in figure 6, which is bounded between 0 and 1.92.

Example 4:[17]. Consider the following Fredholm FIDE:

$$D^{\frac{1}{4}} u(x) = f(x) + \int_0^1 x^2 tu(t)dt, \tag{31}$$

subject to the initial condition $u(0) = 1$, where

$$f(x) = \frac{1}{2} x^{\frac{3}{4}} E_{1,\frac{7}{4}} \left(\frac{x}{2} \right) + 2(\sqrt{e} - 2)x^2,$$

and exact solution $u(x) = e^{\frac{x}{2}}$.
We expand the solution to the equation (31) as a fractional power series representation about $x = 0$ in the form

$$u(x) = \sum_{n=0}^{\infty} c_n \frac{x^{\frac{n}{4}}}{\Gamma(\frac{n}{4}+1)}. \tag{32}$$

Using the initial conditions given in this example by substituting 0 in (32), we find that $c_0 = 1$. Then we apply the recursive formula in (24) to find that:

$$\begin{gathered} c_1 = c_2 = c_3 = c_5 = c_6 = c_7 = 0, \\ c_4 = \frac{1}{2}, \ c_8 = \frac{1}{4}, \ c_9 \approx -1.13828 \times 10^{-2}, \\ c_{n+1} = \begin{cases} \frac{1}{2^{\frac{n+1}{4}}} & \text{if } n = 2m-1, m = 6, 8, 10 \cdots \\ 0 & \text{if } n = 2m-1, m = 5, 7, 9 \cdots \\ 0 & \text{if } n = 2m, m = 5, 6, 7 \cdots \end{cases} \end{gathered} \tag{33}$$

Therefore, we find the residual power series approximation $u_R(x,100) = u(x,100)$.
Now, we solve this example using the proposed scheme in section III by applying the steps mentioned there.
We first replace each $u(x)$ by the $k-th$ truncated series $u(x,k)$ in (31). Accordingly, we have

$$D^{\frac{1}{4}} u(x,k) = f(x) + \int_0^1 x^2 tu(t)dt.$$

Here, we write down some calculations needed for step 2:

1)

$$D^{\frac{1}{4}(n)} k(0,t) = 0, \ \forall n \geq 9$$

2)

$$D^{\frac{1}{4}(n)} \frac{1}{2} x^{\frac{3}{4}} E_{1,\frac{7}{4}} \left(\frac{x}{2} \right) = \sum_{j=\beta_n}^{\infty} \frac{x^{j+\frac{3-n}{4}}}{2^{j+1}\Gamma(j+\frac{7-n}{4})}, \text{ where}$$

$$\beta_n = \begin{cases} \frac{n}{4} & \text{if } n = 2m, m = 2, 4, 6 \cdots \\ \frac{n-2}{4} & \text{if } n = 2m, m = 1, 3, 5 \cdots \\ \frac{n-3}{4} & \text{if } n = 2m-1, m = 2, 4, 6 \cdots \\ \frac{n-1}{4} & \text{if } n = 2m-1, m = 1, 3, 5 \cdots \end{cases}$$

3) Using the previous note, we find that

$$D^{\frac{1}{4}(n)}f(0)=\begin{cases}\frac{1}{2^{\frac{n+1}{4}}} & \text{if } n=2m-1, m=6,8,10..\\ 0 & \text{if } n=2m-1, m=5,7,9..\\ 0 & \text{if } n=2m, m=5,6,7\cdots\end{cases}$$

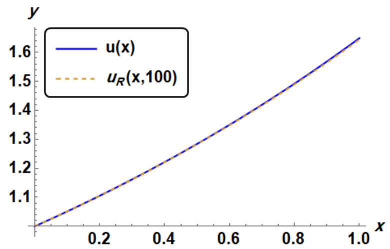
Using the initial conditions given in this example by substituting 0 in (32), we find that $c_0=1$. Now we write the $k\times 1$ equations as mentioned in step 2 of section III as follows:
For $m=1,2,\cdots,8$, we find that:

$$\begin{aligned} c_1=c_2=c_3=c_5=c_6=c_7=0,\ c_4=\frac{1}{2},\ c_8=\frac{1}{4},\\ c_9=4(\sqrt{e}-2)+\sum_{j=0}^{k}\frac{8c_j}{(j+8)\Gamma(1+\frac{j}{4})}.\end{aligned} \tag{34}$$

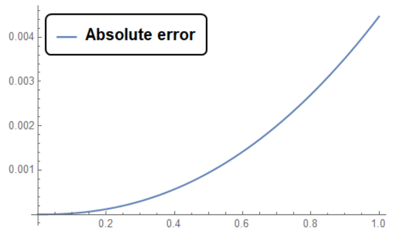
Continuing in this process and using the calculations mentioned from 1 to 3 above, we conclude the following for $n\geq 9$:

$$c_{n+1}=\begin{cases}\frac{1}{2^{\frac{n+1}{4}}} & \text{if } n=2m-1, m=6,8,10\cdots\\ 0 & \text{if } n=2m-1, m=5,7,9\cdots\\ 0 & \text{if } n=2m, m=5,6,7\cdots\end{cases} \tag{35}$$

Now, we sum up the work done above. The unknown coefficient is c_9 which can be easily determined by solving the above equations for it once we chose the level of approximation (the value of k) we want. As k gets larger, the approximation gets better. If we choose $k=100$. Then we find the value of c_9 by solving (34,35) for it. We conclude that $c_9=-3.29558\times 10^{-36}$.

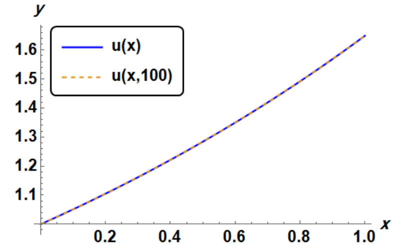


((a)) The graph of $u_R(x,100)$ with the exact solution $u(x)$.

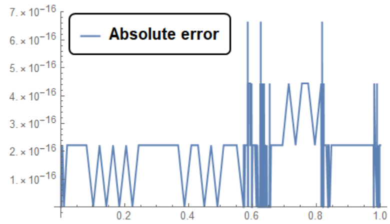


((b)) Graph of the Absolute error $|u_R(x,100)-u(x)|$.

Fig. 7. Behavior of the RPSM approximate solution $u_R(x,100)$ together with exact solution $u(x)$ and the Absolute error for Example 4.



((a)) The graph of $u(x,100)$ with the exact solution $u(x)$.



((b)) Graph of the Absolute error $|u(x,100)-u(x)|$.

Fig. 8. Behavior of the approximate solution $u(x,100)$ using our approach together with exact solution $u(x)$ and the Absolute error for Example 4.

The proposed method in section III yields a high-accuracy approximate solution with Absolute error bounded between 0 and 7×10^{-16} $\forall x\in[0,1]$ as the graph shows in figure 8. The RPSM, on the other hand, provides a good approximation, with an Absolute error range of 0 to 4.5×10^{-3} as shown in figure 7. However, according to the figures, the proposed scheme's approximation is much better than the RPS approximation in this example.

Example 5: Consider the following Fredholm FIDE:

$$D^{\frac{1}{2}}u(x)=f(x)+\int_0^1 xtu(t)dt, \tag{36}$$

subject to the initial condition $u(0)=0$, where

$$f(x)=\frac{\frac{8}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}}{\sqrt{\pi}}+\frac{x}{12},$$

and exact solution $u(x)=x^2-x$.
We expand the solution to the equation (36) as a fractional power series representation about $x=0$ in the form

$$u(x)=\sum_{n=0}^{\infty}c_n\frac{x^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{37}$$

Using the initial conditions given in this example by substituting 0 in (37), we find that $c_0=0$. Then we apply the recursive formula in (24) to find that:

$$\begin{aligned} c_1=0,\ c_2=-1,\ c_3=\frac{-1}{4(1-\frac{8}{21\sqrt{\pi}})},\ c_4=2\\ c_n=0,\ \forall n\geq 5.\end{aligned} \tag{38}$$

Therefore, we find the residual power series approximation $u_R(x,4)=-x-\frac{x^{\frac{3}{2}}}{3(\sqrt{\pi}-\frac{8}{21})}+x^2$.
Now, we solve this example using the proposed scheme in

section III by applying the steps mentioned there.
We first replace each $u(x)$ by the $k-th$ truncated series $u(x,k)$ in (36). Accordingly, we have

$$D^{\frac{1}{2}}u(x,k) = f(x) + \int_0^1 xtu(t,k)dt,$$

Here, we write down some calculations needed for step 2:

1)

$$D^{\frac{1}{2}(n)}f(0) = 0,\ \forall n \geq 4$$

2)

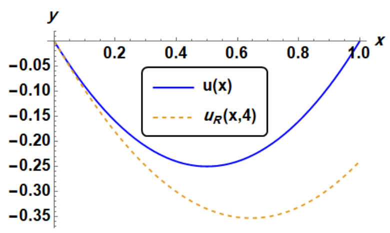
$$D^{\frac{1}{2}(n)}\left(\int_0^1 xtu(t,k)dt\right)|_{x=0} = 0 = 0,\quad \forall n \geq 3.$$

Now we write the $k \times 1$ equations as mentioned in step 2 of section III as follows:
For $m = 1, 2, \cdots, k$, we find that:

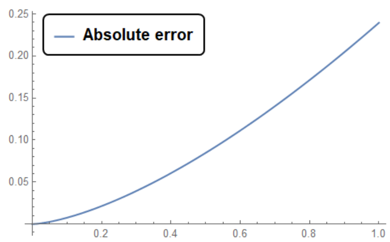
$$c_1 = 0,\, c_2 = -1,\, c_4 = 2,\, c_3 = \frac{1}{12} + \sum_{j=0}^{k} \frac{2c_j}{(j+4)\Gamma(1+\frac{j}{2})},$$
$$c_n = D^{\frac{1}{2}(n-1)}f(0) + D^{\frac{1}{2}(n-1)}\left(\int_0^1 xtu(t,k)dt\right)|_{x=0} = 0,\ \forall n \geq 5. \tag{39}$$

Now, we sum up the work done above. The unknown coefficient is c_3 which can be easily determined by solving the above equations for it. The value of k in this example does not really matter, because $c_n = 0,\ \forall n \geq 5$, as long as $k \geq 4$.

If we choose $k \geq 4$. Then we find the value of c_3 by solving (39) for it. We conclude that $c_3 = 0$ and thus $u(x,4) = x^2 - x$ which is the exact solution.



((a)) The graph of $u_R(x,4)$ with the exact solution $u(x)$.



((b)) Graph of the Absolute error $|u_R(x,4) - u(x)|$.

Fig. 9. Behavior of the RPSM approximate solution $u_R(x,4)$ together with exact solution $u(x)$ and the Absolute error for Example 5.

VI Discussion and Results

This section presents and discusses the most prominent points that emerged from the comparison in the previous subsection V.B. Furthermore, we discuss the characteristics of the proposed scheme in section III and justify why do we think that it is superior to the RPSM.

1) From looking at figure 6 part a, we can see that as x gets larger the residual approximation becomes inaccurate and its graph gradually moves away from the graph of the exact solution, and unaccepted approximation at the large portion of the interval. This effect can be also seen in part b of the same figure which shows that as x gets larger the error increases more. which indicates a big variation between the absolute error at different points in the interval $[0,1]$ as part b of that figure shows. On the other hand, we obtained the exact solution using the method in section III. Moreover, we will be able to obtain the exact solution, using the approach in section III, whenever the unknown function can be written as a finite FPS (That includes polynomials) and its highest degree is known so that we can choose the suitable level of approximation k that enable us to find all the coefficients.
The same results can be addressed to Example 5, which produced unaccepted approximate solution according to Figure 9 using the RPSM and exact solution using the technique in section III.

2) From looking at figure 7 and figure 8, we can see that both methods were able to obtain a good approximation with a better one using the approach in section III according to the b parts of each figure. Moreover, the variation in the absolute error graph of figure 8 is relatively small comparing to the RPSM ones in part b of figure 7. which guarantees more stability and accuracy at each point in the interval $[0,1]$ of the approximate solution. This feature can also be seen in the figures and the tables of the first and second examples.

3) Approximate solutions had been found to the problem in example 5 using several techniques [37, 38, 39, 40, 41]. For instance, the problem was solved [37] using the Least Squares Method (LSM) and Shifted Chebyshev Polynomial by Mohammed and using the LSM and shifted Chebyshev polynomials of the third kind by Mahdy et al. [38]. Also, Mahdy et al. [39] utilized the LSM and shifted Laguerre polynomials pseudo-spectral method to solve that problem. They obtained approximate solutions to the problem in example 5 and graphed it with the exact solution as the only indicator to see how well it agreed with the exact solution, without presenting the numerical results of the errors in the problem. That is not accurate to conclude that they obtained an exact or excellent agreement with the exact solution.
Khongnual et al. in [40] found an approximate solution $u(x) = -x + x^2 - 1.57083 \times 10^{-16}(-x + x^2)$ using a method based on hybrid of block-pulse functions and Taylor polynomials with absolute error bounded between 0 and 4×10^{-17}.

Also, Oyedepo et al. found an approximate solution $u_3(x) = 0.00010800322060.9998062943x + 0.999757556x^2 + 0.000013067x^3$ and a table at some grade points that presents the absolute error bounded above by 3.5524×10^{-5} of standard LSM and 1.4367×10^{-4} of perturbed LSM Absolute error.
If we overlook some of the negatives in presenting the approximate solutions in some of the previous papers and agree on their compatibility with the exact solution, however, we have obtained the exact solution in our method. Which gives a clear indication of the importance of our method in dealing with the fractional Fredholm IDEs despite its simplicity, ease of use, and negligible computational time, unlike the numerical methods which require a major time comparing to the analytical methods.

4) We point out that the comparison between the technique in section III and the RPSM was done at the same level of approximation (k).
5) The limitation of using the approach in section III is clarified in the restrictions written on system 1 in section I in addition to that each $f_i(x)$ can be written as a FPS of powers $n\alpha_i$.
6) The proposed technique may be extended and tested to solve other classes of linear or nonlinear IDEs.

VII Conclusion

In this article, a new technique has been utilized to find an analytic-numeric solution for a system of fractional Fredholhm IDEs. Two examples were solved using the technique illustrated in section III where the obtained solutions were in excellent agreement with the exact solution, and better results can be obtained as k gets bigger. Moreover, the main ideas in the paper [24] were reviewed. In particular, correcting the derivation of the recursive formula, which generates the unknown coefficients c_n of the FPS, was done. Furthermore, the first two examples in the paper [24] were corrected and solved using the RPSM and the proposed technique in section III. The Absolute error figures for these two examples show high accuracy and stability in the obtained solution using the proposed method. The comparison in the figures shows a clear superiority of our method over the RPSM for solving this class of equations as explained in detail in the previous section. Also, a fractional Fredholm IDE, which was investigated in a number of previous studies, has been solved in subsection VI and compared the obtained solution in our approach with the previous studies ones where the obtained solution in our approach was the exact one and was the best solution between all the mentioned studies.
The reason for comparing our method with RPSM is due to the similarity of the two schemes. That is, the two schemes follow the same steps outlined in section III except that the RPSM replace each m by k in step 2 and replace the system 5 by the system of residual functions. But still applying the $D^{(k-1)\alpha_i}|_{x=0}$ for all $i = 1, 2, \cdots, n$ on both sides of the equations of the system (5) result the same as applying it on the system of the residual functions $\forall\ k = 1, 2, \cdots, k$.

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