# Surfaces of finite *III*-type

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**Abstract— In this paper, we consider surfaces of revolution in the 3-dimensional Euclidean space $E^3$ with nonvanishing Gauss curvature. We introduce the finite Chen type surfaces with respect to the third fundamental form of the surface. We present a special case of this family of surfaces of revolution in $E^3$, namely, surfaces of revolution with $R$ is constant, where $R$ denotes the sum of the radii of the principal curvature of a surface.**

**Keywords— Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Beltrami operator, Surfaces of revolution.**

## I. INTRODUCTION

ONE of the most interesting and profound aspects of differential geometry is the idea of surfaces of finite type which was born by B-Y. Chen in the early 1970s and since then, it has become a source of interest for many researchers in this field. The reader can refer to [10] for more details. In the framework of this kind of study the first-named author with S. Stamatakis have given in [18] a new generalization to this area of study by giving a similar definition of surfaces of finite type.

Let $r$ be an isometric immersion of a surface $S$ in the 3-dimensional Euclidean space $E^3$. We represent by $\Delta^J$ the Laplacian operator of $S$ acting on the space of smooth functions $C^\infty(S)$. Then $S$ is said to be of finite $J$-type, $J = I, II, III$, if the position vector $r$ of $S$ can be decomposed as a finite sum of eigenvectors of $\Delta^J$ of $S$, that is

$$r = c + \sum_{i=1}^{k} x_i, \qquad (1)$$

where

$$\Delta^J x_i = \lambda_i x_i, \ i = 1, \ldots, k,$$

$c$ is a fixed vector and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of the operator $\Delta^J$.

Surfaces of finite type according to the second or third fundamental forms became a topic of active research in the last years. Surfaces of finite type in $E^3$ regarding to the second fundamental form were investigated for some important classes of surfaces. More precisely, the class of ruled surfaces where studied in [7], while in [3], H. AL-Zoubi studied tubes in $E^3$. Other classes such as translation surfaces, Quadric surfaces, surfaces of revolution, helicoidal surfaces, cyclides of Dupin and spiral surfaces, the classification of its finite *II*-type surfaces still unknown. According to the third fundamental form, ruled surfaces in [4], tubes in [5] and quadric surfaces are the only classes were investigated in $E^3$.

This type of study can be also extended to any smooth map, not necessary for the position vector of the surface, for example, the Gauss map of a surface. Regarding this see [8].

Another generalization to the above, one can study surfaces in $E^3$ whose position vector $r$ satisfies the following condition

$$\Delta^J r = A r, \quad J = II, III, \qquad (2)$$

where $A \in \mathfrak{R}^{3\times3}$.

Related to this, in [19] it was proved that the spheres and the catenoids are the only surfaces of revolution satisfying the above equation. Similarly, in [1] it was shown that, helicoids and spheres are the only quadric surfaces in $E^3$ that satisfy (2). Next, in [2] condition (2) was studied for the class of translation surfaces. In fact authored ascertained that Scherk's surface is the only translation surface in the Euclidean 3-space that satisfies (2), finally, in [17] the authors studied the class of translation surfaces in Sol$_3$ satisfying (2). Surfaces satisfying condition (2) are said to be of coordinate finite $J$-type [13].

In order to achieve our goal, we briefly introduce a formula for $\Delta^{III} r$ and $\Delta^{III} N$ by using tensors calculations, where $N$ denotes the normal vector field of $S$. Further, in the last section, we contribute to the solution of our main result.

## II. BASIC CONCEPTS

Let $S$ be a smooth surface in $E^3$ given by a patch $r = r(u^1, u^2)$ on a region U: $= (a, b) \times \mathfrak{R}$ of $\mathfrak{R}^2$ in which does not contain parabolic points. We denote by

$$I = g_{ij} du^i du^j, \quad II = b_{ij} du^i du^j, \quad III = e_{ij} du^i du^j$$

the first, second and third fundamental forms of $S$ respectively. For two sufficiently differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ on $S$, the first differential parameter of Beltrami with

respect to the third fundamental form $J$ is defined by [15]

$$\nabla^J(f,g): = d^{ij} f_{/i} \, g_{/j},$$

where $f_{/i}: = \dfrac{\partial f}{\partial u^i}$ and $(d^{ij})$ denotes the inverse tensor of $(g_{ij})$, $(b_{ij})$ and $(e_{ij})$ for $J = I, II$ and $III$ respectively.

The second differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ of $S$ is defined by [10]

$$\Delta^J f: = -d^{ij} \, \nabla^J_i f_{/j}, \qquad (3)$$

where $f$ is sufficiently differentiable function, $\nabla^J_i$ is the covariant derivative in the $u^i$ direction with respect to the fundamental form $J$ [7].

If we apply (3) for the position vector $r$ with respect to the fundamental form $III$, it can be easily verified that [18]

$$\Delta^{III} r = \text{grad}^{III} R - R \, N. \qquad (4)$$

By applying (3) for the normal vector $N$ we get [18]

$$\Delta^{III} N = 2 \, N. \qquad (5)$$

From (5) we find that the Gauss map of every surface $S$ in $E^3$ is of finite $III$-type 1 with eigenvalue $\lambda = 2$ [18].

Let $S^*$ be a parallel surface of $S$ in (directed) distance $\mu = $ const. $\neq 0$, so that $1 - 2\mu H + \mu^2 K \neq 0$, where $K$ is the Gauss curvature and $H$ is the mean curvature of $S$ respectively. Then $S^*$ possesses the position vector $r^* = r + \mu N$.

Denoting by $K^*$ and $H^*$ the Gauss and mean curvature of $S^*$ respectively, we mention the following relations [15]

$$K^* = \frac{K}{1 - 2\mu H + \mu^2 K},$$

$$H^* = \frac{H - \mu K}{1 - 2\mu H + \mu^2 K}.$$

Hence we get

$$R^* = \frac{2H^*}{K^*} := R - 2\mu \qquad (6)$$

On the other hand, the surfaces $S$, $S^*$ have common unit normal vector and spherical image. Thus $III = III^*$ and $\Delta^{III} = \Delta^{III*}$. We prove now the following theorem for later use.

**Theorem 1**. Let $S$ be a minimal surface in $E^3$. Then $S^*$ is parallel surface of $S$ if and only if the sum of the principal radii of curvature $R^*$ of $S^*$ is constant.
*Proof.* Suppose that $S$ is minimal surface in $E^3$, which is defined on a simply connected domain $D$ in the $(u^1, u^2)$-plane. Let

$$S^*: r^* = r + \mu N, \; \mu \neq 0$$

be parallel surface of $S$.
From (6) and taking into account $H = 0$, we find $R^* = -2\mu = $ const.. Hence the first part of the theorem is proved.

Conversely, let $R^* = $ const. $\neq 0$. Then from Theorem (4.4) (see [18]), $S^*$ is of null $III$- type 2. Therefore from (1) there exist nonconstant eigenvectors $x_1(u^1, u^2)$ and $x_2(u^1, u^2)$ defined on the same domain $D$ such that

$$r^* = x_1 + x_2, \qquad (7)$$

where $\Delta^{III} x_1 = \lambda_1 x_1$, $\Delta^{III} x_2 = \lambda_2 x_2$, and here we have $\lambda_1 = 0$ because $S^*$ is of null $III$- type 2.

Once we have $\Delta^{III} r^* = \Delta^{III} x_1 + \Delta^{III} x_2$, it then follows that

$$\Delta^{III} r^* = \lambda_2 x_2. \qquad (8)$$

On the other hand, since $R^* = $ const. $\neq 0$, we get

$$\Delta^{III} r^* = - R^* N. \qquad (9)$$

Thus from (8) and (9), one finds

$$\lambda_2 x_2 = - R^* N$$

or $x_2 = c \, N$, where $c = -\dfrac{R^*}{\lambda_2}$, and then (7) becomes

$$r^* = x_1 + cN. \qquad (10)$$

The differential of the above equation is

$$dr^* = dx_1 + cdN. \qquad (11)$$

Taking the inner product of both sides of (11) with $N$ yields

$$< dx_1, N > = 0. \qquad (12)$$

Now we want to show that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$. It is enough to prove that

$$x_{1/1} \times x_{1/2} \neq 0, \quad \forall (u^1, u^2) \in D,$$

where $\times$ is the Euclidean cross product. We have

$$x_1 = r^* - \mu N^* \qquad (13)$$

Using the Wingarten equations [15]

$$N_{/i} = -b_{ij} g^{jk} r^*_{/k}.$$

and the equation (13), it follows that

$$x_{1/1} \times x_{1/2} = (r^*_{/1} - \mu N_{/1}) \times (r^*_{/2} - \mu N_{/2})$$

$$= (r^*_{/1} \times r^*_{/2}) - \mu (r^*_{/1} \times N_{/2}) + \mu (r^*_{/2} \times N_{/1}) + \mu^2 (N_{/1} \times N_{/2})$$

$$= (1 - 2\mu H + \mu^2 K)(r^*_{/1} \times r^*_{/2}) \neq 0, \; \forall (u^1, u^2) \in D. \qquad (14)$$

Hence, on account of (12) and (14), we conclude that $x_1(u^1, u^2)$ is a regular parametric representation of a surface in $E^3$ with $N$ it's Gauss map.

Since $\Delta^{III} \boldsymbol{x}_1 = \boldsymbol{0}$. Consequently, from Theorem (3.1) (see [18]), $\boldsymbol{x}_1(u^1, u^2)$ is a minimal surface. Thus from (10) we obtain that $S^*$ is parallel surface of a minimal. Now we mention and prove our main theorem.

**Theorem 2.** The only surfaces of revolution in $E^3$ of which the sum of the radii of the principal curvature $R$ is constant are
- parts of spheres which are of finite *III*-type 1,
- catenoids which are of finite null *III*-type 1, and
- the parallel surfaces to the catenoids, which are of finite null *III*-type 2.

### III. PROOF OF THEOREM 2

Let $C$ be a smooth curve lies on the $xz$-plane parametrized by

$$\boldsymbol{x}(u) = (f(u), 0, g(u)), \ u \in J, (J \subset \mathfrak{R}),$$

where $f$, $g$ are smooth functions and $f$ is a positive function. When $C$ is revolved about the $z$-axis, the resulting point set $S$ is called the surface of revolution generated by the curve $C$. In this case, the $z$-axis is called the axis of revolution of $S$ and $C$ is called the profile curve of $S$. On the other hand, a subgroup of the rotation group which fixes the vector $(0, 0, 1)$ is generated by

$$\begin{pmatrix} cosv & -sinv & 0 \\ sinv & cosv & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the position vector of $S$ is given by

$$\boldsymbol{r}(u, v) = (f(u) \cos v, f(\mathrm{u}) \sin v, g(u)), \quad (15)$$
$$u \in J, v \in [0, 2\pi).$$

(For the parametric representation of surfaces of revolution, see [14, 16]).

Without loss of generality, we may assume that $C$ has the arc-length parametrization, i.e., it satisfies

$$(f')^2 + (g')^2 = 1, \quad (16)$$

where

$' := \dfrac{d}{du}$. Furthermore if $f' g' = 0$, then $f = $ const. or $g = $ const. and $S$ would be a circular cylinder or part of a plane, respectively. A case which has been excluded since $S$ would consist only of parabolic points [19].respectively. Hence $S$ would consist only of parabolic points, which has been excluded.

The partial derivatives of (15) are

$$\boldsymbol{r}_u = (f'(u) \cos v, f'(u) \sin v, g'(u)),$$
and
$$\boldsymbol{r}_v = (-f(u) \sin v, f(u) \cos v, 0).$$

The components $g_{ij}$ of the first fundamental form in (local) coordinates are the following

$$g_{11} = <\boldsymbol{r}_u, \boldsymbol{r}_u> = 1, \ g_{12} = <\boldsymbol{r}_u, \boldsymbol{r}_v> = 0, \ g_{22} = <\boldsymbol{r}_v, \boldsymbol{r}_v> = f^2.$$

Denoting by $\kappa$ the curvature of the curve $C$ and $R_1$, $R_2$ the principal radii of curvature of $S$, we have

$$R_1 = \frac{1}{\kappa}, \quad R_2 = \frac{f}{g'},$$

The Gauss curvature and the mean curvature of $S$ are respectively

$$K = \frac{1}{R_1 R_2} = \frac{\kappa g'}{f} = -\frac{f''}{f},$$

$$2H = \frac{1}{R_1} + \frac{1}{R_2} = \kappa + \frac{g'}{f}.$$

The Gauss map $N$ of $S$ is computed as follows

$$N(u,v) = (-g'\cos v, -g'\sin v, -f'). \quad (17)$$

Now, by using the natural frame $\{N_u, N_v\}$ of $S$ defined by

$$N_u = (-g'' \cos v, -g'' \sin v, f'')$$
and
$$N_v = (g' \sin v, -g' \cos v, 0)$$

the components $e_{ij}$ of the third fundamental form in (local) coordinates are the following

$$e_{11} = <N_u, N_u> = (g'')^2 + (f'')^2,$$

$$e_{12} = <N_u, N_v> = 0, \ e_{22} = <N_v, N_v> = (g')^2.$$

The Beltrami operator $\Delta^{III}$ in terms of local coordinates $(u, v)$ of $S$ can be expressed as follows

$$\Delta^{III} = -\frac{1}{\kappa^2} \frac{\partial^2}{\partial u^2} + \frac{g'\kappa' - \kappa g''}{g'\kappa^3} \frac{\partial}{\partial u} - \frac{1}{g'^2} \frac{\partial^2}{\partial v^2}. \quad (18)$$

On account of (16) we put

$$f' = \cos \varphi, \ g' = \sin \varphi,$$
where $\varphi = \varphi(u)$. Then $\kappa = \varphi'$ and the parametric representation (17) of the unit vector $N$ of $S$ becomes

$$N(u, v) = \{-\sin\varphi \cos v, -\sin\varphi \sin v, \cos\varphi\}. \quad (19)$$

Also relation (18) takes the following form

$$\Delta^{III} = -\frac{1}{\varphi'^2} \frac{\partial^2}{\partial u^2} + \left(\frac{\varphi''}{\varphi'^3} - \frac{\cos\varphi}{\varphi'\sin\varphi}\right) \frac{\partial}{\partial u} - \frac{1}{\sin^2\varphi} \frac{\partial^2}{\partial v^2}. \quad (20)$$

For the sum of the principal radii of curvature $R = R_1 + R_2 = \frac{2H}{K}$, one finds

$$R = \frac{f}{\sin\varphi} + \frac{1}{\varphi'}. \tag{21}$$

Taking the derivative of (21) we find

$$R' = -\frac{\varphi''}{\varphi'^2} - \frac{f\varphi'\cos\varphi}{\sin^2\varphi} + \frac{\cos\varphi}{\sin\varphi}. \tag{22}$$

Let $(r_1, r_2, r_3)$ be the coordinate functions of (15). By virtue of (20) we obtain

$$\Delta^{III} r_1 = \Delta^{III}(f\cos v) =$$

$$\left( \frac{\varphi''\cos\varphi}{\varphi'^3} - \frac{1}{\varphi'\sin\varphi} + \frac{2\sin\varphi}{\varphi'} + \frac{f}{\sin^2\varphi} \right)\cos v \tag{23}$$

$$\Delta^{III} r_2 = \Delta^{III}(f\sin v) =$$

$$\left( \frac{\varphi''\cos\varphi}{\varphi'^3} - \frac{1}{\varphi'\sin\varphi} + \frac{2\sin\varphi}{\varphi'} + \frac{f}{\sin^2\varphi} \right)\sin v \tag{24}$$

$$\Delta^{III} r_3 = \Delta^{III}g = -\frac{2\cos\varphi}{\varphi'} + \frac{\varphi''\sin\varphi}{\varphi'^3}. \tag{25}$$

From (21) and (22), equations (23), (24) and (25) become respectively

$$\Delta^{III} r_1 = \left( R\sin\varphi - \frac{R'\cos\varphi}{\varphi'} \right)\cos v, \tag{26}$$

$$\Delta^{III} r_2 = \left( R\sin\varphi - \frac{R'\cos\varphi}{\varphi'} \right)\sin v, \tag{27}$$

$$\Delta^{III} r_3 = -\frac{R'\sin\varphi}{\varphi'} - R\cos\varphi. \tag{28}$$

We obtain the following two cases:

*Case I.* $R \equiv 0$.

Thus $H \equiv 0$. Consequently $S$, being a minimal surface of revolution, is a catenoid [19].

*Case II.* $R = \text{const.} \neq 0$.

From (26), (27) and (28) we obtain

$$\left. \begin{array}{l} \Delta^{III} r_1 = R\sin\varphi\cos v \\ \Delta^{III} r_2 = R\sin\varphi\sin v \\ \Delta^{III} r_3 = -R\cos\varphi \end{array} \right\} \tag{29}$$

Let $(N_1, N_2, N_3)$ be the coordinate functions of $N$. From (18), relations (29) can be written

$$\Delta^{III} r_1 = -RN_1, \; \Delta^{III} r_2 = -RN_2, \; \Delta^{III} e_3 = -RN_3,$$

and hence

$$\Delta^{III} r = -R N. \tag{30}$$

In view of (5) and (30) we have

$$(\Delta^{III})^m r = -(2^{m-1}) R N. \tag{31}$$

Now, if $S$ is of finite type $k$, then there exist real numbers $c_1, c_2, \ldots, c_k$ such that

$$(\Delta^{III})^k r + c_1(\Delta^{III})^{k-1} r + \ldots + c_k r = 0. \tag{32}$$

From (30) and (31) relation (32) becomes

$$-2^{k-1} R N - 2^{k-2} c_1 R N - \ldots - c_{k-1} R N + c_k r = 0,$$

or

$$c N + c_k r = 0, \tag{33}$$

where $c = -R(2^{k-1} + 2^{k-2} c_1 + \ldots + c_{k-1}) = \text{const.}$.

Now, if $c_k \neq 0$, then from (33) we have $r = -\frac{c}{c_k} N$, and hence

we get $|r| = \left| \frac{c}{c_k} \right|$ and so $S$ is a sphere. On account of Theorem (3.3) (see [18]), $S$ is of finite $III$-type 1. If $c_k = 0$, then $S$ is of null type k. Since $R = \text{const.}$, thus according to Theorem (4.4) (see [18]) and Theorem (1), $S$ is of null $III$-type 2 which is parallel surface of a minimal.

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