A Modified Relay-Feedback Parameterization of Time-Delay Models: Theory and Application

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***Abstract*—** The well-known on/off relay-feedback identification test suffers from two imperfections. First, the parameters estimation does not bring a sufficient accuracy; second, a single test enables to quantify only two unknown model parameters. In this paper, two possible approaches dealing with these problems are attacked. Namely, the Autotune Variation Plus (ATV+) procedure introducing an additional artificial delay is utilized for a multiple-points model parameterization, and, the use of a saturation relay and a relay transient experiment with the Discrete Time Fourier Transform (DTFT) aid a better parameters estimation. The novelty of this contribution resides in that these methodologies are utilized with the combination with Linear Time-Invariant Time Delay Systems (LTI TDS) and their models. In our recent papers, theoretical aspects of the techniques were introduced and discussed, separately from other contributions dealing with a laboratory application of the saturation relay based approach. Both, theoretical and practical issues are summarized in this paper in a comprehensive presentation.

***Keywords*—** Autotuning, Fourier Transform, Heating system, Identification, Matlab/Simulink, Numerical optimization, Saturation relay, Time delay systems.

I. Introduction

Autotuning, in the parlance of the modern control theory, constitutes a set of methods which enable the controller to be tuned automatically based on a feedback test with a nonlinear element [1], [2]. In many cases, the test can serve to process model parameters identification or estimation (i.e. model parameterization), see e.g. [3], [4]. It is robust, easy to implement, timesaving, close-loop control which keeps the process close to the setpoint and it has been proved that it is a very useful tool in the industrial practice [5].

The original autotuning experiment proposition - sometimes called ATV (Autotune Variation) [5], [6] - designed in [1] utilizes the simple symmetrical on/off relay with the step-step static characteristics with the abrupt change around the zero point. It, however, suffers from some essential drawbacks: First, this basic test enables to estimate only a single point of the frequency characteristics (i.e. to identify two plant model parameters). Second, since the relay output has the form of rectangular waves that are, subsequently, analyzed using the Fourier series expansion into a sum of harmonic signals, this linear approximation is far from to be exact and from the ideal sinusoidal shape. For instance, there is an error of 23% for a first order unstable system with dominant input-output delay [7]. Last but not least, as mentioned above, a simple on/off relay has an abrupt change around zero input, which means that the actuator is burden with frequent limit changes (which can be undesirable behaviour mainly for mechanical systems).

Hence, many advanced techniques, which should eliminate the above mentioned deficiencies, have been developed [2], [5]. Some of them have been used in our research and are presented in this contribution. Namely, plant model parameters identification is improved by the use of two techniques. The first one is the application of a saturation relay [5], [8]. The second one utilizes the relay transient with DTFT [9], [10]. The advantage of the latter method is that arbitrarily many points on the Nyquist curve can be estimated by a single test. The use of the saturation relay, in this paper, is followed by the so-called Autotune Variation Plus (ATV+) procedure [11] - [13] which is based on the insertion of an artificial (additional) delay element into the open loop – It serves as a tool for finding multiple matching points on the frequency characteristics.

These all techniques are known (even though not very well); however, the novelty of this contribution is the combination of them with models of LTI TDS. Time-delay systems mean a family of processes and systems including delays as a part of their dynamics (besides integrators and derivatives) [14] - [16], and they can be found all around us, not only in industry, but also in economy, biology, etc. Since they belong to the class of infinite-dimensional systems, i.e. with an infinite spectrum with infinitely many modes, it is not easy to analyze and control them [16], [17].

Hence, we introduce here basic theoretical facts about LTI TDS and parameter identification of their models via the above mentioned methodologies and, consequently, the derived results are verified and benchmarked by simulations as well as by real experiments and measurements on a circuit heating laboratory plant with significant internal (state) delays [18], [19]. This paper summarizes and extends by real data some preliminary theoretical and simulation results published in [20]. There are many new outcomes and facts to be found in this contribution.

The paper is organized as follows. A general input-output model of LTI TDS is concisely introduced in Section II.

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Fundaments and the idea of the basic relay autotuning and identification is the issue Section III. In Section IV, advanced techniques of the relay autotuning, namely, the use of a saturation relay, implementation of a relay transient and the utilization of an additional artificial delay are briefly described. The evaluation of the relay feedback test data and some its computational and numerical aspects is the matter of Section V. Section VI provides a mathematical model of the laboratory plant for a benchmark and verification together with a simplified LTI TDS model for relay experiments, and the derivation of eventual formulas for model parameters calculations. Finally, simulation and real experimental results of plant model parameterization are presented in Section VII.

II. LTI TDS Model

LTI TDS are assumed to contain delay elements not only in input-output relations in an LTI system or model but also in its dynamics, which can be modeled by applying both integrators and delay elements either in lumped or distributed form on the left side of a differential equation. Thus, it is no longer ordinary but functional. Theory, models, analyses and/or applications of these systems can be found e.g. in [15], [16], [21] - [23] etc. Obviously, such systems or models are infinite-dimensional due to their infinite spectrum.

The input-output representation (map) of an LTI TDS, which is suitable for the sake of this paper, can be expressed in the form of a transfer function as a result of the application of the Laplace transform on a particular functional differential equation:

$$G(s)=\frac{b(s)}{a(s)} \tag{1}$$

where $a(s)$ expresses a quasipolynomial of degree n and quaipolynomial $b(s)$ can be factorized as $b(s)=b_0(s)\exp(-\tau s),\ \tau\geq 0$ where $b_0(s)$ is a (quasi)polynomial of degree $l\leq n$ of the general form

$$b_0(s)=s^l+\sum_{i=0}^{l}\sum_{j=1}^{h_i}b_{ij}s^i\exp(-\vartheta_{ij}s),\vartheta_{ij}\geq 0 \tag{2}$$

The reader is referred e.g. to [16] for some other general LTI TDS models in the state space and input-output maps.

III. Simple Relay Feedback Autotuning Test

The nascence of autotuning is linked up with the very famous work of Ziegler and Nichols [24] where, besides the proportional-integral-derivative (PID) controller tuning rule, an interesting identification procedure based on the information on the critical gain and the critical frequency was introduced. This is often referred to as the trial-and-error procedure. Historically, other methodologies were investigated as well, for example, the Cohen-Coon method [25], which requires an open-loop test on the process and it is thus inconvenient to apply. The disadvantage of other methods is e.g. the need of large setpoint change, see details in [2].

A. Relay Feedback Experiment

The relay feedback autotuning (identification) test performing limit cycle oscillations was successfully applied to the autotuning of PID controllers in [1] and it is widely used and in practice as a well applicable technique. It is robust, easy to implement, timesaving, easy to use and close-loop control which keeps the process close to the setpoint. The classical relay-feedback loop scheme with a symmetrical on/off relay is depicted in Fig. 1.

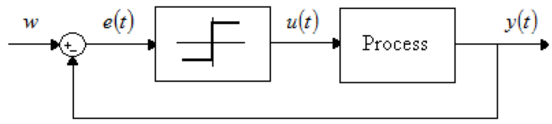

Fig. 1 Basic relay feedback experiment scheme

If the process is stabilizable and has a phase lag of at least π radians, the process input $u(t)$ and output $y(t)$ are logged until the system reaches stationary oscillations, the amplitude A of error $e(t)$ equals the amplitude of $y(t)$ and the overall phase shift between $e(t)$ and $y(t)$ is $-\pi$. Hence, the *ultimate period* T_u is obtained from oscillations, which gives the information about the critical point, together with the *ultimate gain* which is approximately given by

$$k_u=R(A)=\frac{4B}{\pi A} \tag{3}$$

where B is the relay amplitude. The *ultimate* (critical) *frequency* is close to the value of $\omega_u=2\pi/T_u$. Formula (3) comes from the linearization of the relay output via Fourier series approximation when upper harmonic components of the signal are neglected, since a relay is a non-linear element and it can be linearized for linear theory approaches, details can be viewed e.g. in [5].

B. Process Model Parameters Estimation

The relay feedback experiment can be utilized for model parameters identification. Let $G(s)$ be the controlled system (model) transfer function and $R(A)$ the describing (linearized) function of a relay (or a nonlinear element, in general), then for sustained oscillation holds

$$R(A)G(\mathrm{j}\omega_u)=-1+0\,\mathrm{j} \tag{4}$$

or equivalently

$$|R(A)G(\mathrm{j}\omega_u)|=1, \arg[R(A)G(\mathrm{j}\omega_u)]=-\pi \tag{5}$$

which describes one point at the open-loop Nyquist plot giving rise to the estimation of two plant model parameters by the solution of it. Nevertheless, as mentioned above, an estimation of two or more points requires using a special technique.

Dominant input-output delay, say τ, can be estimated as a time lag between the change of $u(t)$ and the maximum (minimum) value of $y(t)$ within the period, see Fig. 2.

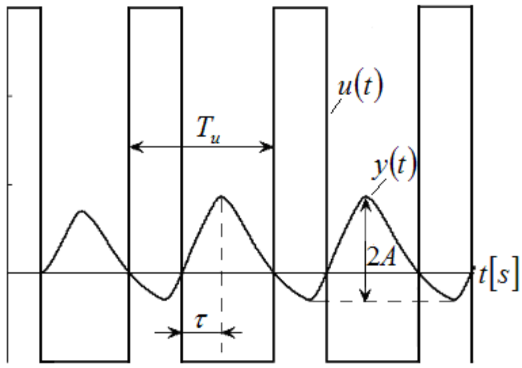

Fig. 2 Limit cycles and dominant delay estimation

IV. Advanced Relay Feedback Experiments

Now, some techniques which improve the plant model parameterization and/or enable to estimate multiple frequency points are introduced.

A. Multiple Parameters Estimation – Artificial Delay

As mentioned above, an estimation of two or more points requires using a special technique. For instance, an analytic expression and evaluation of some quantities in input and output signals was introduced in [26], [27], a decomposition into transient and stationary cycle parts followed by the discrete Fourier transform (DFT) or DTFT, more precisely, or the fast Fourier transform (FFT) can be found in [7], a utilization of a damping element with DTFT in [10], or inserting of an integral or a delay element into the open loop was the topic e.g. of [11], [12], [28], to name just a few methods.

In this paper, the use of an additional (artificial) delay, i.e. the ATV+ technique, element into the feedback loop is utilized. The first step of the ATV+ procedure is a standard relay test. The second step introduces an artificial delay τ^+ between the relay and the process.

The overall phase shift is $-\pi$, however only a part of this is attributed to the process, as τ^+ is characterized by the phase leg $\phi_D = \tilde{\omega}_u \tau^+$ where $\tilde{\omega}_u$ is a new ultimate frequency. The new amplitude $\tilde{A}$ of the output can be read as well. Every next setting of τ^+ determines one point of the Nyquist curve, hence, one needs to set the number $\lceil n/2-1 \rceil$ of various values of τ^+ where n is the number of unknown model parameters.

In [11] the following setting was suggested

$$\tau^+ = \frac{5\pi}{12\omega_u} \tag{6}$$

where ω_u means the ultimate frequency with no artificial delay.

B. More Accurate Parameters Estimation – Saturation Relay

Model parameters estimation can be improved by the use of a saturation relay [5], [8], the static characteristics of which is depicted in Fig. 3.

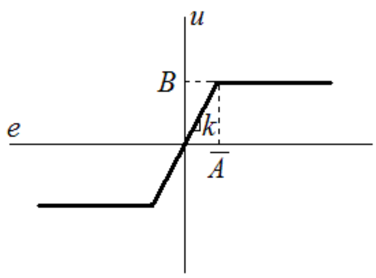

Fig. 3 Static characteristics of a saturation relay

Its advantage lies in the feature that relay output is not stepwise (i.e. with an abrupt slope change at the zero point), but it provides a smooth transient around the zero point. The relay input signal $e(t)$ is multiplied by k up to the limit value $B = k\overline{A}$ of $u(t)$, hence $u(t)$ is ideally in the form of a harmonic waves with an upper and lower limit. The output of the nonlinear element $u(t)$ looks like a truncated sinusoidal wave, as can be seen in Fig. 4. The meaning of $\overline{A}$ is clear from the figure.

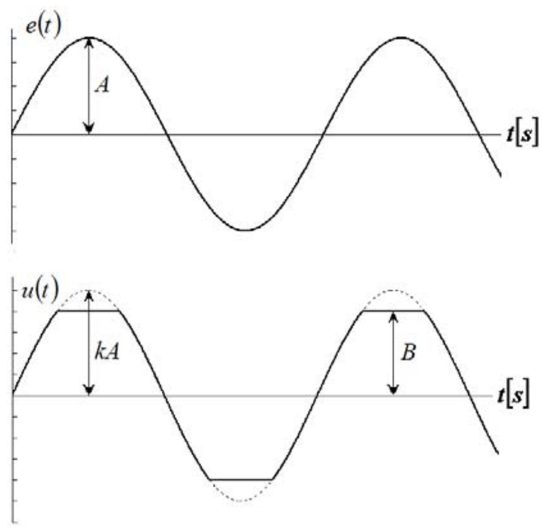

Fig. 4 Feedback signals with a saturation relay

Obviously, the ideal case is that when $u(t)$ has the shape of $e(t)$ while $\overline{A} = A$, where A is the amplitude of $e(t)$. In this case, the ultimate gain equals the value of k exactly. Another limit case arrives when $k \to \infty$, which agrees with the classical on/off relay.

The describing function of the relay can be obtained from the Fourier series expansion and neglecting higher harmonic parts of $u(t)$ and $e(t)$ as follows

$$R(A) = k_u = \frac{2B}{\pi \overline{A}}\left(\arcsin\left(\frac{\overline{A}}{A}\right) + \frac{\overline{A}}{A}\sqrt{1-\left(\frac{\overline{A}}{A}\right)^2}\right) \tag{7}$$

Hence, the aim is to find k (or equivalently $\overline{A}$) such that $\overline{A} \approx A$ for a given *B*, which provides the almost exact critical gain estimation. However, there is also a potential problem that can make the test fail. If the slope of the static characteristics k is too small, or equivalently, if $\overline{A} > A$, limit cycles may not exist. To avoid this, there has been proposed a two-step procedure finding a rough estimation of the lower bound on k, say $k_{\min}$, followed by a saturation relay test [5], [8]:

Algorithm 1 (Saturation relay experiment)

1) Select the height B of the relay (i.e. of the manipulated input).

2) Use an ideal on/off relay (or set the slope of a saturation relay to a large value $k \to \infty$) to estimate k_u according to (3). Set **Chyba! Objekty nemohou být vytvořeny úpravami kódů polí.**.

3) Calculate the slope of the saturation relay $k = 1.4 k_{\min}$.

4) Use the saturation relay with calculated *k*.

5) Find ω_u from the relay feedback test and compute the ultimate gain from (7).

C. Use of a Relay Transient

Some approaches can make improve both, the model parameters estimation and the number of possibly identified model parameters (even under a single test). The procedure proposed in [10] uses a relay transient with the DTFT (FFT) evaluation. Its summary follows.

Using a standard relay test, $u(t)$ and $y(t)$ are recorded from the initial time until the system reaches a stationary oscillation and they are subjected to exponential decaying according to

$$\overline{u}(t) = u(t)\exp(-at), \overline{y}(t) = y(t)\exp(-at) \tag{8}$$

Obviously, $\overline{u}(t)$ and $\overline{y}(t)$ will decay to zero for $a > 0$ and $t \to \infty$.

The Fourier transform applied to (8) results in

$$\begin{aligned}
\overline{U}(\mathrm{j}\omega) &= \int_0^\infty \overline{u}(t)\exp(-\mathrm{j}\omega t)\mathrm{d}t \\
&= \int_0^\infty u(t)\exp(-at)\exp(-\mathrm{j}\omega t)\mathrm{d}t = U(\mathrm{j}\omega + a) \\
\overline{Y}(\mathrm{j}\omega) &= \int_0^\infty \overline{y}(t)\exp(-\mathrm{j}\omega t)\mathrm{d}t \\
&= \int_0^\infty y(t)\exp(-at)\exp(-\mathrm{j}\omega t)\mathrm{d}t = Y(\mathrm{j}\omega + a)
\end{aligned} \tag{9}$$

Hence

$$G(\mathrm{j}\omega + a) = \frac{\overline{Y}(\mathrm{j}\omega)}{\overline{U}(\mathrm{j}\omega)} = \frac{Y(\mathrm{j}\omega + a)}{U(\mathrm{j}\omega + a)} \tag{10}$$

Values of functions $\overline{U}(\mathrm{j}\omega)$ and $\overline{Y}(\mathrm{j}\omega)$ can be computed at discrete frequencies with DTFT as

$$\begin{aligned}
\overline{U}(\mathrm{j}\omega_l) &= \mathrm{DTFT}(\overline{u}(t)) = T\sum_{k=0}^{N-1}\overline{u}(kT)\exp(-\mathrm{j}\omega_l kT),\ l = 1,2,\ldots,m \\
\overline{Y}(\mathrm{j}\omega_l) &= \mathrm{DTFT}(\overline{y}(t)) = T\sum_{k=0}^{N-1}\overline{y}(kT)\exp(-\mathrm{j}\omega_l kT),\ l = 1,2,\ldots,m
\end{aligned} \tag{11}$$

where T is the sampling interval, N means the number of samples and $t_f = (N-1)T$ expresses the final time for which the value of $\overline{u}(t)$ (or $\overline{y}(t)$) is sufficiently small. Usually, $m = N/2$ and $\omega_l = 2\pi l/(NT)$, see e.g. [7]. If, moreover, $N = 2^n, n \in \mathrm{N}$ (where N means a set of natural numbers), then the standard FFT can be used for faster computing.

V. Relay Feedback Test Evaluation

The data from relay experiments ought to be suitable evaluated. An overview of methodologies and numerical tools used in our research follows.

A. Frequency Domain Solution

Plant model parameters can be estimated directly using (4) and (5) with respect to describing functions (3) or (7). In a case of the using of artificial delay, these formulas are considered for every single test separately (i.e. with "new" values of A, ω_u).

If the relay transient is performed, the direct calculation is provided by the evaluation of (10) for every single value of ω_l.

Both the procedures yield sets of nonlinear algebraic equations.

B. Time Domain Solution

In [29] a methodology for limit cycles data evaluation based on a time-domain description instead of frequency one was introduced.

The idea comes from the fact that rectangular (for an on/off relay) or truncated sinusoidal (for a saturation relay) waves on a plant input (i.e. relay output) can be viewed as sinus waves in the lights of linearization (3) or (7).

Hence, if the artificial delay is not used, it is possible to write

$$u(t) \approx R(A) A \sin(t\omega_u) \tag{12}$$

Since neither the ideal nor the saturation relay evokes a phase shift, a plant output is given by

$$y(t) \approx -A \sin(t\omega_u) \tag{13}$$

i.e. $y'(t) = -A\omega_u \cos(t\omega_u)$, $y''(t) = A\omega_u^2 \sin(t\omega_u)$ etc. which is then inserted to a model differential equation. Subsequently, appropriate fixed time values are chosen which yields a set of nonlinear algebraic equations for the unknown model parameters again.

The use of the ATV+ gives rise to a phase lag ϕ_D, i.e.

$$\begin{aligned} &\widetilde{y}(t) = -\widetilde{A} \sin(t\widetilde{\omega}_u + \phi_D), \widetilde{y}'(t) = -\widetilde{A}\widetilde{\omega}_u \cos(t\widetilde{\omega}_u + \phi_D) \\ &\widetilde{y}''(t) = \widetilde{A}\widetilde{\omega}_u^2 \sin(t\widetilde{\omega}_u + \phi_D), \ldots \end{aligned} \tag{14}$$

where $\widetilde{A}$ is the amplitude of perpetual oscillations of $\widetilde{y}(t)$ for an additional delay element.

Again, these expressions are inserted to the appropriate model differential equation with two selected values of t for every addition delay test.

C. Numerical Aspects

Due to, generally, an ill-conditioned set of nonlinear algebraic equations obtained from approaches introduced above and because of multimodality of its solution [29], it would be useful to solve the equations by some advanced methods. For the objective of this research paper we have utilized three methods. Namely, the well-known Levenberg-Marquardt (LM) method (which is close to the Gauss-Newton one), see e.g. [30], the Nelder-Mead (NM) algorithm belonging to the class of comparative (direct search) algorithms, also called irregular simplex search algorithm, originally published in [31], and a standard Microsoft Excel (MS) Solver.

VI. Heating Plant Models and Conditional Equations for Identification

As a benchmark and testing example, a laboratory circuit heating plant with both internal and input-output delays serves as a benchmark process the model parameters of which are to be identified. The appliance was assembled at the Faculty of Applied Informatics, Tomas Bata University in Zlín, Czech Republic [18]. A photo and a sketch of the scheme of it are displayed in Fig. 5.

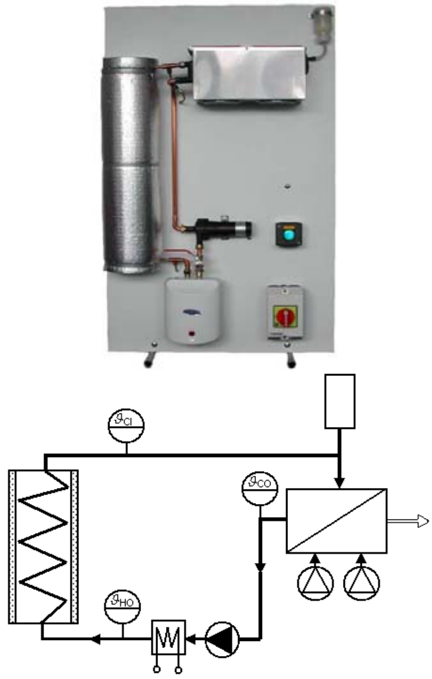

Fig. 5 Laboratory heating appliance with delays

Actually, heating systems are typical representatives of time-delay systems, see e.g. [32]; however, what is unordinary in our case is that the model includes a circuit leading to internal (state) delays.

Let us introduce a description of the plant that can be found e.g. in [18]: The heat transferring fluid (namely distilled water) is transported using a continuously controllable DC pump into a flow heater with maximum power $P_H(t)$ of 750 W. The temperature of a fluid at the heater output is measured by a platinum thermometer giving value of $\vartheta_{HO}(t)$. Warmed liquid then goes through a 15 meters long insulated coiled pipeline which causes the significant delay in the system. The air-water heat exchanger (cooler) with two cooling fans represents a heat-consuming appliance. The speed of the first fan can be continuously adjusted, whereas the second one is of on/off type. Input and output temperatures of the cooler are measured again by platinum thermometers giving $\vartheta_{CI}(t)$ and $\vartheta_{CO}(t)$, respectively. The expansion tank compensates for the expansion effect of the water.

A. Mathematical Models

A rigorous mathematical model of the appliance was presented in [19]. Although there are three continuous-time manipulated inputs ($P_H(t)$, voltage input to the pump, $u_P(t)$, and voltage input to the cooler, **Chyba! Objekty nemohou být vytvořeny úpravami kódů polí.**) and three measured outputs ($\vartheta_{HO}(t)$, $\vartheta_{CI}(t)$, $\vartheta_{CO}(t)$), the intention here is to control $\vartheta_{CO}(t)$ only by means of $P_H(t)$. For this relation, it was derived the following transfer function

$$G(s)=\frac{\vartheta_{CO}(s)}{P_H(s)}=\frac{[b_{0D}\exp(-\tau_0 s)+b_0]\exp(-\tau s)}{s^3+a_2 s^2+a_1 s+a_0+a_{0D}\exp(-\vartheta s)} \tag{15}$$

It was determined that for the working point

$$\begin{aligned}&[u_p,u_c,P_H,\vartheta_{HO},\vartheta_{CI},\vartheta_{CO},\vartheta_A]\\&=[5\,\mathrm{V},3\,\mathrm{V},300\,\mathrm{W},44.1\,^\circ\mathrm{C},43.8\,^\circ\mathrm{C},36.0\,^\circ\mathrm{C},24.0\,^\circ\mathrm{C}]\end{aligned} \tag{16}$$

it holds that

$$\begin{aligned}&b_{0D}=2.334\cdot 10^{-6}, b_0=-2.146\cdot 10^{-7},\\&a_2=0.1767, a_1=0.009, a_0=1.413\cdot 10^{-4},\\&a_{0D}=-7.624\cdot 10^{-5}, \tau_0=1.5, \tau=131, \vartheta=143\end{aligned} \tag{17}$$

Consider model (17) as an "exact" system description for simulations below and let us introduce a simplified model for relay identification experiment. Hence, the simplified LTI TDS model reads

$$G_m(s)=\frac{b_0\exp(-\tau s)}{s+a_0+a_1\exp(-\vartheta s)} \tag{18}$$

B. Conditional Equations for Identification

1) Saturation relay with ATV+ : Frequency-domain solution

Now the task is to find conditional equations for identification of model parameters (i.e. parameterization) by means of a relay feedback test with an on-off and saturation relay. There are five unknown real parameters in the model, i.e. $b_0,a_0,a_1,\tau,\vartheta$; however, two of them can be estimated not from the knowledge of the ultimate gain and frequency. Namely, the static gain $k=b_0/(a_1+a_0)=0.0325$ can be calculated from the step-response, and the value of input-output delay τ can be estimated from Fig. 2.

Hence, in the first step, a biased (asymmetrical) on/off relay with hysteresis is used to estimate these two parameters. Then, a simple (symmetrical) on/off followed by a saturation relay and the use of an artificial delay τ^+ can be utilized to calculate the remaining parameters from (5) and (7). Four conditional equation can be obtained by doing this, therefore one may improve the estimation of k or (preferably) τ. The use of a saturation relay yield

$$\begin{aligned}&f_{11}:=\\&\frac{b_0}{\sqrt{(a_0+a_1\cos(\vartheta\omega_u))^2+(\omega_u-a_1\sin(\vartheta\omega_u))^2}}\\&\frac{2B}{\pi\overline{A}}\left(\arcsin\left(\frac{\overline{A}}{A}\right)+\frac{\overline{A}}{A}\sqrt{1-\left(\frac{\overline{A}}{A}\right)^2}\right)-1=0\end{aligned} \tag{19}$$

$$f_{12}:=-\arctan\frac{\omega_u-a_1\sin(\vartheta\omega_u)}{a_0-a_1\cos(\vartheta\omega_u)}-\tau\omega_u+\pi=0 \tag{20}$$

The use of the ATV+ results in

$$\begin{aligned}&f_{13}:=\\&\frac{b_0}{\sqrt{(a_0+a_1\cos(\vartheta\tilde{\omega}_u))^2+(\tilde{\omega}_u-a_1\sin(\vartheta\tilde{\omega}_u))^2}}\\&\frac{2B}{\pi\tilde{\overline{A}}}\left(\arcsin\left(\frac{\tilde{\overline{A}}}{\tilde{A}}\right)+\frac{\tilde{\overline{A}}}{\tilde{A}}\sqrt{1-\left(\frac{\tilde{\overline{A}}}{\tilde{A}}\right)^2}\right)-1=0\end{aligned} \tag{21}$$

$$f_{14}:=-\arctan\frac{\tilde{\omega}_u-a_1\sin(\vartheta\tilde{\omega}_u)}{a_0-a_1\cos(\vartheta\tilde{\omega}_u)}-\tau\tilde{\omega}_u-\phi_D+\pi=0 \tag{22}$$

2) Saturation relay with ATV+ : Time-domain solution

If we follow the idea of time-domain limit cycles data evaluation presented in Subchapter V-B, with a saturation relay and ATV+ again, the following results are obtained.

Model transfer function (18) agrees with the functional differential equation

$$y'(t)+a_0 y(t)+a_1 y(t-\vartheta)=b_0 u(t-\tau) \tag{23}$$

which implies from (12), (13) etc.

$$\begin{aligned}&-A(\omega_u\cos(t\omega_u)-a_0\sin(t\omega_u)-a_1\sin((t-\vartheta)\omega_u))\\&=b_0 R(A)\sin((t-\tau)\omega_u)\end{aligned} \tag{24}$$

First, fix $t=\omega_u^{-1}(2l\pi)$, $l=0,1,\ldots$, and l be chosen so that $t>\max\{\tau,\vartheta\}$ and the limit cycle is stable (settled). Then

$$\begin{aligned}&f_{21}:=-\omega_u+a_1\sin(\vartheta\omega_u)\\&+b_0\frac{2B}{\pi\overline{A}}\left(\arcsin\left(\frac{\overline{A}}{A}\right)+\frac{\overline{A}}{A}\sqrt{1-\left(\frac{\overline{A}}{A}\right)^2}\right)\sin(\tau\omega_u)=0\end{aligned} \tag{25}$$

As second, let e.g. $t=\omega_u^{-1}((0.5+2l)\pi)$, then

$$\begin{aligned}&f_{22}:=a_0+a_1\cos(\vartheta\omega_u)\\&+b_0\frac{2B}{\pi\overline{A}}\left(\arcsin\left(\frac{\overline{A}}{A}\right)+\frac{\overline{A}}{A}\sqrt{1-\left(\frac{\overline{A}}{A}\right)^2}\right)\cos(\tau\omega_u)=0\end{aligned} \tag{26}$$

Analogously, the use of an additional delay element gives rise to the following conditions

$$f_{23} := -\tilde{\omega}_u + a_1 \sin(\vartheta\tilde{\omega}_u) + b_0 \frac{2B}{\pi\tilde{\bar{A}}}\left(\arcsin\left(\frac{\bar{\tilde{A}}}{\tilde{A}}\right) + \frac{\bar{\tilde{A}}}{\tilde{A}}\sqrt{1-\left(\frac{\bar{\tilde{A}}}{\tilde{A}}\right)^2}\right)\sin(\tau\tilde{\omega}_u + \phi_D) = 0 \quad (27)$$

$$f_{24} := a_0 + a_1 \cos(\vartheta\tilde{\omega}_u) + b_0 \frac{2B}{\pi\tilde{\bar{A}}}\left(\arcsin\left(\frac{\bar{\tilde{A}}}{\tilde{A}}\right) + \frac{\bar{\tilde{A}}}{\tilde{A}}\sqrt{1-\left(\frac{\bar{\tilde{A}}}{\tilde{A}}\right)^2}\right)\cos(\tau\tilde{\omega}_u + \phi_D) = 0 \quad (28)$$

Note that particular conditional equations for the relay transient are not presented here. They can be easily obtained by the direct calculation on (10) and separation the result into real and imaginary parts.

VII. Simulation and Real Experiment Results

Finally, let us present both, Matlab/Simulink and real measurements, results of relay-feedback identification tests via methodologies introduced above.

A. Simulations

We attempted to simplify the original mathematical model (15) by the use of the relay-feedback experiment with model (18), thus, try to identify its parameters.

1) Frequency-domain solution

The relay test was performed with an on-off relay, $B = 200[\mathrm{W}]$, first, The results were the following: $A_1 = 1.9975[°C]$, $T_{u,1} = 364.8[s]$, which gives $k_{u,1} = 127.48 = k_{\min}$, $k = b_0/(a_0 + a_1) = 0.0325$. Dead time was estimated in the accordance with Fig. 2 as $\tau = 136.7$. Then we tried to perform the saturation-relay test with $k_{u,2} = 1.4k_{\min}$; however, the restoration of limit cycles took a long time and there was an obvious margin in the setting of the saturation relay. Thus, the option $k_{u,2} = 1.1k_{\min} = 140.23 \Rightarrow \bar{A}_2 = 1.426$ resulted in $T_{u,2} = 373.4$, $A_2 = 1.9245$. These results enable to estimate two model parameters.

Hence, introduce an artificial delay element with $\tau^+ = 5\pi/(12\omega_u) = 5/24T_{u,2} = 77.8$. Again, the procedure started with a (symmetrical) on-off relay $B = 200$ yielding $\tilde{A}_1 = 3.1, \tilde{T}_{u,1} = 555.3, \tilde{k}_{u,1} = \tilde{k}_{\min} = 82.14$. Since $\tilde{k}_{u,2} = 1.1\tilde{k}_{\min}$ did not cause limit cycles, $\tilde{k}_{u,2} = 1.4\tilde{k}_{\min} = 115 \Rightarrow \bar{\tilde{A}}_2 = 1.7391$ were taken for the saturation-relay test which gave $\tilde{A}_2 = 2.52, \tilde{T}_{u,2} = 597.8$.

Solutions of the set (19) – (22) via various techniques are introduced in Table 1. The static gain value $k = 0.0325$ is fixed and the initial parameters estimation reads $a_0 = a_1 = 0.5/T_{u,2} = 0.013$, $\tau = \vartheta = 136.7$.

Table 1 Frequency-domain solution with saturation relay and artificial delay (Simulations)

	LM method	NM method	Excel Solver
a_0	$1.751567 \cdot 10^{-2}$	$9.9301645 \cdot 10^{-3}$	$4.422509 \cdot 10^{-2}$
a_1	$-9.08719 \cdot 10^{-3}$	$-3.9951297 \cdot 10^{-3}$	$-2.4479 \cdot 10^{-2}$
τ	102.83	159.83	140.92
ϑ	131.49	130.75	155.2
e	$1.27 \cdot 10^{-2}$	$3.25 \cdot 10^{-20}$	$8.06 \cdot 10^{-4}$

Note that NM algorithm and the MS Excel Solver minimize the sum of squares of the left-hand sides of (19) – (22), which agrees with error e in the table.

2) Time-domain solution

Simulation experiment results from the preceding subchapter can be used for alternative, time-domain, solution of the relay identification problem given by the set (25) – (28). Again, the results are summed up in Table 2.

Table 2 Time-domain solution with saturation relay and artificial delay (simulations)

	LM method	NM method	Excel Solver
a_0	$9.9301645 \cdot 10^{-3}$	$9.9301645 \cdot 10^{-3}$	$9.9301645 \cdot 10^{-3}$
a_1	$-3.9951297 \cdot 10^{-3}$	$-3.9951297 \cdot 10^{-3}$	$-3.9951297 \cdot 10^{-3}$
τ	159.83	159.83	159.83
ϑ	130.75	130.75	130.75
e	$1.82 \cdot 10^{-13}$	$3.38 \cdot 10^{-25}$	$2.11 \cdot 10^{-21}$

Obviously, the three computational techniques provide (almost) the same results identical with that obtained by the frequency-domain solution via NM algorithm.

3) Use of relay transient

Finally, try to use the relay transient introduced in Subsection IV-A-c. Limit cycles from the experiment with on-off relay were utilized. Exponential decaying function was chosen as $\exp(-0.01t)$, the sampling period for the DTFT was set to $T_0 = 0.1$ and the final time was taken as $t_f = 2000$. These values give rise to discrete frequencies $\omega_l = 0.0031l, l \in \mathrm{N}$ on which the DTFT is calculated and, subsequently, model parameters are estimated according to (9) – (11).

For $l = 1$, $\bar{U}(\mathrm{j}\omega_1) = -1.1139 \cdot 10^4 + 3.3348 \cdot 10^2\,\mathrm{j}$, $\bar{Y}(\mathrm{j}\omega_1) = 20.97 + 16\mathrm{j}$ and $\bar{Y}(\mathrm{j}\omega_1)/\bar{U}(\mathrm{j}\omega_1) = (1.924 - 1.379\mathrm{j}) \cdot 10^{-3}$.

For $l = 3$, $\bar{U}(\mathrm{j}\omega_3) = -1.285243 \cdot 10^4 + 2.52541 \cdot 10^3\,\mathrm{j}$, $\bar{Y}(\mathrm{j}\omega_3) = 11.208 + 25.188\mathrm{j}$ and $\bar{Y}(\mathrm{j}\omega_3)/\bar{U}(\mathrm{j}\omega_3) = (-1.8087 - 0.49819\mathrm{j}) \cdot 10^{-3}$.

Substituting these values into (10) for model (18), optimization techniques yield results introduced in Table 3.

TABLE 3 SOLUTION BY THE USE OF THE RELAY TRANSIENT (SIMULATIONS)

	LM method	NM method	Excel Solver
a_0	$3.06598277 \cdot 10^{-2}$	$3.06598277 \cdot 10^{-2}$	$7.410418 \cdot 10^{-3}$
a_1	$-1.7487959 \cdot 10^{-2}$	$-1.7487959 \cdot 10^{-2}$	$-2.607269 \cdot 10^{-3}$
τ	143.42	143.42	136.7
ϑ	158.08	158.08	136.69
e	$3.19 \cdot 10^{-14}$	$1.59 \cdot 10^{-15}$	$1.29 \cdot 10^{-8}$

Thus, LM and NM techniques provided comparable results.

4) *Comparison of the results*

To sum up the relay experiment, results from Tables 1 – 3 are compared via step responses and Nyquist plots of original and approximating models. Concurrently, the well-known Integral Squared Error (ISE) and Integral Squared Time Error (ISTE) criteria are calculated for step responses for the time range $t \in [0,2000]$ with the step $\Delta t = 0.1$, and Nyquist plots are assessed using the criterion

$$J_{Nyq} = \sum_{\omega_i} \left| G(\omega_i) - \overline{G}(\omega_i) \right| \tag{29}$$

where G is the original model (15), $\overline{G}$ means the approximating one (18) and ω_i are discrete frequencies, here $\omega \in [0,0.1]$ with $\Delta\omega = \omega_i - \omega_{i-1} = 10^{-4}$.

Figs. 6 and 7 provide a graphical comparison, whereas Table 4 gives criterial results.

Results from Tables 1 – 3 are labelled as follows:

a) “Result 1” – NM from Table 1, all results from Table 2.

b) “Result 2” – LM from Table 1

c) “Result 3” – MS Excel Solver from Table 1

d) “Result 4” – LM and NM from Table 3

e) “Result 5” – MS Excel Solver from Table 3

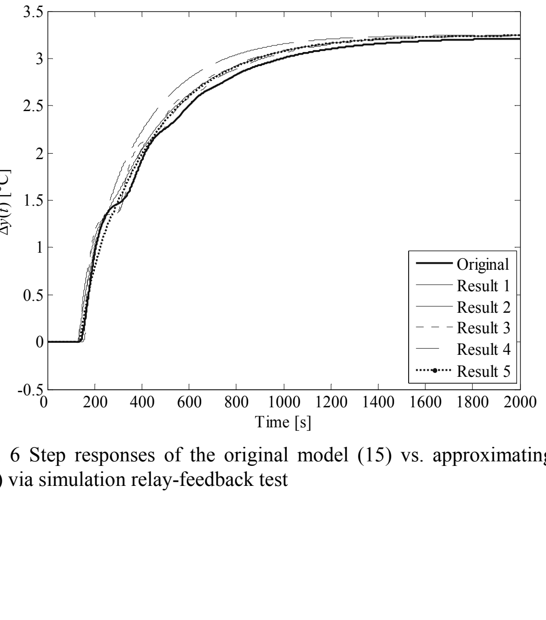

Fig. 6 Step responses of the original model (15) vs. approximating models (18) via simulation relay-feedback test

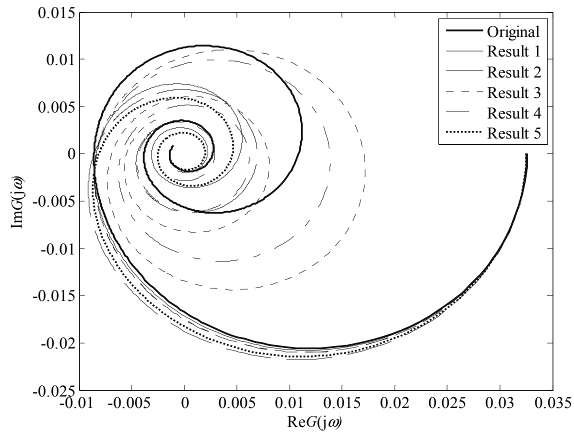

Fig. 7 Nyquist plots of the original model (15) vs. approximating models (18) via simulation relay-feedback test

TABLE 4 COMPARISON OF RELAY EXPERIMENTAL SIMULATION RESULTS

Result	J_{ISE}	J_{ISTE}	J_{Nyq}
Result 1	1.363	845.2	3.227
Result 2	6.304	3704	4.475
Result 3	1.795	1063.4	4.472
Result 4	0.607	450.2	2.835
Result 5	1.015	713.7	2.852

As can be seen from Table 4, the use of the relay transient solved by the LM and NM methods gives the best result. Especially, the Nyquist curves of the original model and the approximating one obtained by this way almost coincide for low frequencies (up to the ultimate frequency). The time-domain solution and the NM technique, generally, provide good approximation as well.

B. *Laboratory Appliance Measurements*

Let us use the relay test performed on the laboratory plant to identify parameters of model (18). A comparison of simulated and measured data from the relay test with a symmetrical and biased on-off relay, saturation relay and those with an artificial delay element are presented in Table 5.

There emerges a problem of a shifted stationary component (y_0) of limit cycles, see Fig. 8, here that is caused likely by process nonlinearity and nonsymmetrical dynamics of heating and cooling. It brings about inconveniencies mainly for the relay transient test since it is not clear whether to take $y_0 = 36$ (i.e. the steady state before the entrance of a symmetrical relay output) or $y_0 = 35.38$ which is the arithmetical mean value of maximum and minimum outputs within the period of limit cycles. Both possibilities are benchmarked within the relay transient procedure below.

In the lights of simulation results, only time-domain a relay transient solutions have been calculated and the NM technique was used.

TABLE 5 COMPARISON OF SIMULATED AND MEASURED RELAY TESTS DATA

Quantity	Measured data	Simulated data
A_1 [°C]	1.85	1.9975
$T_{u,1}$ [s]	377.9	364.8
$k_{u,1}$ [W·°C^{-1}]	137.65	127.48
τ [s]	129.6	136.7
$k_{u,2}$ [W·°C^{-1}]	151.42	140.23
$\overline{A}_2$ [°C]	1.32	1.426
A_2 [°C]	1.59	1.9245
$T_{u,2}$ [s]	380.6	373.4
τ^+ [s]	78.7	77.8
$\widetilde{A}_1$ [°C]	2.72	3.1
$\widetilde{T}_{u,1}$ [s]	579.9	555.3
$\widetilde{k}_{u,1}$ [W·°C^{-1}]	93.62	82.14
$\widetilde{k}_{u,2}$ [W·°C^{-1}]	131.07	115
$\overline{\overline{A}}_2$ [°C]	2.55	2.52
$\widetilde{A}_2$ [°C]	1.53	1.7391
$\widetilde{T}_{u,2}$ [s]	616.7	597.8

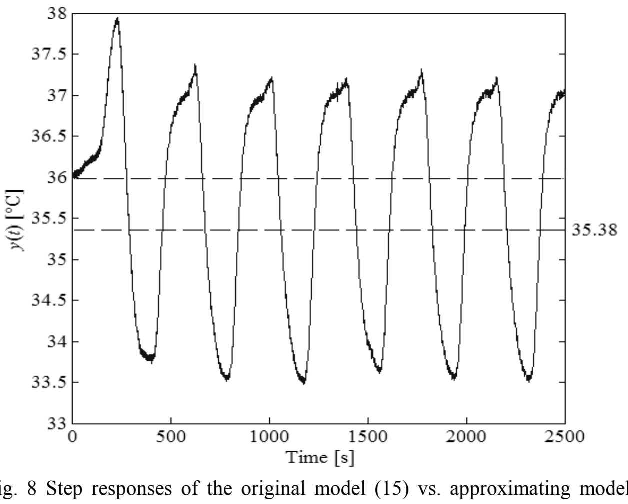

Fig. 8 Step responses of the original model (15) vs. approximating models (18) via simulation relay-feedback test

1) Time-domain solution

Time-domain solution of limit cycles data via the NM method (starting from the initial parameters estimation $a_0 = a_1 = 0.5 / T_{u,2} = 0.013$, $\tau = \vartheta = 129.6$) is introduced in Table 6, where i stands for the iteration step. The steps are chosen so that they provide substantially diverse model parameters estimations. Note that all parameters sets give stable models and $k = 0.0325$ is taken from a step response.

TABLE 6 TIME DOMAIN SOLUTION VIA NM METHOD

i	8	20	40	1800
$a_0 \cdot 10^{-2}$	2.1923	1.6926	1.5704	3.8155
$a_1 \cdot 10^{-2}$	-1.7542	-1.0696	-8.5139	-2.6778
τ	137.71	134.59	134.45	78.88
ϑ	137.71	139.34	139.84	132.55
e	$1.04 \cdot 10^{-3}$	$8.03 \cdot 10^{-5}$	$3.56 \cdot 10^{-6}$	$1.47 \cdot 10^{-33}$
J_{ISE}	96.95	5.601	0.321	2.291
J_{ISTE}	82175	3341.2	183.85	1331.1

Apparently, the estimation for $i = 40$ gives a better result than the converged one, see Fig. 9 for the comparison of step responses, that is comparable to the best simulated result (in Table 2).

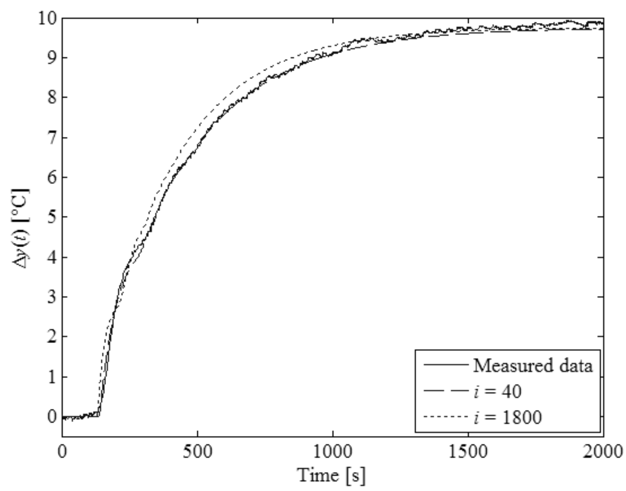

Fig. 9 Step responses comparison of measured data vs. relay based model using the time-domain solution

2) Use of relay transient

Consider now the use of the relay transient with the same settings as in Subchapter VI-A-3. Data for $y_0 = 36$ then (using the NM optimization) provide parameters estimations that mostly give unstable plant models (e.g. for $i = 20, 40, 1800$). Exceptional "stable" values are presented in Table 7.

If it is considered that $y_0 = 35.38$, the NM method converges as well; however, almost all estimations give unstable models except for $i = 12$, with $a_0 = 0.3007063$, $a_1 = -0.2233959$, $\tau = 157.15$, $\vartheta = 101.79$, $e = 1.64 \cdot 10^{-3}$, $J_{ISE} = 16.165$, $J_{ISTE} = 9900$.

Step responses for the best results of both variants ($y_0 = 36$ vs. $y_0 = 35.38$) compared to the measured response are pictured in Fig. 10. It seems that $y_0 = 36$ is a more suitable choice.

TABLE 7 RESULTS OF THE USE OF THE RELAY TRANSIENT WITH $y_0 = 36$

i	8	20
$a_0 \cdot 10^{-2}$	9.3969	5.5768
$a_1 \cdot 10^{-2}$	–6.9056	–3.1011
τ	177.95	165.2
ϑ	158.17	145.21
e	$1.09 \cdot 10^{-2}$	$6.69 \cdot 10^{-3}$
J_{ISE}	324.28	4.002
J_{ISTE}	26016	1591

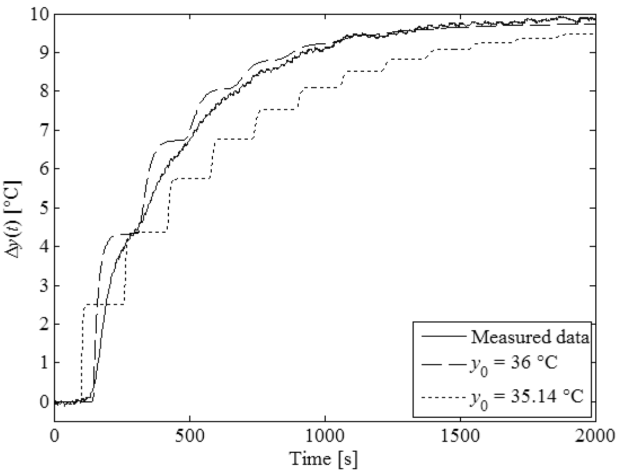

Fig. 10 Step responses comparison of measured data vs. relay based model using the relay transient

3) Comparison of the results

As it is clear from Figs. 9 and 10, the time-domain evaluation of limit cycles from on-off and saturation relay tests gives better plant model parameters estimations in comparison with the use of relay transient, although simulation benchmark has given rather different results. The letter methodology is sensitive to signal noise and the estimation of a stationary component of the signal.

VIII. CONCLUSION

Both the theoretical and practical aspects of (advanced) relay-feedback identification for (linear time-invariant) time delay systems have been presented in this paper. A methodology utilizing a simple on/off relay followed by the use of a relay with saturation and the philosophy of a relay transient has been used, and – what is a novelty – this approach is performed to time delay systems. A saturation relay and a relay transient with DTFT can improve model parameters estimation. Moreover, a relay transient can serve to multiple frequency points estimation. For a saturation relay, the ATV+ technique incorporating additional delay element in the loop has been utilized to solve this task.

Limit cycles data have been then evaluated using a standard frequency-based approach and via an unusual methodology working on the functional differential equations in the time domain.

The consequent example has demonstrated and proofed the usability of proposed methodologies on model parameters identification of a real (laboratory) system heating system with delays. Both, simulation and real-measurement results have been provided. While a relay transient provides the best simulation results, the practical implementation indicates the suitability of the use of a saturation relay with time-domain evaluation.

In the future, the eventual models can be used to real-time control of the laboratory appliance for verification of several control algorithms for time delay systems.

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