The Hamiltonian Dynamical System Associated to the Isotropic Harmonic Oscillator - A Numerical Study of Conservation Laws

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Abstract—In this paper we study the interplay between dynamical systems geometrical theory and computational calculus of dynamical systems. The viewpoint is geometric and we also compute and characterize objects of dynamical significance, in order to understanding the mathematical properties observed in numerical computation for dynamical models arising in mathematics, science and engineering. We use appropriate numerical methods for the study of conservation laws of Hamiltonian dynamical systems associated to the isotropic harmonic oscillator for one and two dimensional case.

Keywords—Conservation law, Noether theorem, Runge-Kutta methods, symmetry.

I. INTRODUCTION

Symmetries play a key role in mathematics, physics and very specially in mechanics. The study of those quantities which are conserved by a mechanical system is highly relevant both from a theoretical and from a practical point of view ([8], [28]). Different methods for finding such conserved quantities are known, those based on Noether theory ([26]), which addresses the invariance of the action functional under infinitesimal transformations, have proved powerful and widely used. For theoretical geometrical models, numerical analysis of models and more computation details see also [4], [5], [12], [25], [21], [22].

The inverse problem of Lagrangian mechanics is the question ([11], [30]): given a system of second-order ordinary differential equations, under what circumstances does there exist a regular Lagrangian function, such that the corresponding Lagrange equations are equivalent (i.e. have the same solutions) as the original equations. Locally, the question can be translated immediately into more precise terms as follows: considering a given second-order system in normal form \( \ddot{q} = f(q, \dot{q}) \), which (for the time being)

we take to be autonomous for simplicity, what are the conditions for the existence of a symmetric, non-singular multiplier matrix \((g_{ij}(q, \dot{q}))\) such that

\[
g_{ij} \left( \ddot{q}^i - f^j(q, \dot{q}) \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i}
\]

for some \(L\). Clearly \((g_{ij})\), if it exists, will become the Hessian of the Lagrangian \(L\). The literature on this problem is extensive ([8]); the conditions for the existence of \(L\) are usually referred to as the Helmholtz conditions, but these can take many different forms depending on the mathematical tools one uses and on the feature one focusses on ([27]).

In this paper we will study symmetries, conservation laws and relationship between this in the geometric framework of Classical Mechanics ([1], [2], [8], [23], [24]). Also, we will use appropriate numerical methods for the study of the behavior of dynamical system associated to the isotropic harmonic oscillator for one dimensional case and two dimensional case. For theoretical geometrical models, numerical analysis of models and more computation details see also [4], [5], [12], [21], [22].

There is a very well-known way to obtain conservation laws for a system of differential equations given by a variational principle: the use of the Noether Theorem ([26]) which associates to every symmetry a conservation law and conversely. However, there is a method introduced by G.L. Jones ([19]) and M. Crăşmăreanu ([10]) which can be obtained new kinds of conservation laws, without the help of a Noether’s type theorem.

In the second section we recall the basic notions and results for the geometrical study of a Hamiltonian dynamical systems, we present the classical Noether Theorem ([26]), the Theorem of Jones-Crăşmăreanu ([19], [10]) and, finally, we focus on two examples: 1D and 2D isotropic harmonic oscillator ([10], [30]).

In the third section, we will make a numerical study for the isotropic harmonic oscillator. More precisely, constructing a Matlab-based numerical code, we are looking to approximate and characterize different types of invariants and also to extract statistical information on the dynamical behavior and perform some comparisons for different initial conditions associated to the considered problem.

All manifolds are real, paracompact, connected and \(C^\infty\). All maps are \(C^\infty\). Sum over crossed repeated indices is understood.

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II. CLASSICAL RESULTS

A. Symmetries and Conservation Laws for Dynamical Systems Let \( M \) be a smooth, \( n \)-dimensional manifold, \( \mathcal{C}^\infty(M) \) the ring of real-valued smooth functions, \( \mathcal{X}(M) \) the Lie algebra of vector fields and \( \mathcal{A}^p(M) \) the \( \mathcal{C}^\infty(M) \)-module of \( p \)-differential forms, \( 1 \leq p \leq n \). For \( X \in \mathcal{X}(M) \) with local expression \( X = X^i(x) \frac{\partial}{\partial x^i} \), we consider the system of ordinary differential equations which give the flow \{\Phi_t\}_t\) of \( X \), locally,

\[
\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i(x^1(t), \ldots, x^n(t)).
\]

A dynamical system is a couple \((M, X)\), where \( M \) is a smooth manifold and \( X \in \mathcal{X}(M) \). A dynamical system is denoted by the flow of \( X \), \{(\Phi_t)\}_t\), or by the system of differential equations (1).

A function \( f \in \mathcal{C}^\infty(M) \) is called conservation law for dynamical system \((M, X)\) if \( f \) is constant along the every integral curves of \( X \), that is

\[
L_X f = 0,
\]

where \( L_X f \) means the Lie derivative of \( f \) with respect to \( X \).

A diffeomorphism \( \Phi : M \to M \) is said to be a symmetry of \((M, X)\) if \( \Phi \) maps integral curves of \( X \) onto integral curves of \( X \), i.e., \( T\Phi(X) = X \).

\( Y \in \mathcal{X}(M) \) is called dynamical symmetry (or, shortly, symmetry) for \((M, X)\) if its flow \{\( \Phi_t \)\}_t\) consists of symmetries of \((M, X)\), or, equivalently, \([Y, X] = L_X Y = 0\).

Let us recall that \( \omega \in \mathcal{A}^p(M) \) is called invariant form for \((M, X)\) if \( L_X \omega = 0 \).

B. Hamiltonian Systems If \((M, \omega)\) is a symplectic manifold then the dynamical system \((M, X)\) is said to be a dynamical Hamiltonian system (or, shortly, Hamiltonian system) if there exists a function \( H \in \mathcal{C}^\infty(M) \) (called the Hamiltonian) such that

\[
i_X \omega = -dH,
\]

where \( i_X \) denotes the interior product with respect to \( X \). The vector field \( X \) from (3) is called the Hamiltonian vector field associated to \( H \) (also denoted by \( X_H \)) and

\[
X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}
\]

with respect to natural local coordinates \((x^i, p_i)\), given by Darboux Theorem, ([23], [8]).

The equation (3) is equivalent, locally, with Hamilton-Jacobi equations:

\[
\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}.
\]

A regular Lagrangian \( L \) on \( TM \) is a smooth function \( L : TM \to \mathbb{R} \), for which the fundamental metric tensor \( g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} \) has rank \( n \) on \( TM \setminus \{0\} \). Let us remember the Legendre map associated to \( L \), \( FL : TM \to T^* M \), which is a local diffeomorphism given by \((x^i, y^i) \mapsto (\dot{x}^i, p_i)\), where \( p_i = \frac{\partial L}{\partial \dot{x}^i} \).

Let us consider the Hamiltonian system \((TM, S_L)\) on the symplectic manifold \((TM, \omega_L)\), where \( S_L \) is the canonical semispray associated to the regular Lagrangian \( L \), locally, given by

\[
S_L = y^i \frac{\partial}{\partial x^i} - G^i \frac{\partial}{\partial x^i},
\]

where \( G^i = g^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial y^j} \right) \) and \( g^{ij} \) being the inverse of the fundamental metric of \( L \), \( g_{ij}, [23] \).

In fact, the Euler-Lagrange equations associated to \( L \),

\[
E_i(L) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^i} \right) - \frac{\partial L}{\partial y^i} = 0
\]

are equivalently with the system of ODE’s,

\[
\frac{dy^i}{dt} = -G^j \frac{dx^j}{dt} = y^i,
\]

which give the flow of vector field \( S_L \) on \( TM \).

\( \omega_L = d\theta_L \) is the Cartan 2-form of \( L \), \( \theta_L = J^*(dL) \), \( J^* \) being the adjoint of the natural tangent structure \( J \) on \( TM \) and

\[
H = E_L = \frac{\partial L}{\partial \dot{y}^i} y^i - L
\]

is the energy of \( L \) or the Hamiltonian of \((TM, S_L)\).

A Cartan symmetry for Lagrangian \( L \) is a vector field \( X \in \mathcal{X}(TM) \) characterized by \( L_X \omega_L = 0 \) and \( L_X H = 0 \). It is known that that any Cartan symmetry for Lagrangian \( L \) is a symmetry for the canonical semispray \( S \) of \( L \).

For each Cartan symmetry \( X \) for \((M, L)\) we have \( dL_X \theta_L = 0 \), which implies that \( L_X \theta_L \) is a closed 1-form. If \( L_X \theta_L \) is an exact 1-form, then we say that \( X \) is exact Cartan symmetry for \((M, L)\).

Obviously, the canonical semispray of \( L \) is an exact Cartan symmetry for Lagrangian \( L \), [8], [23], [28].

C. Noether Theorem

In the classical case \((k = 1)\), we know that Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse.

**Theorem 1** ([26]) If \( X \) is an exact Cartan symmetry with \( L_X \theta_L = df \), then

\[
P_X = J(X)L - f
\]

is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian \( L \).
Conversely, if \( F \) is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian \( L \), then the vector field \( X \) uniquely defined by
\[
i_X \omega_L = -dF
\]
is an exact Cartan symmetry.

D. Nonclassical Conservation Laws

If \( Z \in \mathcal{X}(M) \) is fixed, then \( Y \in \mathcal{X}(M) \) is called \( Z \)-pseudosymmetry for \( (M, X) \) if there exists \( f \in C^\infty(M) \) such that \( L_X Y = fZ \).

A \( X \)-pseudosymmetry for \( X \) is called pseudosymmetry for \( (M, X) \).

**Example:** ([11],[13]) The Nahm’s system from the theory of static SU(2)-monopoles:
\[
\frac{dx^1}{dt} = x^2 x^3, \quad \frac{dx^2}{dt} = x^3, \quad \frac{dx^3}{dt} = x^1 x^2.
\]
The vector field \( X = x^2 x^3 \frac{\partial}{\partial x^3} + x^3 x^1 \frac{\partial}{\partial x^1} + x^1 x^2 \frac{\partial}{\partial x^2} \) satisfies \( [Y, X] = X \), where \( Y = \sum_i x^i \frac{\partial}{\partial x^i} \).

This means that \( Y \) is a \( X \)-pseudosymmetry for \( X \).

The next theorem gives the association between pseudosymmetries and conservation laws. In this way, we will obtain new kinds of conservation laws, nonclassical, without the help of Noether’s type theorem.

**Theorem 2** ([10],[19]) Let \( X \in \mathcal{X}(M) \) be a fixed vector field and \( \omega \in \Lambda^p(M) \) be an invariant \( p \)-form for \( X \). If \( Y \in \mathcal{X}(M) \) is a symmetry for \( X \) and \( S_1, \ldots, S_{p-1} \in \mathcal{X}(M) \) are \((p-1)\) \( Y \)-pseudosymmetries for \( X \) then
\[
\Phi = \omega(X, S_1, \ldots, S_{p-1})
\]
or, locally,
\[
\Phi = S_1^{i_1} \cdots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 \ldots i_{p-1} i_p}
\]
is a conservation laws for \( (M, X) \).

Particularly, if \( Y, S_1, \ldots, S_{p-1} \) are symmetries for \( X \) then \( \Phi \) given by (5) is conservation laws for \( (M, X) \).

Now, we can apply this result to the dynamical Hamiltonian systems.

**Proposition 3** Let be \( (M, X_H) \) a Hamiltonian system on the symplectic manifold \( (M, \omega) \), with the local coordinates \( (x^i, p_i) \). If \( Y \in \mathcal{X}(M) \) is a symmetry for \( X_H \) and \( Z \in \mathcal{X}(M) \) is a \( Y \)-pseudosymmetry for \( X_H \) then
\[
\Phi = \omega(Y, Z)
\]
is a conservation law for the Hamiltonian system \( (M, X_H) \).

Particularly, if \( Y \) and \( Z \) are symmetries for \( X_H \) then \( \Phi \) from (7) is a conservation law for \( (M, X_H) \).

If \( Y = Y^k \frac{\partial}{\partial x^k} + \tilde{Y}_k \frac{\partial}{\partial p_k} \) and \( Z = Z^k \frac{\partial}{\partial x^k} + \tilde{Z}_k \frac{\partial}{\partial p_k} \) then (7) becomes
\[
\Phi = \left( Y^k \tilde{Y}_k \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} Z^k \\ \tilde{Z}_k \end{array} \right) = \tilde{Y}_k Z^k - \tilde{Z}_k Y^k.
\]

**Corollary 4** If \( Y \in \mathcal{X}(M) \) is a \( X_H \)-pseudosymmetry for \( X_H \) then
\[
\Phi = \omega(X_H, Y) = -L_Y H
\]
or
\[
\Phi = \frac{\partial H}{\partial x^k} Y^k + \frac{\partial H}{\partial p_k} \tilde{Y}_k
\]
is a conservation law for \( (M, X_H) \).

E. Nonclassical Conservation Laws for Regular Lagrangian Systems

If we consider the Hamiltonian system \( (TM, S_L) \) on the symplectic manifold \( (TM, \omega_L) \), where \( S_L \) is the canonical semispray and \( \omega_L \) the canonical 2-form associated to a regular Lagrangian \( L \) on \( TM \), then we have:

**Corollary 5** If \( Y = Y^k \frac{\partial}{\partial x^k} + \tilde{Y}_k \frac{\partial}{\partial p_k} \in \mathcal{X}(TM) \) is a \( S_L \)-pseudosymmetry for \( S_L \) then
\[
\Phi = \omega_L(S_L, Y) = -L_Y E_L
\]
or
\[
\Phi = \frac{\partial E_L}{\partial x^k} Y^k + \frac{\partial E_L}{\partial p_k} \tilde{Y}_k
\]
is a conservation law for \( (TM, S_L) \).

An immediately consequence of this last result is the following:

**Corollary 6** If the canonical semispray \( S_L \) associated to the regular Lagrangian \( L \) is \( 2 \)-positive homogeneous with respect to velocity \( (S_L \text{ is a spray}) \) and \( g_{ij} \) is the metric tensor of \( L \), then \( \Phi = g_{ij} Y^i \tilde{Y}^j \) is a conservation law for \( (TM, S_L) \).

The canonical semispray \( S_L \) is a spray if and only if \( [S_L, C] = S_L \), that is \( L_{S_L} C = S_L \). So, in this case, the canonical vector field \( C = y^j \frac{\partial}{\partial x^j} \) is a pseudosymmetry for \( S_L \) and using the last corollary we obtain the conservation of the kinetic energy
\[
T = \frac{1}{2} g_{ij} y^i y^j.
\]
This is the case when \( L = F^2 \), where \( F \) is a Finsler function like in [23].

F. Examples

**Example:** 1D harmonic oscillator

The dynamics of the 1-dimensional harmonic oscillator is given by second-order ordinary differential equation
\[
\ddot{q} + \omega^2 q = 0,
\]
where $\omega^2 = \frac{k}{m}$, $m$ is the mass of the oscillator (usually we take a standard, $m = 1$) and $k$ is the elastic constant. The period of the oscillations is $T = \frac{2\pi}{\omega}$.

The Lagrangian is $L(q, \dot{q}) = T - V = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$, where $T$ is the kinetic energy $T = \frac{1}{2} \dot{q}^2$ and $V$ is the potential of the elastic force, $V = \frac{1}{2} k q^2$. The total energy of $L$, $E_L = q \frac{d}{dq} L = \frac{1}{2}(\dot{q}^2 + \omega^2 q^2)$ is a conservation law for (13) and represents the Hamiltonian function of the dynamical system (13). The canonical semispray of $L$ is $S_L = \frac{1}{\omega^2} q \frac{d}{d\dot{q}}$.

Next, we have $L$, $\dot{q}$ the Euler-Lagrange equation (13) with initial conditions $(13)$ represents the Hamiltonian function of the dynamical system (13). The canonical semispray of $L$ is $S_L = \frac{1}{\omega^2} q \frac{d}{d\dot{q}}$.

The period of the oscillations is $T(0) = \frac{2\pi}{\omega}$, where $\omega = \sqrt{\frac{k}{m}}$.

Let us remark that $\Phi_4$ is a nonclassical conservation law, obtained by symmetries, and $\Phi_4$ represent the energy of a new Lagrangian of $\mathcal{L}' = q_i^2 q^2 - \omega^2 q^2 q_i^2$ (30).

Like in the 1D case, $q_i(t) = \tilde{A}_i \cos \omega t + \frac{B_i}{\omega} \sin \omega t$, $i = 1, 2$ is the solutions of the Euler-Lagrange equations (14) with initial conditions $q(0) = A_i, \dot{q}(0) = B_i$, $i = 1, 2$, and we have

\[
L = -\frac{1}{2} \left( (A_i^2 + A_j^2) \omega^2 + B_i^2 + B_j^2 \right) \cos 2\omega t - AB \sin 2\omega t,
\]

or

\[
H = \frac{1}{2} \left( A_i^2 \omega^2 + B_i^2 \right),
\]

and

\[
T = \frac{1}{2} \left( A_i^2 \omega^2 \sin^2 \omega t + B_i^2 \cos^2 \omega t \right).
\]

**Example: 2D isotropic harmonic oscillator**

Let the 2-dimensional isotropic harmonic oscillator

\[
\begin{align*}
\dot{q}_1^1 + \omega^2 q_1^1 &= 0 \\
\dot{q}_2^2 + \omega^2 q_2^2 &= 0
\end{align*}
\]  

(14)

is a toy model for many methods to finding conservation laws. The Lagrangian is

\[
L = \frac{1}{2} \left[ (q_1^1)^2 + (q_2^2)^2 \right] - \omega^2 \left[ (q_1^1)^2 + (q_2^2)^2 \right] - \omega^2 \left( q_1^1 q_2^2 \right)
\]

Next, by applying the conservation of energy we have two conservation laws $\Phi_1 = (q_1^1)^2 + \omega^2 (q_1^1)^2$, $\Phi_2 = (q_2^2)^2 + \omega^2 (q_2^2)^2$. Let us observe that $\frac{1}{2} (\Phi_1 + \Phi_2) = H$, the total energy of the system.

A straightforward computation give that the complet lift of $X = q_1^1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2}$ is an exact Cartan symmetry with $f = 0$ and then the associated classical Noetherian conservation law is

\[
\Phi_3 = P_X = J(X)L = X^i \frac{\partial L}{\partial q^i} = q_2^1 \dot{q}_1^1 - q_1^1 \dot{q}_2^2.
\]

Taking into account that the canonical spray of $L$ is

\[
S = \dot{q}_1^1 \frac{\partial}{\partial q_1} + q_2^2 \frac{\partial}{\partial q_2} - \omega^2 q_1^1 \frac{\partial}{\partial q_1} - \omega^2 q_2^2 \frac{\partial}{\partial q_2}
\]

and

\[
Y = q_1^1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} - \omega^2 q_1^1 \frac{\partial}{\partial q_1} - \omega^2 q_2 \frac{\partial}{\partial q_2}
\]

is a symmetry for $S$, it result that

\[
Z = q_1^1 \frac{\partial}{\partial q_1} + q_2^2 \frac{\partial}{\partial q_2} + q_1^1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2}
\]

is a symmetry for $S$.

Next, we have $L_Y H = 0$, $L_Z H = 2H$ and then $\Phi_3 = \omega_L(S, Y) = 0$, $\Phi_4 = \omega_L(S, Z) = 2H$, that means that we not have new conservation law applying main Theorem 2. But $\Phi_4 = \omega_L(Y, Z) = \dot{q}_1^2 q^2 + \omega^2 q_1^2 q_2^2$ is a new conservation law given by the main Theorem 2 or by their corollaries.

III. NUMERICAL STUDY

In this section, constructing a Matlab-based numerical code, we are looking to approximate and characterize different types of invariants and also to extract informations on the dynamical behavior and perform comparisons for both different initial conditions associated to the considered problem and for different values of the parameter $\omega$.

In the first stage we focus on the numerical solving of the initial value problem given by the ordinary differential equation with a prescribed initial condition, by appropriate numerical methods, such as Runge-Kutta methods. Numerical integration is a mature subject, but due to the todays high computer efficiency it is still very active especially with regard to algorithms designed for special classes of equations, [17], [29], [18]. Thus we obtain the numerical solution represented by the approximate values of the exact solution for a discrete set of data points.

In the second stage, using this approach we perform a numerical analysis of the conservation laws and main sizes, like the Lagrangian, the Hamiltonian, the kinetic energy, the kinetic momentum.

A. One Dimensional Harmonic Oscillator

We consider the initial value problem (IVP) given by:

\[
\begin{align*}
\ddot{q} + \omega^2 q &= 0 \\
q(0) &= 1 \\
\dot{q}(0) &= 0
\end{align*}
\]
(a) $\omega = 0.59$

Fig 1: Numerical solution on a period

Fig 2: Lagrangian $L$, function of $t$ and $q$

Fig 3: Lagrangian $L$, as function of $q$, respectively function of $t$

Fig 4: Total energy $H$, function of $t$ and $q$

Fig 5: Total energy $H$, as function of $q$, respectively as function of $t$

Fig 6: Kinetic energy $T$, function of $t$ and $q$

Fig 7: Kinetic energy $T$, as function of $q$, respectively as function of $t$

(b) $\omega = 7.8$
Fig 8: Numerical solution on a period

Fig 9: Lagrangian $L$, function of $t$ and $q$

Fig 10: Lagrangian $L$, as function of $q$, respectively as function of $t$

Fig 11: Total energy $H$, function of $t$ and $q$

Fig 12: Total energy $H$, as function of $q$, respectively as function of $t$

Fig 13: Kinetic energy $T$, function of $t$ and $q$

Fig 14: Kinetic energy $T$, as function of $q$, respectively as function of $t$
We present a comparison concerning the kinetic energy $T$ for three different values of the parameter $\omega$:

![Fig 15: Profiles of the kinetic energy $T$, for three different values of $\omega$](image1)

Next we give the same comparison concerning the comportment of the kinetic energy, but for another initial value problems obtained from the considered second order differential equation characterizing the 1D harmonic oscillator and different initial conditions.

More precisely considering the following IVP:

\[
\begin{align*}
\ddot{q} + \omega^2 q &= 0 \\
q(0) &= 1 \\
\dot{q}(0) &= 1
\end{align*}
\]

the graphic profiles of kinetic energy are:

![Fig 16: Profiles of the kinetic energy $T$, for three different values of $\omega$](image2)

and also for the following IVP:

\[
\begin{align*}
\ddot{q} + \omega^2 q &= 0 \\
q(0) &= 1 \\
\dot{q}(0) &= -1
\end{align*}
\]

the kinetic energy presents the graphic profiles plotted in the next figure:

![Fig 17: Profiles of the kinetic energy $T$, for three different values of $\omega$](image3)

B. Two Dimensional Harmonic Oscillator

We consider the initial value problem given by:

\[
\begin{align*}
\ddot{q}_1 + \omega^2 q_1 &= 0 \\
\ddot{q}_2 + \omega^2 q_2 &= 0 \\
q_1(0) &= 1 \\
\dot{q}_1(0) &= 0 \\
q_2(0) &= 1 \\
\dot{q}_2(0) &= 0
\end{align*}
\]

(a). $\omega = 0.59$

![Fig 18: Profile of the numerical solution $q_1$ on a period](image4)

![Fig 19: Profile of the numerical solution $q_2$ on a period](image5)
Fig 20: Lagrangian $L$, function of $t$

Fig 21: Lagrangian $L$, function of $q_1$, $q_2$

Fig 22: Total energy $H$, function of $q_1$, $q_2$

Fig 23: Kinetic energy $T$, function of $q_1$, $q_2$

Fig 24: First conservation law $\Phi_1$, function of $q_1$, $q_2$

Fig 25: Second conservation law $\Phi_2$, function of $q_1$, $q_2$
Fig 26: Third conservation law $\Phi_3$, function of $q_1$, $q_2$

Fig 27: Fourth conservation law $\Phi_4$, function of $q_1$, $q_2$

Fig 28: Lagrangian $\tilde{L}$, function of $q_1$, $q_2$

(b) $\omega = 7.8$

Fig 29: Profile of the numerical solution $q_1$ on a period

Fig 30: Profile of the numerical solution $q_2$ on a period

Fig 31: Lagrangian $L$, function of $t$
Fig 32: Lagrangian $L$, function of $q_1$, $q_2$

Fig 33: Total energy $H$, function of $q_1$, $q_2$

Fig 34: Kinetic energy $T$, function of $q_1$, $q_2$

Fig 35: First conservation law $\Phi_1$, function of $q_1$, $q_2$

Fig 36: Second conservation law $\Phi_2$, function of $q_1$, $q_2$

Fig 37: Third conservation law $\Phi_3$, function of $q_1$, $q_2$
Observation 7 The range of the values of the Lagrangian $L$ is symmetric around zero and the extreme values are directly proportional with the values of parameter $\omega$. This can also be observed for the new Lagrangian $\tilde{L}$.

The extreme values of the kinetic energy $T$ are also directly proportional with the values of parameter $\omega$. The dependence of the kinetic energy upon the initial conditions, for the case of 1D harmonic oscillator, can be observed in Fig.16 and Fig.17, while the influence of the values of the parameter $\omega$ on the comportment of the kinetic energy can be analysed in Fig.15.

The total energy $H$ takes constant values for different values of $\omega$, as we presented in Fig.5, Fig.12, Fig.22 and Fig.33. Thus we are in agreement with the property to be a conservation law.

The others sizes as $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Phi_4$, which are also laws of conservation, are characterized through our numerical study, by constant values, depending on the values of $\omega$, Fig.24-27 and Fig.35-38.

IV. CONCLUSION
We perform a numerical study of this mathematical models, in order to approximate different types of invariants and main sizes, through numerical codes based on appropriate numerical calculus techniques. Thus, starting from certain initial value problems associated to our models, we obtain the numerical solution and we develop the numerical characterization of the main sizes previously analysed from the geometrical point of view. Thus we are able to make different comparisons between these studied quantities for different values of parameters, for different initial conditions etc.

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