Practical Approach for Modeling Chaotic Maps related to Mobius Transformation

Taeil Yi[†]

Abstract— It has been very difficult to find a dense orbit point and densely many periodic points of a chaotic map. One of the reasons is the complexity of the most popular definition of chaos made by Devaney. There have been several attempts to replacing Devaney's definition with simpler one, and one of them is using topological properties only, i.e., it uses the transitivity and the densely many periodic points properties of the function. In this paper, using these properties, we present a chaotic maps on $\Pi_{i=1}^{n} S^1$. We produce a sequence space on the n symbols, and show that the (left) shift map is a chaotic map on it. Then, by building a continuous bijective map between the sequence space and S^1, we show that the angle multiplying map is a chaotic map on S^1. From this we show that a product of angle multiplying maps on $\Pi_{i=1}^{n} S^1$ becomes a chaotic map by constructing densely many periodic points and a dense orbit. We also show that the function has infinitely many dense orbits, and the Möbius transformation produces a chaotic map on T^2.

Keywords— angle multiplying map, chaotic map, Mobius transformation.

I. Introduction

Devaney introduced a definition of chaotic function in [7] as follows: A continuous map $f : X \to X$ is said to be *chaotic* on a metric space X if f is (topologically) transitive, the periodic points of f are dense in X, and f has sensitive dependence on initial conditions. We say that f is *(topologically) transitive* if for all non-empty open subsets U and V of X there exists a positive integer k such that $f^k(U) \cap V$ is nonempty. We also say that f has *sensitive dependence on initial conditions* if there is a positive real number δ (a *sensitivity constant*) where, for every neighborhood N of arbitrary point x in X, there exists a point y in N and a nonnegative integer n such that the nth iterates $f^n(x)$ and $f^n(y)$ of x and y respectively, are more than distance δ apart.

J. Banks and others showed in [3] that if $f : X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions, i.e., chaos rely on topological properties, not on metric. Since having a dense orbit implies transitive, a continuous map f on a metric space X is chaotic if f has a dense orbit and densely periodic points.

T. Yi is with the Department of Mathematics, University of Texas at Brownsville, Brownsville, TX 78520 USA (phone: 956-882-6621; fax: 956-882-6637; e-mail: taeil.yi@utb.edu).

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In this paper we show that a product of angle multiplying maps on $\Pi_{i=1}^{n} S^1$ becomes chaotic by constructing a dense orbit and densely many periodic points.

II. Shift map on (Σ_n, d)

For positive integers $n(>1)$ and k there exist n^k n-ary sequences of length k such as

$$\begin{gathered} \langle 0,\cdots,0,0\rangle,\cdots,\langle 0,\cdots,0,n-1\rangle, \\ \langle 0,\cdots,1,0\rangle,\cdots,\langle 0,\cdots,1,n-1\rangle, \\ \ddots \\ \langle n-1,\cdots,n-1,0\rangle,\cdots,\langle n-1,\cdots,n-1,n-1\rangle. \end{gathered}$$

For every $1 \le j \le n^k$ there is unique finite sequence $\langle s_1, s_2, ..., s_k\rangle$ where $s_i \in \mathbf{Z}_n$ with $i \in \{1, 2, ..., k\}$ such that $j = 1 + \sum_{i=1}^{k} \left(s_i \times n^{k-i}\right)$. We denote the sequence $\langle s_1, s_2, ..., s_k\rangle$ by $c(n, k, j)$. Clearly, for any $1 \le j \le n^k$, $c(n, k, j)$ is the j-th n-ary sequences of length k.

Example 1 *Let $n = 4$ and $k = 3$. Then there are $4^3 = 64$ 4-ary sequences of length 3 as the following:*

$$\begin{array}{cccc} \langle 0,0,0\rangle, & \langle 0,0,1\rangle, & \langle 0,0,2\rangle, & \langle 0,0,3\rangle, \\ \langle 0,1,0\rangle, & \langle 0,1,1\rangle, & \langle 0,1,2\rangle, & \langle 0,1,3\rangle, \\ \cdots & \cdots & \cdots & \cdots \\ \langle 3,3,0\rangle, & \langle 3,3,1\rangle, & \langle 3,3,2\rangle, & \langle 3,3,3\rangle. \end{array}$$

For example, since $1 + (1 \cdot 4^{3-1}) + (0 \cdot 4^{3-2}) + (0 \cdot 4^{3-3}) = 1 + 16 = 17$, we get $c(4, 3, 17) = \langle 1, 0, 0\rangle$. That is, we have $c(4, 3, 1) = \langle 0, 0, 0\rangle$, $c(4, 3, 4) = \langle 0, 0, 3\rangle$, $c(4, 3, 5) = \langle 0, 1, 0\rangle$, $c(4, 3, 17) = \langle 1, 0, 0\rangle$, and $c(4, 3, 64) = \langle 3, 3, 3\rangle$, etc.

Now we define an operation to combine any two finite sequences. For any finite sequences $\langle s_1, ..., s_u\rangle$ and $\langle t_1, ..., t_v\rangle$, we define that $\langle s_1, ..., s_u\rangle \oplus \langle t_1, ..., t_v\rangle = \langle s_1, ..., s_u, t_1, ..., t_v\rangle$. Thus, for example, $c(4, 3, 1) \oplus c(4, 3, 4) = \langle 0, 0, 0\rangle \oplus \langle 0, 0, 3\rangle = \langle 0, 0, 0, 0, 0, 3\rangle$ and $c(4, 3, 5) \oplus c(4, 3, 59) = \langle 0, 1, 0\rangle \oplus \langle 3, 2, 3\rangle = \langle 0, 1, 0, 3, 2, 3\rangle$, etc.

We also define that $c(n, k, j)^m = \oplus_{i=1}^{m} c(n, k, j)$, i.e., $\langle s_1, s_2, ..., s_k\rangle^m = \oplus_{i=1}^{m} \langle s_1, s_2, ..., s_k\rangle$. For any finite sequence $\langle s_1, s_2, ..., s_k \rangle$ we define two infinite sequences as follows:

$$\begin{aligned} \langle s_1, s_2, ..., s_k \rangle^\infty &= \oplus_{i=1}^{\infty} \langle s_1, s_2, ..., s_k \rangle \\ &= \langle s_1, s_2, ..., s_k, s_1, ..., s_k, ... \rangle, \text{ and} \\ \langle s_1, s_2, ..., s_k \rangle^0 &= \langle s_1, s_2, ..., s_k \rangle \oplus \langle 0 \rangle^\infty \\ &= \langle s_1, s_2, ..., s_k, 0, 0, 0, ... \rangle . \end{aligned}$$

For a positive integer $n > 1$ let

$$\begin{aligned} \Sigma_n = &\{\langle s_1, s_2, s_3, ... \rangle \,|s_j \in \mathbf{Z}_n\} \\ &-\{\langle s_1, s_2, ..., s_k \rangle \oplus \langle n-1 \rangle^\infty \,|k \in N \text{ and } s_j \in \mathbf{Z}_n\} \end{aligned}$$

be the *sequence space* on the n symbols where $\langle n-1 \rangle^\infty = \langle 0 \rangle^\infty$. We define a distance, d, between two sequences $s = \langle s_1, s_2, s_3, ... \rangle$ and $t = \langle t_1, t_2, t_3, ... \rangle$ in Σ_n with $\theta_s = \sum_{i=1}^{\infty} \frac{s_i}{n_i} \leq \sum_{i=1}^{\infty} \frac{t_i}{n_i} = \theta_t$ by

$$d(s,t) = \begin{cases} \theta_t - \theta_s & \text{if } \theta_t - \theta_s \leq 1/2 \\ 1 - (\theta_t - \theta_s) & \text{if } \theta_t - \theta_s > 1/2 \end{cases}$$

Since $|s_i - t_i| \in \mathbf{Z}_n$ for every i, the infinite series is dominated by the geometric series $\sum_{i=1}^{\infty} \frac{n-1}{n^i} \leq 1$. Hence we have the following:

Proposition 2 *(Σ_n, d) is a metric space.*

Proof: Clearly, $d(s,t) \geq 0$ for any $s, t \in \Sigma_n$, and $d(s,t) = 0$ if and only if $s_i = t_i$ for all i. Since $|s_i - t_i| = |t_i - s_i|$, it follows that $d(s,t) = d(t,s)$. If $r, s, t \in \Sigma_n$, then $d(r,s) + d(s,t) \geq d(r,t)$, because $|r_i - s_i| + |s_i - t_i| \geq |r_i - t_i|$. □

For every n the *(left) shift map* $\sigma : (\Sigma_n, d) \to (\Sigma_n, d)$ defined by

$$\sigma(\langle s_1, s_2, s_3, ... \rangle) = \langle s_2, s_3, s_4, ... \rangle$$

is clearly onto, since there are n pre-images under σ for any $s \in \Sigma_n$. For instance, for $\langle 1, 0, 1, 1, ... \rangle \in \Sigma_2$, we get $\sigma^{-1}(\langle 1, 0, 1, 1, ... \rangle) = \{\langle 0, 1, 0, 1, 1, ... \rangle, \langle 1, 1, 0, 1, 1, ... \rangle\}$. That is, for any $\langle s_1, s_2, s_3, ... \rangle \in \Sigma_n$, we get

$$\sigma(\langle m, s_1, s_2, s_3, ... \rangle) = \langle s_1, s_2, s_3, ... \rangle$$

for any $m \in \{0, 1, 2, ..., n-1\}$.

For any element $s = \langle s_1, s_2, s_3, ... \rangle \in \Sigma_n$ and $1 \leq i \leq j$, $\langle s_i, s_{i+1}, s_{i+2}, ..., s_j \rangle$ is called the (i,j)-*cylinder* of s, and denoted by $s(i,j)$.

Proposition 3 *The shift map $\sigma : (\Sigma_n, d) \to (\Sigma_n, d)$ is continuous.*

Proof: For an arbitrary $\varepsilon > 0$ and $s = \langle s_1, s_2, s_3, ... \rangle \in \Sigma_n$, there is a positive integer k such that $\frac{n-1}{n^k} < \varepsilon$. Then, for any $t = \langle t_1, t_2, t_3, ... \rangle$ satisfies $d(s,t) < \frac{n-1}{n^{k+1}}$, we have $s(1, k+2) = t(1, k+2)$. Hence $d(\sigma(s), \sigma(t)) \leq \frac{n-1}{n^k} < \varepsilon$. That is, the shift map $\sigma : (\Sigma_n, d) \to (\Sigma_n, d)$ is continuous on (Σ_n, d). □

We need another operation, *difference*, on (Σ_n, d) by the following:
For every $s = \langle s_1, s_2, s_3, ... \rangle$ and $t = \langle t_1, t_2, t_3, ... \rangle$, we define the difference $s - t$ as

$$s - t = \langle s_1 - t_1, s_2 - t_2, s_3 - t_3, ... \rangle$$

where, if

$$s - t = \langle s_1 - t_1, s_2 - t_2, s_3 - t_3, ..., s_N - t_N \rangle \oplus \langle n-1 \rangle^\infty$$

with $s_N - t_N < n - 1$ for some positive integer N, then

$$s - t = \langle s_1 - t_1, s_2 - t_2, s_3 - t_3, ..., s_N - t_N + 1 \rangle^0 .$$

It is sufficient to show that there are densely many periodic points and a dense orbit of σ in (Σ_n, d) to prove that the shift map $\sigma : (\Sigma_n, d) \to (\Sigma_n, d)$ is a chaotic map for any integer $n > 1$ (See [1]). First, we will show that there are densely many periodic points.

Proposition 4 *There are densely many periodic points of σ in (Σ_n, d).*

Proof: For any finite sequence $c(n,k,j) = \langle s_1, s_2, , ..., s_k \rangle$ with some positive integers k and $j \leq n^k$, let

$$s = c(n,k,j)^\infty = \langle s_1, s_2, , ..., s_k, s_1, s_2, , ..., s_k, ... \rangle .$$

Clearly, $s \in \Sigma_n$, and $\sigma^k(s) = s$. That is, s is a periodic point of σ.
For any $\varepsilon > 0$ there is a positive integer K such that $\sum_{i=K+1}^{\infty} \frac{n-1}{n^i} < \epsilon$. Clearly, for any $t = \langle t_1, t_2, ..., t_l, ... \rangle \in \Sigma_n$, there is a positive integer $J(\leq n^K)$ such that the J-th n-ary sequence is the $(1, K)$-cylinder of t, i.e., $c(n, K, J) = \langle t_1, t_2, ..., t_K \rangle = t(1, K)$. Since we will have 0s at least for the first K terms in $t - c(n, K, J)^\infty$, i.e., $t - c(n, K, J)^\infty = \langle 0, 0, ..., 0, t_{K+1} - t_k, ... \rangle$ for some $1 \leq k \leq K$, we have that

$$d\left(t - c(n, K, J)^\infty\right) \leq \sum_{i=K+1}^{\infty} \frac{|t_i - t_k|}{n^i} \leq \sum_{i=K+1}^{\infty} \frac{n-1}{n^i} < \varepsilon$$

with some $1 \leq k \leq K$. Therefore there are densely many periodic points in Σ_n. □

For instance, for $s = C(4, 3, 18)^\infty = \langle 1, 0, 1, 1, 0, 1, ... \rangle$, we get $\sigma^3(s) = \langle 1, 0, 1, 1, 0, 1, ... \rangle = s$. That is, s is a periodic point of period 3.

Now, we need to construct a point which will have a dense orbit. For any positive integer p let

$$D_p = \langle d_1, d_2, d_3, ... \rangle = \oplus_{k \geq 1} \left(\oplus_{j=1}^{n^k} c(n,k,j)^p\right) \in \Sigma_n.$$

For instance, if $n = 2$ and $p = 1$, then

$$\begin{aligned} D_1 &= \oplus_{k\geq 1}\left(\oplus_{j=1}^{2^k} c(2,k,j)^1\right) \\ &= \left(\oplus_{j=1}^{2^1} c(2,1,j)\right) \\ &\quad \oplus \left(\oplus_{j=1}^{2^2} c(2,2,j)\right) \\ &\quad \oplus \left(\oplus_{j=1}^{2^3} c(2,3,j)\right) \oplus \cdots \\ &= (c(2,1,1) \oplus c(2,1,2^1)) \\ &\quad \oplus (c(2,2,1) \oplus c(2,2,2) \oplus c(2,2,3) \oplus c(2,2,2^2)) \\ &\quad \oplus (c(2,3,1) \cdots c(2,3,2^3)) \oplus \cdots \\ &= \langle 0,1\rangle \\ &\quad \oplus \langle 0,0,0,1,1,0,1,1\rangle \\ &\quad \oplus \langle 0,0,0,\cdots,1,1,1\rangle \oplus \cdots \\ &= \langle 0,1,0,0,0,1,1,0,1,1,0,0,0,\cdots,1,1,1,\cdots\rangle \\ &\in \Sigma_2. \end{aligned}$$

Proposition 5 *For any positive integer* p,

$$D_p = \langle d_1, d_2, d_3, ...\rangle = \oplus_{k\geq 1}\left(\oplus_{j=1}^{n^k} c(n,k,j)^p\right)$$

has a dense orbit under σ *on* (Σ_n, d).

Proof: For any $\varepsilon > 0$ there is a positive integer K such that $\sum_{i=K+1}^{\infty} \frac{n-1}{n^i} < \varepsilon$. For any $t = \langle t_1, t_2, t_3, ...\rangle \in \Sigma_n$ and each positive integer p there is a positive integer M_p such that $(M_p, M_p + K - 1)$-cylinder of D_p is same as $c(n, K, J)$ for some J which is the $(1, K)$-cylinder of t, that is,

$$\langle d_{M_p}, d_{M_p+1}, d_{M_p+2}, ..., d_{M_p+K-1}\rangle = \langle t_1, t_2, t_3, ..., t_K\rangle .$$

Then $d\left(\sigma^{M_p-1}(D_p), t\right) < \sum_{i=K+1}^{\infty} \frac{n-1}{n^i} < \varepsilon$. So, for every positive integer p, we have that $Orb_{\sigma}^{+}(D_p)$ is dense in Σ_n. □

From the construction of D_p we can get a surprising result as the following.

Proposition 6 *There are infinitely many dense orbits of* σ *in* Σ_n.

Proof: It is clear that, if $p \neq q$, then we have $\sigma^k(D_p) \neq D_q$ for any integer $k \geq 0$. Therefore there are infinitely many dense orbits of σ in Σ_n. □

For example, let $n = 2, p = 1$ and $q = 2$. Then we have

$$\begin{aligned} D_1 &= \oplus_{k\geq 1}\left(\oplus_{j=1}^{2^k} c(2,k,j)^1\right) \\ &= \langle 0,1,0,0,0,1,1,0,1,1,0,0,0,\cdots,1,1,0,\cdots\rangle \end{aligned}$$

$$\begin{aligned} D_2 &= \oplus_{k\geq 1}\left(\oplus_{j=1}^{2^k} c(2,k,j)^2\right) \\ &= (c(2,1,1)^2 \oplus c(2,1,2^1)^2) \\ &\quad \oplus (c(2,2,1)^2 \oplus c(2,2,2)^2 \oplus c(2,2,3)^2 \oplus c(2,2,2^2)^2) \\ &\quad \oplus (c(2,3,1)^2 \cdots c(2,3,2^3)^2) \oplus \cdots \\ &= \langle 0,0,1,1,0,0,0,0,0,1,0,1,\cdots,1,1,0,\cdots\rangle \end{aligned}$$

For $\epsilon = 0.2$, there is $k = 3$ such that

$$\sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^4} + \frac{1}{2^5} + \cdots < \frac{1}{2^3} < 0.2 = \epsilon.$$

For an arbitrary sequence $t \in \Sigma_2$, say $t = \langle 1,1,0,0,1,\cdots\rangle$, the $(1,3)$-cylinder of t, $\langle 1,1,0\rangle$, is the $(29,31)$-cylinder of D_1 and the $(57,59)$-cylinder of D_2. We get

$$\begin{aligned} &d(\sigma^{28}(D_1), t) \\ &\quad = d\left(\langle \mathbf{1},\mathbf{1},\mathbf{0},1,1,1\cdots\rangle, \langle \mathbf{1},\mathbf{1},\mathbf{0},0,1,\cdots\rangle\right) \\ &\quad < \Sigma_{i=4}^{\infty}\tfrac{1}{2^i} = \tfrac{1}{8} < 0.2 = \epsilon. \end{aligned}$$

$$\begin{aligned} &d(\sigma^{56}(D_2), t) \\ &\quad = d\left(\langle \mathbf{1},\mathbf{1},\mathbf{0},1,1,0\cdots\rangle, \langle \mathbf{1},\mathbf{1},\mathbf{0},0,1,\cdots\rangle\right) \\ &\quad < \Sigma_{i=4}^{\infty}\tfrac{1}{2^i} = \tfrac{1}{8} < 0.2 = \epsilon. \end{aligned}$$

Hence D_1 and D_2 have dense orbits in σ_2, and it is true for any positive integer p.

Since there are densely many periodic points (Proposition 4) and a (actually infinitely many) dense orbit(s) of σ on (Σ_n, d) for every integer $n > 1$ (Propositions 5 and 6), the (left) shift map $\sigma : (\Sigma_n, d) \to (\Sigma_n, d)$ is a chaotic map for any n.

III. CHAOTIC MAPS ON S^1

For any integer $n > 1$ we define the *angle multiplying map*, $\psi_n : S^1 \to S^1$, by $\psi_n(z) = z^n$. Clearly, an angle multiplying map ψ_n is an onto map on S^1 with $\psi_n(e^{2\pi i\theta}) = e^{2\pi in\theta}$ for $0 \leq \theta < 1$.

For a positive integer k, let $e^{2\pi i\theta} \in S^1$ be a periodic point of ψ_n with period k, i.e., $e^{2\pi i\theta} = \psi_n^k(e^{2\pi i\theta}) = e^{2\pi in^k\theta}$. Then θ should satisfy the equation $\theta \equiv n^k\theta$ (mod 1). That is, $(n^k - 1)\theta \equiv 0$ (mod 1). Since $0 \leq \theta < 1$, for any

$$\theta \in \left\{\frac{i}{n^k - 1} | i = 0, 1, 2, ..., n^k - 2\right\},$$

we get $\psi_n^k(e^{2\pi i\theta}) = e^{2\pi i\theta}$. That is, θ is a periodic point with a period $\leq k$. Hence, for any $\varepsilon > 0$ and any point $z = e^{2\pi i\tau} \in S^1$, there is a sufficiently large positive integer K s.t. $|\tau - \frac{i}{n^K-1}| < \varepsilon$ for some $i \in \{0, 1, 2, ..., n^K - 2\}$. Thus, for any positive integer k and $i \in \{0, 1, 2, ..., n^k - 2\}$, the set of periodic points, including fixed points, of ψ_n,

$$\left\{e^{2\pi i\theta} \in S^1 | \theta = \frac{i}{n^k - 1}\right\}$$

is dense in S^1. That is, there are densely periodic points of an angle multiplying map ψ_n in S^1 for any integer $n > 1$.

Example 7 *For* $n = 2$ *and* $k = 2$, $\frac{1}{3}$ *and* $\frac{2}{3}$ *are the images of each other under the angle doubling map* ψ_2. *That is,* $\{\frac{1}{3}, \frac{2}{3}\}$ *is a periodic orbit. If* $k = 3$, *then there are two periodic orbits such as* $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ *and* $\{\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\}$. *Note that* 0 *is a fixed point for both.*
On the other hand, if $n = 3$ *and* $k = 2$, *then*

$$\theta = \frac{i}{n^k - 1} = \frac{i}{3^2 - 1} = \frac{i}{8}.$$

So, we have three orbits, $\{\frac{1}{8}, \frac{3}{8}\}, \{\frac{2}{8}, \frac{6}{8}\}, \{\frac{5}{8}, \frac{7}{8}\}$ *of period 2, and two fixed points,* 0 *and* $\frac{1}{2}$.

We are ready to define a map between S^1 and (Σ_n, d). For every integer $n > 1$ each point $\theta \in [0,1) = S^1$ can be denoted by the n-ary expansion such as $\theta = \sum_{i=1}^{\infty} \frac{t_i}{n^i}$ with $t_i \in \{0, 1, 2, ..., n-1\}$.

Now we define a map $\varphi_n : S^1 \to \Sigma_n$ by

$$\varphi_n(z) = \varphi_n\left(e^{2\pi i\theta}\right) = \langle t_1, t_2, t_3, ...\rangle \in \Sigma_n$$

where $z = e^{2\pi i\theta} \in S^1$ with $\theta = \sum_{i=1}^{\infty} \frac{t_i}{n^i}$.

It is trivial that $\varphi_n : S^1 \to \Sigma_n$ is 1-1. Since, for any $\langle t_1, t_2, t_3, ...\rangle \in \Sigma_n$ we get $\sum_{i=1}^{\infty} \frac{t_i}{n^i} \in [0,1) = S^1$, the map $\varphi_n : S^1 \to \Sigma_n$ is onto.

We need a distance, d', on S^1 for a fixed n. For any $z = e^{2\pi i\theta}$ and $w = e^{2\pi i\tau}$ with $\theta = \sum_{i=1}^{\infty} \frac{s_i}{n^i}$ and $\tau = \sum_{i=1}^{\infty} \frac{t_i}{n^i}$, let $s = \langle s_1, s_2, \cdots\rangle$ and $t = \langle t_1, t_2, \cdots\rangle$ in Σ_n. We define the distance d' by $d'(z, w) = d(s, t)$. Than we have the following.

Proposition 8 *For any integer $n > 1$, $\varphi_n : S^1 \to \Sigma_n$ and $\varphi_n^{-1} : \Sigma_n \to S^1$ are continuous.*

Proof: For any integer $k > 0$ and $j = 0, 1, 2, ..., n^k - 1$, denote $A_{kj} = \left[\frac{j}{n^k}, \frac{j+1}{n^k}\right)$. Then $l(A_{kj}) = \frac{1}{n^k}$, and for any $x = \sum_{i=1}^{\infty} \frac{a_i}{n^i} \in A_{kj}$, $a_1, a_2, ..., a_k$ are fixed. Hence for any $s = \langle s_1, s_2, ...\rangle$ and $t = \langle t_1, t_2, ...\rangle \in \Sigma_n$ with $s_i = t_i$ for $i = 1, 2, ..., k$, we have $d(s, t) < \frac{1}{n^k}$. Then there exist $j \in \{0, 1, 2, 3, ..., n^K - 1\}$ s.t. $\varphi_n^{-1}(s)$, $\varphi_n^{-1}(t) \in A_{kj}$. Therefore $d(\varphi_n^{-1}(s), \varphi_n^{-1}(t)) < l(A_{kj}) = \frac{1}{n^k}$, and $\varphi_n^{-1} : \Sigma_n \to S^1$ is continuous.
By the similar way, $\varphi_n : S^1 \to \Sigma_n$ is continuous. □

Hence it is clear that the diagram Figure 1 is commutative, i.e.: $\sigma \circ \varphi_n = \varphi_n \circ \psi_n$.

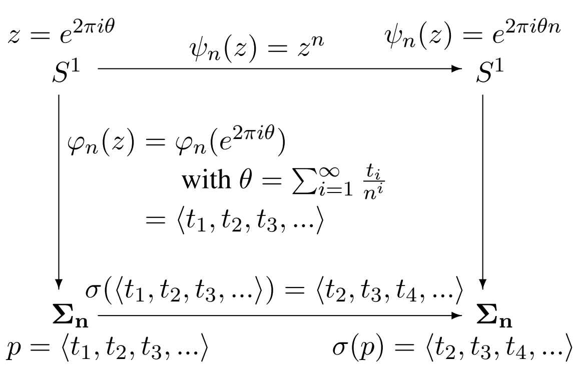

Fig. 1 Commutative Diagram: $\sigma \circ \varphi_n = \varphi_n \circ \psi_n$

From Propositions 3 and 8 the map

$$\psi_n = \varphi_n^{-1} \circ \sigma \circ \varphi_n : S^1 \to S^1$$

is continuous. By Propositions 4 and 6 it is clear that there are densely many periodic points and infinitely many dense orbits of the map ψ_n in S^1 for every positive integer $n > 1$. That is, the preimages of dense orbits and periodic points of σ on (Σ_n, d) under φ_n become dense orbits and periodic points of ψ_n on S^1, respectively. Therefore ψ_n is a chaotic map on S^1 for any integer $n > 1$.

IV. Conclusion

Let M be an arbitrary positive integer, and $n_i (> 1)$ a positive integer for each $1 \le i \le M$. Define a map $\Psi = \Pi_{i=1}^{M} \psi_{n_i}$ on $\Pi_{i=1}^{M} S^1$ with

$$\psi_{n_i} = \varphi_{n_i}^{-1} \circ \sigma \circ \varphi_{n_i} : S^1 \to S^1.$$

To show the map Ψ is a chaotic map on $\Pi_{i=1}^{M} S^1$ we need the following two lemmas.

Lemma 9 *There exist densely many periodic points of $\Psi = \Pi_{i=1}^{M} \psi_{n_i}$ on $\Pi_{i=1}^{M} S^1$.*

Proof: Let $\varepsilon > 0$ and $\Pi_{j=1}^{M} z_j = \Pi_{j=1}^{M} e^{2\pi i\theta_j}$ an arbitrary point in $\Pi_{j=1}^{M} S^1$ with $\theta_j = \sum_{i=1}^{\infty} \frac{t_{(j,i)}}{n_j^i}$ where $t_{(j,i)} \in \{0, 1, 2, ..., n_j - 1\}$. Then, for $j = 1, 2, ..., M$, there are K_j such that

$$d\left(\varphi_{n_j}(z_j), \left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^{\infty}\right)^2 < \varepsilon^2/M.$$

Hence we get

$$\sqrt{\Sigma_{j=1}^{M} d\left(\varphi_{n_j}(z_j), \left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^{\infty}\right)^2} < \varepsilon.$$

Then $\Pi_{j=1}^{M} \varphi_{n_j}^{-1} \left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^{\infty}$ is a periodic point of Ψ with the period of $lcm(K_1, K_2, ..., K_M)$ and

$$d'\left(\Pi_{j=1}^{M} z_j, \Pi_{j=1}^{M} \varphi_{n_j}^{-1} \left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^{\infty}\right) < \varepsilon. \quad \square$$

For instance, let $M = 2, \varepsilon = 0.01, n_1 = 3$ and $n_2 = 2$. From $\varepsilon^2/M = (0.01)^2/2 = 0.00005$, we pick $K_1 = 5$ and $K_2 = 8$ since

$$1/3^5 \approx 0.000017 < 0.00005 < 0.000152 \approx 1/3^4 \text{ and}$$
$$1/2^8 \approx 0.000015 < 0.00005 < 0.000061 \approx 1/2^7.$$

For an arbitrary point in $S^1 \times S^1$, say (z_1, z_2), from $z_1 = e^{2\pi i\theta_1}$ where $\theta_1 = 0.7 = \sum_{i=1}^{\infty} \frac{t_{(1,i)}}{3^i}$ with $t_{(1,i)} \in \{0, 1, 2\}$ and $z_2 = e^{2\pi i\theta_2}$ where $\theta_2 = 0.12 = \sum_{i=1}^{\infty} \frac{t_{(2,i)}}{2^i}$ where $t_{(2,i)} \in \{0, 1\}$, we have

$$\begin{aligned} \theta_1 &= \textstyle\sum_{i=1}^{\infty} \frac{t_{(1,i)}}{3^i} \\ &= \frac{2}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{0}{3^6} + \cdots, \quad \text{and} \\ \theta_2 &= \textstyle\sum_{i=1}^{\infty} \frac{t_{(2,i)}}{2^i} \\ &= \frac{0}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \frac{1}{2^9} + \cdots. \end{aligned}$$

Hence we get

$$\begin{aligned} \varphi_3(z_1) &= \langle 2, 0, 0, 2, 2, 0, \cdots\rangle, \\ \varphi_2(z_2) &= \langle 0, 0, 0, 1, 1, 1, 1, 0, 1, \cdots\rangle, \end{aligned}$$

and

$$d\left(\varphi_3(z_1), \left\langle t_{(1,1)}, t_{(1,2)}, ..., t_{(1,5)}\right\rangle^\infty\right)^2$$
$$= d\left(\langle 2,0,0,2,2,0,0,2,\cdots\rangle, \langle 2,0,0,2,2\rangle^\infty\right)^2$$
$$= d\left(\langle 2,0,0,2,2,0,0,2,\cdots\rangle, \langle 2,0,0,2,2,2,0,0\cdots\rangle\right)^2$$
$$\leq \left(\sum_{i=6}^\infty \frac{2}{3^i}\right)^2 \leq \left(\frac{1}{3^5}\right)^2 \approx 0.000017 < 0.00005,$$

$$d\left(\varphi_2(z_2), \left\langle t_{(2,1)}, t_{(2,2)}, ..., t_{(2,8)}\right\rangle^\infty\right)^2$$
$$= d\left(\langle 0,0,0,1,1,1,1,0,1,\cdots\rangle, \langle 0,0,0,1,1,1,1,0\rangle^\infty\right)^2$$
$$= d\left(\langle 0,0,0,1,1,1,1,0,1,\cdots\rangle, \langle 0,0,0,1,1,1,1,0,0,\cdots\rangle\right)^2$$
$$\leq \left(\sum_{i=9}^\infty \frac{2}{2^i}\right)^2 \leq \left(\frac{1}{2^8}\right)^2 \approx 0.000015 < 0.00005.$$

Hence we have

$$\sqrt{\Sigma_{j=1}^M d\left(\varphi_{n_j}(z_j), \left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^\infty\right)^2}$$
$$\leq \sqrt{\left(\sum_{i=6}^\infty \frac{2}{3^i}\right)^2 + \left(\sum_{i=9}^\infty \frac{2}{2^i}\right)^2}$$
$$< \sqrt{0.00005 + 0.00005} = 0.01 = \varepsilon.$$

Clearly,

$$\Pi_{j=1}^2 \varphi_{n_j}^{-1}\left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^\infty$$
$$= \left(\varphi_3^{-1}\left\langle t_{(1,1)}, t_{(1,2)}, ..., t_{(1,5)}\right\rangle^\infty, \varphi_2^{-1}\left\langle t_{(2,1)}, t_{(2,2)}, ..., t_{(2,8)}\right\rangle^\infty\right)$$

is a periodic point in $S^1 \times S^1$ with the period of $40 = lcm(K_1, K_2) = lcm(5, 8)$, and

$$d'\left((z_1, z_2), \Pi_{j=1}^2 \varphi_{n_j}^{-1}\left\langle t_{(j,1)}, t_{(j,2)}, ..., t_{(j,K_j)}\right\rangle^\infty\right) < \varepsilon.$$

Lemma 10 *For any positive integer M and $n_i > 1$ for $i = 1, 2, ..., M$, there is a point $(z_1, ..., z_M) \in \Pi_{i=1}^M S^1$ having a dense orbit of $\Psi = \Pi_{i=1}^M \psi_{n_i}$, that is, $Orb_\Psi^+((z_1, ..., z_M))$ is dense in $\Pi_{i=1}^M S^1$.*

Proof: For every positive integer k, let

$$N_{t+}^k = \Pi_{i=t+1}^M n_i^k \quad \text{and} \quad N_{t-}^k = \Pi_{i=1}^{t-1} n_i^k$$

with

$$N_{1-}^k = 1 \quad \text{and} \quad N_{M+}^k = 1.$$

For each $t = 1, 2, ..., M$ define

$$z_t = \varphi_{n_t}^{-1}\left(\oplus_{k=1}^\infty \left(\oplus_{j=1}^{n_t^k} c(n_t, k, j)^{N_{t+}^k}\right)^{N_{t-}^k}\right) \in S^1.$$

Since, for every $t = 1, 2, ..., M$, z_t contains all the n_t-ary sequences of length k with $k = 1, 2, ...$, it is clear that $Orb_{\psi_{n_t}}^+(z_t)$ is dense in S^1.

For an arbitrary point $(s_1, s_2, ..., s_M)$ in $\Pi_{i=1}^M S^1$ and any $\varepsilon > 0$, there exist least positive integers K_t and j_t such that

$$d\left(\varphi_{n_t}(s_t), c(n_t, K_t, j_t)^0\right) < \varepsilon^2/M.$$

Let $K = max\{K_1, K_2, \cdots, K_M\}$. Then there is a positive integer J_t such that $c(n_t, K_t, j_t) \oplus \langle 0\rangle^{K-K_t} = c(n_t, K, J_t)$, i.e.,

$$d\left(\varphi_{n_t}(s_t), c(n_t, K, J_t)^0\right) = d\left(\varphi_{n_t}(s_t), c(n_t, K_t, j_t)^0\right) < \varepsilon^2/M.$$

The first n_t-ary sequence of length K will appear after $\sum_{k=1}^{K-1} k\left(\prod_{i=1}^M n_i^k\right)$ terms in $\varphi_{n_t}(z_t)$ for every $t = 1, 2, ..., M$. For each $t = 1, 2, ..., M-1$, $c(n_t, K, J_t)$ will appear after $\sum_{t=1}^{M-1}\left((J_t - 1)K \prod_{i=t+1}^M n_i^K\right)$ terms later from the starting of the first n_t-ary sequence of length K in z_t. And $c(n_M, K, J_M)$ appears after J_M terms later from the starting of the first n_M-ary sequence of length K in z_M, Therefore, for

$$L = \sum_{k=1}^{K-1}\left(k \prod_{i=1}^M n_i^k\right) + \sum_{t=1}^{M-1}\left((J_t - 1)K \prod_{i=t+1}^M n_i^K\right) + J_M,$$

we have $c(n_t, K, J_t) = c(\varphi(z_t), K, J_t))$ and

$$d'\left(s_t, \psi^L(z_t)\right) = d\left(\varphi_{n_t}(s_t), c(n_t, K, J_t)^0\right) < \varepsilon^2/M$$

for each $t = 1, 2, ..., M$. Hence we have

$$\left|(s_1, s_2, ..., s_M) - \Psi^L\left((z_1, z_2, ..., z_M)\right)\right|$$
$$\leq \sqrt{\sum_{t=1}^M d'\left(s_t, \psi^L(z_t)\right)}$$
$$< \sqrt{\sum_{t=1}^M d\left(\varphi_{n_t}(s_t), c(n_t, K, J_t)^0\right)}$$
$$< \sqrt{\sum_{t=1}^M \varepsilon^2/M} = \varepsilon.$$

Therefore the point $(z_1, ..., z_M) \in \Pi_{i=1}^M S^1$ has a dense orbit of $\Psi = \Pi_{i=1}^M \psi_{n_i}$, that is, $Orb_\Psi^+((z_1, ..., z_M))$ is dense in $\Pi_{i=1}^M S^1$. □

For example, let $M = 3, n_1 = 5, n_2 = 2, n_3 = 3, \varepsilon = 0.1$ and

$$(s_1, s_2, s_3) = \left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i\theta_3}\right) \in \prod_{i=1}^M S^1$$

with $\theta_1 = 0.635, \theta_2 = 0.22$ and $\theta_3 = 0.582$. From $\varepsilon^2/M = 0.01/3 = 1/300$ and $1/5^4 = 1/625 < 1/300, 1/2^9 = 1/512 < 1/300, 1/3^6 = 1/720 < 1/300$, we get $K_1 = 4, K_2 = 9$ and $K_3 = 6$. That is,

$$K = max\{4, 9, 6\} = 9.$$

We also find that $\frac{3}{5} + \frac{4}{5^3} + \frac{1}{5^4} = 0.6336, \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^9} = 0.2207, \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^4} + \frac{1}{3^6} = 0.5816$, and $|\theta_1 - 0.6336| < 1/300, |\theta_2 - 0.2207| < 1/300, |\theta_3 - 0.5816| < 1/300$. For $\langle t_1, t_2, \cdots, t_k\rangle = c(n, k, j)$ we have $j = 1 + \sum_{i=1}^k\left(t_i \times n^{k-i}\right)$. So, from

$$\langle 3,0,4,1,0,0,0,0,0\rangle = c(5, 9, J_1),$$
$$\langle 0,0,1,1,1,0,0,0,1\rangle = c(2, 9, J_2), \text{ and}$$
$$\langle 1,2,0,2,0,1,0,0,0\rangle = c(3, 9, J_3),$$

we get

$$\begin{aligned} J_1 &= 1+\sum_{i=1}^{9}\left(t_i \times 5^{9-i}\right) \\ &= 1+3\cdot 5^8+4\cdot 5^6+1\cdot 5^5 = 1237501, \\ J_2 &= 1+\sum_{i=1}^{9}\left(t_i \times 2^{9-i}\right) \\ &= 1\cdot 2^6+1\cdot 2^5+1\cdot 2^4+1 = 114, \\ J_3 &= 1+\sum_{i=1}^{9}\left(t_i \times 3^{9-i}\right) \\ &= 1\cdot 3^8+2\cdot 3^7+2\cdot 3^5+1\cdot 3^3 = 11449. \end{aligned}$$

Hence we get

$$\begin{aligned} L &= \sum_{k=1}^{K-1} k\left(\prod_{i=1}^{M} n_i^k\right) \\ &\quad + \sum_{t=1}^{M-1}\left((J_t-1)K\prod_{i=t+1}^{M} n_i^K\right) \\ &\quad + J_M \\ &= \sum_{k=1}^{8} k\left(\prod_{i=1}^{3} n_i^k\right) \\ &\quad + \sum_{t=1}^{2}\left((J_t-1)9\prod_{i=t+1}^{3} n_i^9\right) \\ &\quad + J_3 \\ &= \left[1\prod_{i=1}^{3} n_i^1 + 2\prod_{i=1}^{3} n_i^2+\cdots+8\prod_{i=1}^{3} n_i^8\right] \\ &\quad \left[1237500\cdot 9\prod_{i=2}^{3} n_i^9 + 113\cdot 9\prod_{i=3}^{3} n_i^9\right] \\ &\quad +11449 \\ &= [1\cdot 10+2\cdot 38+\cdots+8\cdot 397442] \\ &\quad [1237500\cdot 9\cdot 20195+113\cdot 9\cdot 19683] \\ &\quad +11449 \\ &= 3861578+224941830111+11448 \\ &= 224945703137, \end{aligned}$$

and the $(224945703137, 224945703137+8)$-cylinders of z_1, z_2 and z_3 are $c(5,9,J_1), c(2,9,J_2)$ and $c(3,9,J_3)$. Therefore we have

$$\begin{aligned} &\left|(s_1,s_2,s_3)-\Psi^{224945703137}\left((z_1,z_2,z_3)\right)\right| \\ &\quad \le \sqrt{\sum_{t=1}^{3} d'\left(s_t, \psi^{224945703137}(z_t)\right)} \\ &\quad < \sqrt{\sum_{t=1}^{3} d\left(\varphi_{n_t}(s_t), c(n_t,9,J_t)^0\right)} \\ &\quad < \sqrt{\sum_{t=1}^{3} 1/300} < 0.1 = \varepsilon. \end{aligned}$$

That is, (z_1,z_2,z_3) has a dense orbit in $\Pi_{i=1}^{M} S^1$.

From Lemma 9 and 10 we have the main theorem:

Theorem 11 *For any positive integer M let $n_i > 1$ be an integer for every $1 \le i \le M$. Then $\Psi = \Pi_{i=1}^{M}\psi_{n_i}$ is a chaotic map on $\Pi_{i=1}^{M} S^1$.*

Corollary 12 and 13 follows the main theorem 11 and Lemma 9 and 10.

Corollary 12 *For an arbitrary positive integer M there are infinitely many dense orbits of $\Psi = \Pi_{i=1}^{M}\psi_{n_i}$ on $\Pi_{i=1}^{M} S^1$.*

For any $t = 1,2,...,M$ let $n_t > 1$ be a positive integer. For any positive integer u let $w^u = (w_1^u, w_2^u, \cdots, w_M^u)$ with

$$\begin{aligned} w_t^u &= \varphi_{n_t}^{-1}\left(\oplus_{k=1}^{\infty}\left(\oplus_{j=1}^{n_t^k}\left[c(n_t,k,j)\oplus\langle 0\rangle^u\right]^{N_{t+}^k}\right)^{N_{t-}^k}\right) \\ &\in S^1. \end{aligned}$$

It is clear that $Orb_\Psi^+(w^u) = Orb_\Psi^+((w_1^u,...,w_M^u))$ is dense in $\Pi_{i=1}^{M} S^1$ and $w^u = (w_1^u,...,w_M^u) \notin Orb_\Psi^+((w_1^v,...,w_M^v)) = Orb_(\Psi)^+(w^v)$ if $u \ne v$. Therefore there are infinitely many dense orbits in $\Pi_{i=1}^{M} S^1$.

Corollary 13 *For every two positive integers $n_1, n_2 > 1$, the Möbius transformation $\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$ is a chaotic map on T^2.*

For every two positive integers $n_1, n_2 > 1$, the Möbius transformation $\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$ can be defined by $\Psi = \Pi_{i=1}^{M}\psi_{n_i} = \psi_{n_i}\times\psi_{n_2}$, which is a chaotic map.

Using the revised definition of chaotic map ([1] and [3]) we show that the (left) shift map on the sequence set (Σ, d) is a chaotic map by finding densely many periodic points and a dense orbit of the shift map on (Σ, d). After making a 1-1, onto and continuous map between S^1 and (Σ, d), we show that an angle multiplying map on S^1 is a composition of the shift map on (Σ, d) and the map between S^1 and (Σ, d), which become a chaotic map on S^1. We prove that any combination of angle multiplying maps on $\Pi_{i=1}^{M} S^1$ become a chaotic map by finding densely many periodic points and a dense orbit of it. Through the construction of such chaotic map on $\Pi_{i=1}^{M} S^1$ in the main theorem (11) and lemmas (9 and 10), we have found that there are infinitely many dense orbits and the Möbius transformation is a chaotic map on T^2.

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