A note on Generalized Bessel Functions

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***Abstract*—** In this paper we will discuss the treat involving various forms of generalized Bessel functions of two-variable, in particular by outlining the linking between the second order differential equation of Bessel type and the different kinds of Bessel functions themselves. Furthermore, by using the formalism of the shift operators, we will present the Bessel operator and we will show how it can be useful to simplify the study of many properties related to Bessel functions.

***Keywords*—** Bessel Functions, Bessel Operator, Orthogonal Polynomials, Hermite Polynomials, Generating Functions.

I. INTRODUCTION

In many problems of pure and applied mathematics the use of generalized functions of special nature offer the possibility to express relevant relations in a particular concise form, which sometimes provides a deeper understanding of the problem itself. This is indeed what has happened in the theory of synchrotron radiation in which the use of multi-variable Bessel functions has provided a unique tool to derive the spectral details of the radiation emitted by relativistic electrons in magnetic structure in a analytical form not achievable with conventional means. Other problems of this type have benefited from the use of new families of special functions which sporadically appeared in the mathematical literature and then, for some reasons, totally forgotten. The number of problems demanding for an extension of the family of ordinary special functions is now growing.

Systematic investigation to frame the new families within a coherent framework are now in progress.

The theory of ordinary Bessel [1,2] functions is sometimes formulated starting from the generating function method. Accordingly, we introduce the two-variable one-parameter cylinder generalized Bessel function (GBF), using the following generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\sum_{n=-\infty}^{+\infty}t^n J_n(x,y;\tau)\,, \tag{1}$$

Where $x, y \in \mathbb{R}$ and $t,\tau \in \mathbb{R}$, such that $0<|t|\neq|\tau|<+\infty$. It is immediately recognized that for $y=0$, the function in the previous relation, reduces to the well-known generating function of the one-variable cylinder Bessel function $J_n(x)$ and we can also immediately nothing that, the generalized two-variable Bessel function $J_n(x,y)$, can be viewed as particular case of the GBF of the parameter $\tau=1$. It is easily checked that the function $J_n(x,y;\tau)$ can be exploited by means of the converging series:

$$J_n(x,y;\tau)=\sum_{l=-\infty}^{+\infty}\tau^l J_{n-2l}(x)J_l(y) \tag{2}$$

after nothing that:

$$\begin{aligned}&\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\\&=\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]\exp\left[\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right],\end{aligned} \tag{3}$$

where:

$$\begin{aligned}&\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{m=-\infty}^{+\infty}t^m J_m(x)\\&\exp\left[\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\sum_{l=-\infty}^{+\infty}t^{2l}\tau^l J_l(y)\end{aligned} \tag{4}$$

By setting $t=e^{i\theta}$ and $\tau=e^{i\phi}$, where $\theta,\phi\in(0,2\pi)$, in the relation (1), we can immediately obtain the Jacobi-Anger expansion in the form:

$$\exp\left[i\left(x\sin\theta+y\sin\left(2\theta+\phi\right)\right)\right]=\sum_{n=-\infty}^{+\infty}e^{in\theta}J_n(x,y;e^{i\phi}) \tag{5}$$

The symmetry properties of the GBF $J_n(x,y;\tau)$ can be inferred from its explicit form, stated in the equation (2); using the similar relation for the ordinary Bessel function [3], we get indeed:

$$J_{-n}(x,y;\tau)=(-1)^n J_n\left(x,y;-\frac{1}{\tau}\right)=J_n\left(-x,-y;\frac{1}{\tau}\right) \tag{6}$$

$$J_n(x,y;-\tau)=J_n\left(x,-y;\tau\right) \tag{7}$$

$$J_n(-x,-y;-\tau)=(-1)^n J_n\left(x,y;\tau\right) \tag{8}$$

We can state the recurrence relations for the GBF $J_n(x,y;\tau)$, by using the generating function expression (1),

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taking the derivative of both sides with respect to *x*, *y*, *t* and τ. We have:

$$\frac{\partial}{\partial x} J_n(x,y;\tau) = \frac{1}{2}\left[J_{n-1}(x,y;\tau) - J_{n+2}(x,y;\tau)\right], \tag{8}$$

$$\frac{\partial}{\partial y} J_n(x,y;\tau) = \frac{1}{2}\left[\tau J_{n-2}(x,y;\tau) - \frac{1}{\tau} J_{n+2}(x,y;\tau)\right], \tag{9}$$

$$\begin{aligned} 2nJ_n(x,y;\tau) &= x\left[J_{n-1}(x,y;\tau) + J_{n+1}(x,y;\tau)\right] + \\ &+2y\left[\tau J_{n-2}(x,y;\tau) + \frac{1}{\tau} J_{n+2}(x,y;\tau)\right], \end{aligned} \tag{10}$$

$$\frac{\partial}{\partial \tau} J_n(x,y;\tau) = \frac{y}{2}\left[J_{n-2}(x,y;\tau) + \frac{1}{\tau^2} J_{n+2}(x,y;\tau)\right], \tag{11}$$

Unlike the ordinary case, the recurrences of $J_n(x,y;\tau)$ do not link the nearest neighbor index only. This is indeed due to the form of the generating function which involves the

$$t^{\pm 1} \text{ and } t^{\pm 2}$$

terms.

The second order differential equation satisfied by the ordinary Bessel function $J_n(x)$ can be derived from the corresponding recurrences relation using a kind of operational techniques involving shift operators. We use a generalization of this method to derive the partial differential equations satisfied by the GBF $J_n(x,y;\tau)$.

By noting, in fact, that the last two recurrences stated above (eq. (10), (11)) for the $J_n(x,y;\tau)$, can be combined to obtain:

$$\begin{aligned} 2nJ_n(x,y;\tau) &= x\left[J_{n-1}(x,y;\tau) + J_{n+1}(x,y;\tau)\right] + \\ &+4\tau \frac{\partial}{\partial \tau} J_n(x,y;\tau), \end{aligned} \tag{12}$$

which, can be rewritten as:

$$\begin{aligned} &\left(n - 2\tau\frac{\partial}{\partial \tau}\right) J_n(x,y;\tau) = \\ &= \frac{x}{2}\left[J_{n-1}(x,y;\tau) + J_{n+1}(x,y;\tau)\right]. \end{aligned} \tag{13}$$

From the recurrence stated in (8), we can write:

$$\frac{1}{2} J_{n-1}(x,y;\tau) = \frac{\partial}{\partial x} J_n(x,y;\tau) + \frac{1}{2} J_{n+1}(x,y;\tau), \tag{14}$$

$$\frac{1}{2} J_{n+1}(x,y;\tau) = \frac{1}{2} J_{n-1}(x,y;\tau) - \frac{\partial}{\partial x} J_n(x,y;\tau). \tag{15}$$

The above relations can be combined in the previous recurrences to obtain the following relations:

$$J_{n-1}(x,y;\tau) = \left[\left(\frac{n}{x} + \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}\right] J_n(x,y;\tau), \tag{16}$$

$$J_{n+1}(x,y;\tau) = \left[\left(\frac{n}{x} - \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}\right] J_n(x,y;\tau). \tag{17}$$

The operators in the square brackets can be viewed as shift operators, in the sense that they act on the GBF's shifting the index by one unit. We write therefore:

$$\hat{E}_{-} = \left(\frac{n}{x} + \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}, \tag{18}$$

$$\hat{E}_{+} = \left(\frac{n}{x} - \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}. \tag{19}$$

By noting that:

$$\hat{E}_{+}\left[\hat{E}_{-}\left(J_n(x,y;\tau)\right)\right] = J_n(x,y;\tau), \tag{20}$$

it can immediately proved that $J_n(x,y;\tau)$ satisfies the partial differential equation:

$$\begin{aligned} &\left[\left(\frac{n-1}{x} - \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}\right]\left[\left(\frac{n}{x} + \frac{\partial}{\partial x}\right) - \frac{2\tau}{x}\frac{\partial}{\partial \tau}\right] J_n(x,y;\tau) = \\ &= J_n(x,y;\tau), \end{aligned} \tag{21}$$

and in more explicit form:

$$\left\{x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + \left(x^2 - n^2\right) - 4\tau\left[(1-n) + \tau\frac{\partial}{\partial \tau}\right]\frac{\partial}{\partial \tau}\right\} J_n(x,y;\tau) = 0 \tag{22}$$

II. Bessel Operator

In the previous section we have seen just an example of partial differential equation admitting the function $J_n(x,y;\tau)$ as solution. The further use of the shift operator technique allows the derivation of other partial differential equations satisfied by the Generalized Bessel Function [4]. Using the procedure outlined before, we can combine the recurrence relations stated in equations (8, 9, 10, 11) to introduce different operators. In fact, by manipulating the recurrence:

$$\frac{\partial}{\partial y} J_n(x,y;\tau) = \frac{1}{2}\left[\tau J_{n-2}(x,y;\tau) - \frac{1}{\tau} J_{n+2}(x,y;\tau)\right], \tag{23}$$

we immediately have:

$$J_{n-2}(x,y;\tau)=\frac{2}{\tau}\left[\frac{\partial}{\partial y}J_n(x,y;\tau)+\frac{1}{2\tau}J_{n+2}(x,y;\tau)\right], \qquad (24)$$

$$J_{n+2}(x,y;\tau)=2\tau\left[\frac{\tau}{2}J_{n-2}(x,y;\tau)-\frac{\partial}{\partial y}J_n(x,y;\tau)\right]. \qquad (25)$$

After noting that the last relation contained in the equations (11) can be written as:

$$\frac{1}{y}\frac{\partial}{\partial\tau}J_n(x,y;\tau)=\frac{1}{2}J_{n-2}(x,y;\tau)+\frac{1}{2\tau^2}J_{n+2}(x,y;\tau), \qquad (26)$$

we can finally obtain the following recurrence relations for the generalized Bessel function $J_n(x,y;\tau)$:

$$J_{n-2}(x,y;\tau)=\left(\frac{1}{y}\frac{\partial}{\partial\tau}+\frac{1}{\tau}\frac{\partial}{\partial y}\right)J_n(x,y;\tau), \qquad (27)$$

$$J_{n+2}(x,y;\tau)=\left(\frac{\tau^2}{y}\frac{\partial}{\partial\tau}-\tau\frac{\partial}{\partial y}\right)J_n(x,y;\tau), \qquad (28)$$

By following the same procedure used to introduce the shift operators of the previous section, we can use the above relations to introduce the operators:

$$\hat{E}_-^{2}=\frac{1}{y}\frac{\partial}{\partial\tau}+\frac{1}{\tau}\frac{\partial}{\partial y}, \qquad (29)$$

$$\hat{E}_+^{2}=\frac{\tau^2}{y}\frac{\partial}{\partial\tau}-\tau\frac{\partial}{\partial y}, \qquad (30)$$

which allow us to write the following equation:

$$\hat{E}_+^{2}\left[\hat{E}_-^{2}\left(J_n(x,y;\tau)\right)\right]=J_n(x,y;\tau), \qquad (31)$$

that in explicit form, provides the partial differential equation:

$$\left[\left(y^2\frac{\partial^2}{\partial y^2}+y\frac{\partial}{\partial y}+y^2\right)-\tau\left(1+\tau\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau}\right]J_n(x,y;\tau)=0. \qquad (32)$$

We can now combine the above equation, obtained by using the second type shift operators, and the similar partial differential equation, stated with the help of the shift operators defined in the previous section. We get:

$$\left[\hat{B}_n(x)-4\hat{B}_0(y)\right]J_n(x,y;\tau)=4n\tau\frac{\partial}{\partial\tau}J_n(x,y;\tau), \qquad (33)$$

where, with $\hat{B}_m(\alpha)$ we denote the Bessel Operator defined as:

$$\hat{B}_m(\alpha)=\alpha^2\frac{\partial^2}{\partial\alpha^2}+\alpha\frac{\partial}{\partial\alpha}+\left(\alpha^2-m^2\right), \qquad (34)$$

where $m=0,n$.

By deriving the first of recurrence relations with respect to x and then combining the result with recurrence involving the y-derivative, we can introduce the operators:

$$\hat{E}_+^{2'}=\frac{2\tau}{(\tau^2+1)}\left[\tau\left(2\frac{\partial^2}{\partial x^2}+1\right)-\frac{\partial}{\partial y}\right], \qquad (35)$$

$$\hat{E}_-^{2'}=\frac{2\tau}{(\tau^2+1)}\left[\frac{1}{\tau}\left(2\frac{\partial^2}{\partial x^2}+1\right)+\frac{\partial}{\partial y}\right], \qquad (36)$$

and by following the same procedure of the previous calculation, we can immediately state:

$$\hat{E}_+^{2'}\left[\hat{E}_-^{2'}\left(J_n(x,y;\tau)\right)\right]=J_n(x,y;\tau). \qquad (37)$$

Once we exploit the operators involving in the above equation, we obtain the third partial differential equation satisfied by the Bessel function of the type $J_n(x,y;\tau)$; that is:

$$\left[\left(2\frac{\partial^2}{\partial x^2}+1\right)^2+\left(\tau-\frac{1}{\tau}\right)\left(2\frac{\partial^2}{\partial x^2}+1\right)\frac{\partial}{\partial y}-\frac{\partial^2}{\partial y^2}\right]J_n(x,y;\tau)= \\ =\frac{\left(\tau^2+1\right)^2}{4\tau^2}J_n(x,y;\tau). \qquad (38)$$

Two sub-cases of the above equation are particularly interesting: $\tau=1$ and $\tau=i$.

When $\tau=1$, the above equation became:

$$\left[\left(2\frac{\partial^2}{\partial x^2}+1\right)^2-\frac{\partial^2}{\partial y^2}\right]J_n(x,y;1)=J_n(x,y;1). \qquad (39)$$

Since $J_n(x,y;1)$ is the generalized Bessel function of two-variable, that is $J_n(x,y)$, it is evident to observe that it is the related differential equation associated to its.

For $\tau=i$, we obtain:

$$-\left(2\frac{\partial^2}{\partial x^2}+1\right)J_n(x,y;i)=i\frac{\partial}{\partial y}J_n(x,y;i), \qquad (40)$$

that is a Schrodinger-type equation [5].

III. OPERATIONAL RESULTS

By using the relation stated in equation (1), that is the generating function of the two-variable Bessel function, we can immediately write:

$$\exp\left[\frac{\lambda x}{2}\left(t-\frac{1}{t}\right)+\frac{\mu y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\sum_{n=-\infty}^{+\infty} t^n J_n(\lambda x,\mu y;\tau) \quad (41)$$

where $\lambda,\mu\in\mathbb{R}$. By denoting with A the argument of the exponential, we have:

$$A=\frac{x}{2}\left(\lambda t-\frac{\lambda}{t}\right)+\frac{y}{2}\left(\mu t^2\tau-\frac{\mu}{t^2\tau}\right),$$

$$A=\frac{x}{2}\left(\lambda t-\frac{1}{\lambda t}\right)+\frac{x}{2}\left(\frac{1-\lambda^2}{\lambda t}\right)+\frac{y}{2}\left(\mu t^2\tau-\frac{1}{\mu t^2\tau}\right)+\frac{y}{2}\left(\frac{1-\mu^2}{\mu t^2\tau}\right)$$

By substituting in the equation (41), we can write:

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty} t^n J_n(\lambda x,\mu y;\tau)=\exp\left[\frac{x}{2}\left(\lambda t-\frac{1}{\lambda t}\right)\right]\\ &\exp\left[\frac{y}{2}\left(\mu t^2\tau-\frac{1}{\mu t^2\tau}\right)\right]\exp\left[\frac{x}{2}\left(\frac{1-\lambda^2}{\lambda t}\right)\right]\exp\left[\frac{y}{2}\left(\frac{1-\mu^2}{\mu t^2\tau}\right)\right].\end{aligned} \quad (42)$$

Exploited the exponentials and considering the expressions of the ordinary Bessel functions, we get:

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty} t^n J_n(\lambda x,\mu y;\tau)=\sum_{r=-\infty}^{+\infty}(\lambda t)^r J_r(x)\sum_{s=-\infty}^{+\infty}\mu^s t^{2s}\tau^s J_s(y)\\ &\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\frac{1-\lambda^2}{\lambda t}\right)^k\left(\frac{x}{2}\right)^k\sum_{v=0}^{+\infty}\frac{1}{v!}\left(\frac{1-\mu^2}{\mu t^2\tau}\right)^v\left(\frac{y}{2}\right)^v.\end{aligned} \quad (43)$$

By manipulating the r.h.s of the above relation, setting $m=r+2s$ and $p=k+2v$, we obtain:

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty} t^n J_n(\lambda x,\mu y;\tau)=\sum_{m=-\infty}^{+\infty}(\lambda t)^m\sum_{s=-\infty}^{+\infty}\left(\frac{\mu\tau}{\lambda^2}\right)^s J_{m-2s}(x)J_s(y)\\ &\sum_{p=0}^{+\infty}\left(\frac{1}{\lambda t}\right)^p\sum_{v=0}^{\left[p/2\right]}\frac{\left(1-\lambda^2\right)^{p-2v}\left(1-\mu^2\right)^2}{(p-2v)!v!}\left(\frac{\lambda^2}{\mu\tau}\right)^v\left(\frac{x}{2}\right)^{p-2v}\left(\frac{y}{2}\right)^v.\end{aligned} \quad (44)$$

By noting that, the multiplication theorem related to generalized two-variable Bessel function, reads as:

$$J_n(\lambda x,\mu y)=\lambda^n\sum_{p=0}^{+\infty}J_{n+p}\left(x,y;\frac{\mu}{\lambda^2}\right)F_p\left(x,y;\frac{\lambda^2}{\mu}\right), \quad (45)$$

where:

$$\begin{aligned}&F_p\left(x,y;\frac{\lambda^2}{\mu}\right)=\\ &=\sum_{v=0}^{\left[p/2\right]}\frac{\left(1-\lambda^2\right)^{p-2v}\left(1-\mu^2\right)^2}{(p-2v)!v!}\left(\frac{\lambda^2}{\mu}\right)^v\left(\frac{x}{2}\right)^{p-2v}\left(\frac{y}{2}\right)^v,\end{aligned} \quad (46)$$

we can finally state the important multiplication result for the generalized two-variable, one-parameter Bessel function:

$$J_n(\lambda x,\mu y;\tau)=\lambda^n\sum_{p=0}^{+\infty}J_{n+p}\left(x,y;\frac{\mu\tau}{\lambda^2}\right)F_p\left(x,y;\frac{\lambda^2}{\mu\tau}\right). \quad (47)$$

IV. CONCLUDING REMARKS

The two-variable extension of Hermite polynomials [6,7], defined by:

$$H_n(x,y)=n!\sum_{r=0}^{\left[n/2\right]}\frac{y^r x^{n-2r}}{r!(n-2r)!}. \quad (48)$$

with generating function:

$$\exp(xt+yt^2)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n(x,y). \quad (49)$$

can be used to obtain a different expression of the operational results related to generalized Bessel functions, presented in the previous section. The Hermite polynomials are an important tool to facilitate the study of many classes of orthogonal polynomials as the Chebyshev polynomials [8], and of a wide class of special functions [9-12].
We start to note that the two-variable Bessel function of the type $J_n(x,y)$, can be written in the form:

$$J_n(x,y)=\sum_{r=0}^{+\infty}\frac{1}{(n+r)!r!}H_{n+r}\left(\frac{x}{2},\frac{y}{2}\right)H_r\left(-\frac{x}{2},-\frac{y}{2}\right), \quad (50)$$

and, symmetrically:

$$J_{-n}(x,y)=\sum_{r=0}^{+\infty}\frac{1}{(n+r)!r!}H_{n+r}\left(-\frac{x}{2},-\frac{y}{2}\right)H_r\left(\frac{x}{2},\frac{y}{2}\right), \quad (51)$$

after noting that:

$$J_{-n}(x,y)=J_n(-x,-y). \quad (52)$$

By using these results, we can immediately state the following important expression for the two-variable, one-parameter Bessel function:

$$J_n(\lambda x, \mu y; \tau) = = \lambda^n \sum_{p=0}^{+\infty} \frac{1}{p!} J_{n+p}\left(x, y; \frac{\mu\tau}{\lambda^2}\right) H_p\left((1-\lambda^2)\frac{x}{2}, (1-\mu^2)\frac{\lambda^2}{\mu\tau}\frac{y}{2}\right) . \tag{53}$$

In this paper we have seen the properties and the related applications of the family of the generalized Bessel functions of two-variable and one-parameter. We can now conclude this discussion to give an example of another family of Bessel-type functions which can be explored in a future article. By following the same procedure outlined for the Bessel function of the type $J_n(x, y; \tau)$, we can introduce the generalized two-index Bessel function, by setting:

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)\left(uv-\frac{1}{uv}\right)\right] = = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} u^n v^n J_{n,m}(x) , \tag{54}$$

where $x \in \mathbb{R}$ and $u, v \in \mathbb{R}$, such that $0 < |u| \neq |v| < +\infty$.

The explicit form of this type of Bessel function can be immediately write:

$$J_{n,m}(x) = \sum_{s=-\infty}^{+\infty} J_{n-s}(x) J_{m-s}(x) J_s(x) . \tag{55}$$

As stated in the introduction, Bessel functions are a powerful tool adopted to solve different classes of problems in the area of physics and engineering. For instance, the relations:

$$\int_{-1}^{+1} \left(1-x^2\right)^{\nu-1/2} C_n^\nu(x) e^{iux} dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n!\Gamma(\nu)} \frac{J_{n+\nu}(u)}{u^\nu} , \tag{56}$$

$$\int_{-\infty}^{+\infty} \frac{J_{n+\nu}(u)}{u^\nu} e^{-iux} du = = \begin{cases} \dfrac{2^\nu n!\Gamma(\nu)}{i^n \Gamma(2\nu+n)} \left(1-x^2\right)^{\nu-1/2} C_n^\nu(x), & |x| < 1 \\ 0, & |x| > 1 \end{cases} \tag{57}$$

where $C_n^\nu(x)$ are Gegenbauer polynomials [8], are used to study mono-dimensional electromagnetic problems in presence of edges [13-14]. Two-variables Hermite polynomials can be used to extend the above integrals to solve bi-dimensional problems too.

In a forthcoming paper we will discuss deeply these arguments related to Bessel functions and two-variables Hermite polynomials.

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