A Model for Herd Behavior of Agents in Financial Markets: Analysis and Simulation

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***Abstract*—**In this paper we consider a stochastic agent model with herd behavior which is related to the construction of price processes as jump processes in continuous time which exhibit heavy-tailed increments. We analyze the model theoretically and by performing numerical simulations for a large number of agents and for different values of the aggregation parameter. We further discuss the approximation properties of a nonlinear mean field model, including the case that the aggregation parameter approaches the maximal value of 1 in dependence of the number of agents. The discussion is backed up by numerical simulations and the outcome is compared to existing results in the literature.

***Keywords*—** agent-based models, coagulation-fragmentation, herding behavior, numerical simulations

I. Introduction

MOTIVATED by the statistical analysis of empirical market data, several models for price processes were considered in order to explain the presence of heavy tails of their short-time variations as deviations from the normal distribution [5], [4]. The approach consists of agent-based models which exhibit a herd behavior. The many agents present in the market do not act independently, but coordinated, gathered in groups which share the same information. The variations of the price returns in one single trading step are then proportional to the size of the group which performs a buying or selling action. The model includes therefore a component which describes aggregation and fragmentation of the agent groups and another component, which is influenced by the trading actions, and describes the evolution of the price process.

The approach presented in [5] considers N agents which are represented as vertices in a network. A vertex can be either isolated, or belonging to a connected component which describes a group of agents which act in a unitary way. The state Φ_j of the agent j belongs to the set $\{0,-1,1\}$, where 0 stands for inactive, -1 for selling and 1 for buying. The dynamics of the network can be described as follows. Initially all agents are isolated and inactive. At every (discrete) time-step an agent j is selected at random. With probability a a trading action is performed, while with the complementary probability (aggregation parameter) $1-a$ an aggregation process takes place.

In the trading step the state of the selected agent becomes active, taking with equal probabilities the values -1 or $+1$, this behavior being followed instantaneously by all $n(j)$ members of the group of agent j. The returns as variation of the logarithm of the price process have in this case the form $R(t_{i+1}) = \log(P(t_{i+1})) - \log(P(t_i)) = n(j)/\lambda$, where the parameter λ describes the liquidity of the market. The trading step ends with the fragmentation of the whole group and with resetting all its agents into an isolated, inactive state. This happens since the jointly shared information has been used for the current trading action and any further activity of the involved agents has to start anew.

During the aggregation step no trading action takes place, but instead another vertex k of the network is choosen uniformly and a link between the two vertices is added. This means that the two corresponding groups (if different) are merged together to form a group of size $n(j)+n(k)$. In the mentioned reference, simulations with the numerical values $N=10^4, a=0.01, 0.1, 0.3, \lambda=5\cdot 10^4$ are performed in order to investigate the behavior of the distribution of the returns in dependence of the herding parameter $h=1/a-1$. It was noticed that for values below a critical treshold $h<h^*$ the distributions obey a power-law with exponential cutoff, while for a more intense herding behavior $h>h^*$ the profiles change qualitatively and exhibit heavy tails, which imply increased probabilities for market crashes.

The paper [4] considers a nonlinear mean field version of the model above which is given by a system of coagulation equations with total fragmentation. Denote by $N(k)$ the number of all clusters of size k and by $u_k = N(k)/N$ the corresponding density. Letting $N\to\infty$ we formally obtain the dynamics

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$$\frac{\partial u_1}{\partial t} = -2(1-a)u_1\sum_{i=1}^{\infty} iu_i + a\sum_{i=2}^{\infty} i^2 u_i \quad (1)$$

$$\frac{\partial u_k}{\partial t} = (1-a)\sum_{i+j=k} iju_iu_j - 2(1-a)ku_k\sum_{i=1}^{\infty} iu_i - aku_k \quad (k>1)$$

In [4] the stationary solution $\{\bar{u}_k\}_{k\geq 1}$ of this system is computed explicitely and the returns are related to the distribution given by $\{\bar{u}_k\}_{k\geq 1}$. The heavy-tailed property is verified by computing its variance and kurtosis, also in the case $a \to 0$ which means an extreme aggregation behavior of the system.

The purpose of this paper is to construct an agent model with herd behavior based on continuous-time Markov jump processes (instead of discrete Markov chains as in [5]) and to analyze the approximation properties of the mean field equations (1) for large N. Several properties of the model are observed by numerical simulations performed with up to $N = 10^9$ agents, among others also the asymptotic convergence for large times towards the stationary solution computed in [4]. Both mentioned references consider in some sense the extreme aggregation behavior of the system, either by taking $a = N^{-1/2}$ (for $N = 10^4$) as in [5], which is exactly the case which is pointed out to exhibit a different qualitative behavior, or by considering $a \to 0$ as in [4]. In this situation, where $a = a(N) \to 0$ as $N \to \infty$ we investigate numerically if the behavior of the system (convergence to the equilibrium state or order of magnitude of the moments) is conserved within this limiting approach.

The organization of this paper is as follows. In Section II. we introduce the model in continuous time based on Markov jump processes and compute some relevant quantities which also indicate the possible mean field approximation in the case of a very large number of agents. This infinite-dimensional system of differential equations is presented in Section III. within the more general context of coagulation-fragmentation equations. Inside this frame it poses difficult mathematical problems which at the moment are not solved: rigorous convergence of the stochastic model towards the mentioned equations, existence and uniqueness of solutions or convergence to equilibrium at large times. The results of the numerical simulations are presented in Section IV., while in Section V. we investigate the properties of the model in the situation of an extreme herding behavior. The Appendix contains basic facts about Markov jump processes in continuous time which are needed within this paper.

The mathematical problems raised in this paper are not completely solved and are a challenge for future research. However, our theoretical considerations point out some important aspects and the results of the numerical simulations give plausible indications on the behavior of the system and on the formulation of its mathematical properties.

II. The Agent Model and the Associated Price Process

Consider a number of N agents which may be divided into several groups and the parameters $a \in [0,1]$ and $\sigma > 0$. For $k \in \mathbb{N}$ denote by $N(k)$ the number of groups of size k and by $u_k = N(k)/N$ the corresponding density. The state space of the Markov process consists of the pairs (u,R) with $u = (u_1, u_2, ..., u_N)$ with $u_k \in \{i/N \mid i = 0,...,N\}$ for all k, satisfying additionally $\sum_k ku_k = 1$. The component $R \in \mathbb{R}$ describes the log of the price process. The transitions of the Markov process and the corresponding rates are the following:

$$(u,R) \to \left(u - \frac{1}{N}e_k + \frac{k}{N}e_1, R + \frac{k\sigma}{\sqrt{N}}\right) \text{ at } \frac{1}{2}Naku_k \quad (2)$$

$$(u,R) \to \left(u - \frac{1}{N}e_k + \frac{k}{N}e_1, R - \frac{k\sigma}{\sqrt{N}}\right) \text{ at } \frac{1}{2}Naku_k \quad (3)$$

$$(u,R) \to \left(u - \frac{1}{N}e_i + \frac{1}{N}e_j + \frac{1}{N}e_{i+j}, R\right) \text{ at rate } K_{ij} \quad (4)$$

For $i, j = 1,...,N$ where e_i is the *i*-th unit vector in $\mathbb{R}^N$ and

$$K_{ij} = N(1-a)iu_i ju_j - \delta_{ij}(1-a)i^2u_i. \quad (5)$$

Relevant functionals of the ensemble of agents structured in groups are the moments of order r: $\mathrm{M}_r(u(t)) := \sum_k k^r u_k(t)$. Since the (finite) number of agents is always conserved, we have $\mathrm{M}_1(t) \equiv 1$. Taking into account this conservation property and assuming boundedness of $\mathrm{M}_2(t)$ (independent on N), since we are interested to approximate the dynamics (1), the probability of choosing one of the transition steps (2) and (3), i.e. a trading action is approaching a for large N. Conditioned by this, the probability that the chosen agent belongs to a group of size k is $kN(k)/N = ku_k$. The changes of the components of the Markov process model in this case the variation of the parameter R by the increment $k\sigma/\sqrt{N}$, that is we take $\lambda = \sqrt{N}/\sigma$ for the liquidity of the market, while the changes in the group structure illustrate the fragmentation of a group of size k into isolated agents. On the other hand, the probability of an aggregation step of type (4) is approximately $1-a$. During this transition we choose first a group of size i with probability proportional to $iN(i)$. A second (different) group is then chosen with probability proportional to $jN(j)$ for $j \neq i$ and $i(N(i)-1)$ if $j = i$, and the two groups are then merged in order to obtain a group of size $i+j$. This reasoning explains the form of the rates in (5). The factors of N in front of the transitions rates are necessary in order to obtain the correct time scaling. The waiting time Δt between two transition steps is exponentially distributed with parameter

$$\tilde{\lambda} = \sum_k Naku_k + \sum_{i,j} N(1-a)iu_i ju_j - (1-a)\sum_i i^2 u_i$$
$$= Na + N(1-a) - (1-a)\mathrm{M}_2(u) = N - (1-a)\mathrm{M}_2(u)$$

Basics about the construction of continuous-time Markov jump processes, their characterization by infinitesimal generators and martingale calculus can be found in the Appendix. By formula (A.2) we obtain the dynamics

$$u_1(t) = u_1(0) + \int_0^t \{-2(1-a)u_1 \sum_{i\ge1} iu_i + a\sum_{i\ge2} i^2 u_i + \frac{2(1-a)}{N}u_1\}ds + M_1(t)$$

$$u_k(t) = u_k(0) + \int_0^t \{(1-a)\sum_{i+j=k} iju_iu_j - 2(1-a)ku_k\sum_{i\ge1} iu_i - aku_k + \frac{(1-a)}{N}(2k^2u_k - (1-r_k)k'^2 u_{k'})\}ds + M_k(t)$$

$$(k = 2,\dots,N) \quad (6)$$

with the martingales $M_i(t), i = 1,\dots,N$ and where $k = 2k' + r_k$ with $r_k \in \{0,1\}$.

We note that these stochastic equations contain as deterministic trend exactly the dynamics (1) which is perturbed by the martingale (trendless) terms $M_i(t)$ and by some terms of order at most $O(k/N)$. In order that (1) yields a mean field approximation for the stochastic agent model as $N \to \infty$, it is necessary that the random fluctuations (the martingales) vanish in the limit. Moreover, by requesting that equations (1) make sense, the term $\mathrm{M}_2(u(t)) = \sum_{k=1}^{\infty} k^2 u_k(t)$ should be finite for the solution of this system, which implies that the terms containing $k^2 u_k / N$ will also vanish from the limiting dynamics. The mean field model (1) can be therefore interpreted (at this stage at least formally) as the limit of some Markov processes as previously described.

Using (A.3), since $\mathrm{M}_2(u(t)) \le N$ we compute (given the initial condition $u(0) = x$):

$$E_x[M_k^2(t)] = O(1/N) \text{ for } k = 2,\dots,N$$

$$E_x[M_1^2(t)] = O(1/N) + \frac{a}{N}\int_0^t E_x[\mathrm{M}_3(u(s))]ds.$$

One can notice that the only critical term is $\int_0^t E_x[\mathrm{M}_3(u(s))]ds$ which should be bounded from above by a quantity of order $o(N)$ in order to obtain the convergence property also for the martingale part of the first component.

By (A.1) we compute

$$\mathrm{M}_2(u(t)) = \mathrm{M}_2(u(0)) + \int_0^t \{2(1-a)\mathrm{M}_2^2(u(s)) - 2\frac{(1-a)}{N}\mathrm{M}_2(u(s)) + a\mathrm{M}_2(u(s)) - a\mathrm{M}_3(u(s))\}ds + \tilde{M}_2(t)$$

and therefore

$$a\int_0^t E_x[\mathrm{M}_3(u(s))]ds = E_x[\mathrm{M}_2(u(0))] - E_x[\mathrm{M}_2(u(t))] + \int_0^t \{2(1-a)E_x[\mathrm{M}_2^2(u(s)) - \frac{1}{N}\mathrm{M}_2(u(s))] + aE_x[\mathrm{M}_2(u(s))]\}ds. \quad (7)$$

When discussing the properties of the stochastic system, an upper bound of at most $o(N)$ for $E_x[\mathrm{M}_2^2(u(t))]$ would suffice to imply vanishing fluctuations of the u_1-component of the Markov process.

A proof of this fact in the general case is open. However, in the following we will show that this statement holds at least for sufficiently large values of a. Using Jensen's inequality and the conservation property $\sum_k ku_k = 1$ we compute

$$\mathrm{M}_2^2(u(t)) = \left(\sum_k k \cdot ku_k\right)^2 \le \sum_k k^2 \cdot ku_k = \mathrm{M}_3(u(t)).$$

We therefore have from (7):

$$(3a-2)\int_0^t E_x[\mathrm{M}_3(u(s))]ds \le C - E_x[\mathrm{M}_2(u(t))] + \left(a - \frac{2(1-a)}{N}\right)\int_0^t E_x[\mathrm{M}_2(u(s))]ds, \quad (8)$$

with $C = E_x[\mathrm{M}_2(u(0))]$. Since $\mathrm{M}_2(u(s)) \le \mathrm{M}_3(u(s))$, assuming $a \ge 2/3$ this implies:

$$E_x[\mathrm{M}_2(u(t))] \le C + 2(1-a)(1-\frac{1}{N})\int_0^t E_x[\mathrm{M}_2(u(s))]ds.$$

By Gronwall's inequality we obtain the exponential bounds:

$$E_x[\mathrm{M}_2(u(t))] \le C\exp(2(1-a)t). \quad (9)$$

Inserting back into (8) we obtain an estimate of the type $\int_0^t E_x[\mathrm{M}_3(u(s))]ds \le C'(t)$ independent on N, and therefore vanishing fluctuations of the u_1 – component.

Note that in general the convergence to 0 in mean square of the martingale terms is not a sufficient condition for convergence of the Markov jump process towards the solution of a deterministic equation of type (1). For example, in the pure aggregation case $a = 0$, the solution of (1) may exhibit *gelation*, that is a loss of mass at infinity. More precisely, we have $\sum_k k u_k(t) < 1$ for $t > t_{gel}$, this phenomenon being related to the explosion of the second moment $\mathrm{M}_2(u(t))$, see [13], [6], [7]. Moreover, in the latter reference is shown that after the gelation time the deterministic limit dynamics of the stochastic model is different from the formal one, taking also into account the loss of mass in the system. At the level of the Markov process, where the mass is *always* conserved, this is illustrated by the appearance in finite time of a cluster of size $O(N)$ which in the limit $N \to \infty$ disappears from the system at infinity due to the possible loss of mass induced by the vague convergence of measures [7]. The exponential bounds (9) for the second moment obtained for $a \geq 2/3$ (i.e. a situation of relatively small aggregation) imply however mass conservation in the limit equations and may allow the use of the methodology of [6] or [7] in order to prove a convergence result of the Markov process towards the mean field equations (1) as $N \to \infty$.

A direct approach for a rigourous convergence proof for arbitrary values of a (especially for $a < 2/3$) using for example techniques as in [6] or [7] as well as a direct proof of an estimate of the type $E_x[\mathrm{M}_2^2(u(t))] \leq o(N)$ by the tools listed in the Appendix fails, since the use of straightforward upper bounds does not capture the details (the correct sign) of the fragmentation dynamics. One should also take into account the ergodicity properties of the embedded Markov chain, which are ensured exactly by the fact that any two possible states are connected via the total fragmentation step through the state of completely isolated agents. This implies also the convergence for large times towards an invariant distribution and therefore probabilistic bounds for the moments of arbitrary order, at least in configurations which are not very far from equilibrium. However, a rigorous proof of these facts is still open. For theoretical details on ergodic properties of Markov processes see for example [10].

Concerning the price process $P(t) = P(0)\exp(R(t))$, we approximate its expectation and variance using Proposition 1 in the Appendix, taking for f the exponential of the R-component of our process. We further assume that the Markov process is near equilibrium and that N is very large, such that we can replace the moments which appear in the computations by the values corresponding to the stationary mean field model which are given explicitly in [4]. The numerical simulations from Section IV. suggest the validity of this approach. We obtain in this way for large N:

$$E_x[P(t)] \approx P(0)\exp\left(\sigma^2 t \frac{2 - 2a + a^2}{2a^2}\right)$$

and

$$Var_x[P(t)] \approx$$

$$P(0)^2 \exp\left(\sigma^2 t \frac{2-2a+a^2}{a^2}\right)\left(\exp\left(\sigma^2 t \frac{2-2a+a^2}{a^2}\right) - 1\right).$$

For $a = 1$, i.e. if all agents are isolated, we obtain the known values for the geometric Brownian motion.

III. The Mean Field Model

Equations (1) are a special case of coagulation equations with total fragmentation

$$\frac{\partial u_1}{\partial t} = -u_1 \sum_{i=1}^{\infty} K(1,i)u_i + a\sum_{i=2}^{\infty} i^2 u_i$$

$$\frac{\partial u_k}{\partial t} = \frac{1}{2}\sum_{i+j=k} K(i,j)u_i u_j - u_k \sum_{i=1}^{\infty} K(k,i)u_i - aku_k \quad (10)$$

for $k > 1$, where groups of sizes i, j coalesce at rate $K(i, j)$ and groups of size k fragment into k single individuals at rate ak. In our model we assumed the multiplicative coalescent, that is $K(i, j) = Cij$. A similar model with constant kernel $K(i, j) = C$ arises in the context of the paper [1], but also without a rigorous convergence proof of the stochastic processes towards the solution of the deterministic equations.

The main difficulty lies in the presence of the total fragmentation term which, especially in the case of fragmentation of large clusters into isolated individuals, induces a strong irregularity in the dynamics of the problem which is hard to control mathematically. In may be hoped that the use of techniques for finite Markov chains can lead into the right direction. The existence of an invariant distribution for large times of the embedded finite Markov chain implies also convergence of the single components towards corresponding equilibrium values, since they are proportional to the probabilities of the total fragmentation events. This fact is also backed up by the numerical simulations from Section IV..

However, the considerations above apply in first instance only for the Markov process, not directly for the mean field model. For the latter, in the case $K(i, j) = Cij$, the equilibrium solution of (10) is computed in [4].

The problem of the trend to equilibrium for general coagulation-fragmentation equations

$$\frac{\partial u_1}{\partial t} = -\sum_{i=1}^{\infty}(K(1,i)u_i u_1 - F(1,i)u_{1+i})$$

$$\frac{\partial u_k}{\partial t} = \frac{1}{2}\sum_{i+j=k}(K(i,j)u_i u_j - F(i,j)u_k) -$$

$$-\sum_{i=1}^{\infty}(K(k,i)u_i u_k - F(k,i)u_{k+i}) \quad (11)$$

for $k > 1$ is solved only under additional assumptions on the fragmentation coefficients $F(i, j)$, which denote the rate of binary fragmentation of a cluster of size $i + j$ in two clusters of sizes i and j. Existence, uniqueness and mass-conservation results are given in [3]. A survey of these equations with further references can be found in [13]. Most papers on this topic deal with the binary fragmentation of clusters as described above. The paper [12] considers the case of multiple fragmentation, which includes also the situation relevant in this paper. However, existence and uniqueness results are shown only under the assumption $K(i, j) \le C(i + j)$ which is the growth condition under which uniqueness is proved also in the pure coagulation case.

The trend to equilibrium is strongly related to the *detailed balance condition* $K(i, j)Q_iQ_j = f(i, j)Q_{i+j}$ for some positive constants Q_i, which essentially describes the reversibility of the dynamics of coagulation of clusters of sizes i, j. That is, the flux of formation of $i + j$-clusters by coagulation of i-clusters and j-clusters equals the flux of formation these types of clusters by fragmentation from $i + j$-clusters, see [2], [13]. Without this property the trend to equilibrium is proved only in special cases, under restrictive technical assumptions [8]. To the knowledge of the authors, the trend to equilibrium in the case of coagulation with multiple fragmentation, which includes also our model, has not been investigated rigorously up to now.

IV. Numerical Simulations

In this section we discuss results of numerical simulations of the dynamics (2)-(4) for large N using the techniques from [9] adapted to the present model. From the analysis performed in Section II. it turns out that certain estimates needed to prove convergence of the Markov processes towards the mean field model as $N \to \infty$ are problematic in the case of the u_1-component, due to the presence of the second moment M_2 in the corresponding equation. For this reason we will present in Figures 1 and 2 the results of numerical simulations for these functionals. The horizonatal dashed lines represent the equilibrium values of (1) $\bar{u}_1 = 1/(2 - a)$ and of the second moment $\overline{\mathrm{M}}_2 = 1/a$ as computed in [4]. The computations are performed with $N = 10^8$ agents for $a = 0.01, 0.1, 0.6, 0.9$ for two different initial conditions: the monodisperse one, where $u_1(0) = 1, u_k(0) = 0$ for $k \ge 2$ and a polydisperse one.

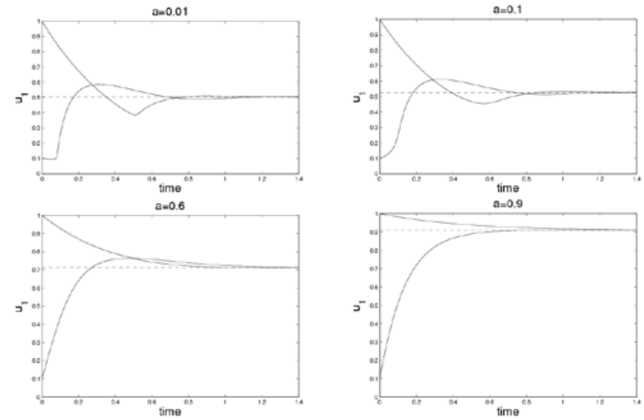

Figure 1: Evolution of $u_1(t)$ for different initial conditions and N=10^8 agents

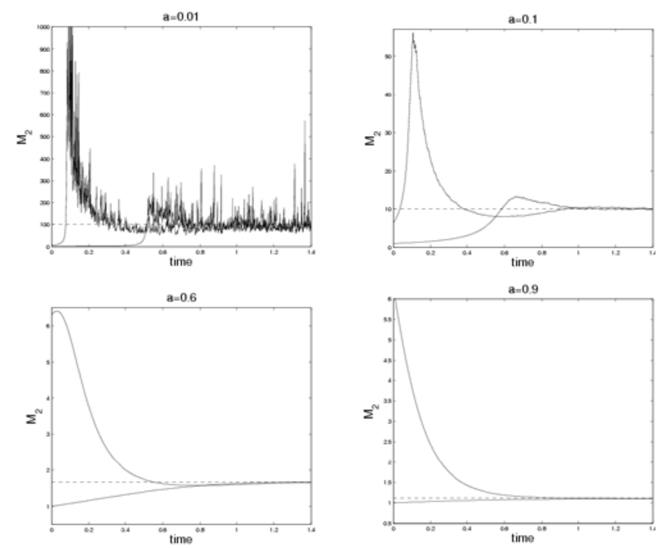

Figure 2: Evolution of $\mathrm{M}_2(t)$ for different initial conditions and N=10^8 agents

In the case of u_1 (as well as for $u_k, k \ge 2$ which are not plotted here) one observes for all choices of the parameter a a convergence towards the equilibrium values corresponding to the mean field model.

A similar behavior is noted in the simulations of the second moment M_2 depicted in Figures 2 and 3. An estimate of the type $\int_0^t E_x[\mathrm{M}_2^2(u(s))]ds \le o(N)$ seems therefore to be likely to hold. In the case $a = 0.01$, with a strong aggregation behavior, the fluctuations of the second moment for $N = 10^8$ are still high, but the simulations performed with $N = 10^9$ agents plotted in Figure 3 show that such a result may hold, provided a is fixed and N sufficiently large. (The case where $a \to 0$ in dependance of N will be investigated in the next section). As problematic appear configurations far from equilibrium, where due to the accentuated aggregation behavior the second moment exhibits a strong increase. This growth leads to instable configurations, followed by a quick and strog decay and large fluctuations. Exactly this regime turns out to provide difficulties for the theoretical estimates.

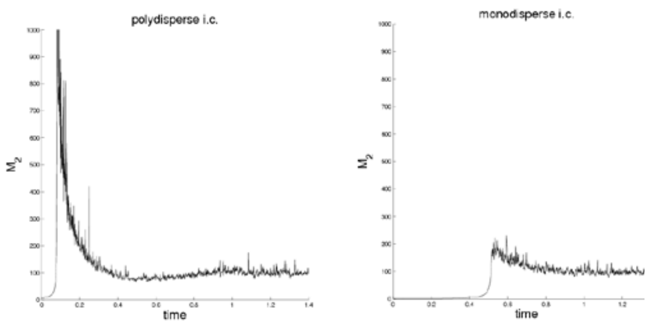

Figure 3: Evolution of $M_2(t)$ for a=0.01 and N=10^8 agents

We investigate next the behavior of the price process for different values of a starting always close to the equilibrium distribution of the agent system. This allows a better comparison of the influence of the aggregation parameter on the properties of the price process and eliminates possible influences due to the initial conditions. The results are plotted in Figure 4. The computation of the price process starts only at $t=1$, since at this time the agent system is near equilibrium, as shown in Figures 1, 2.

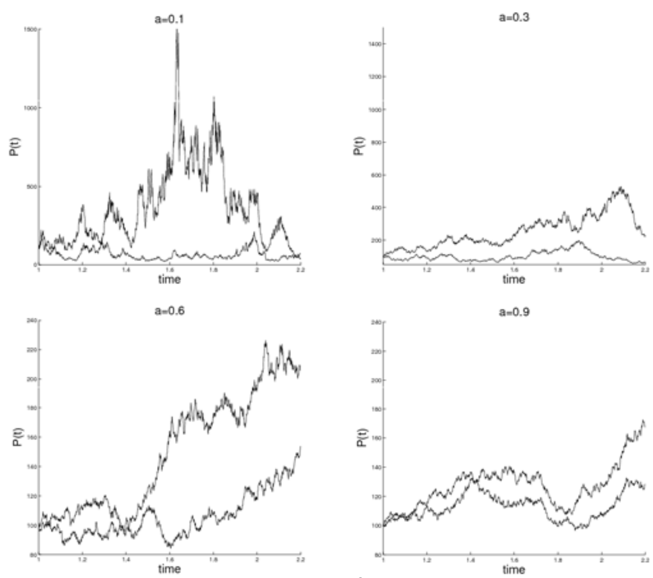

Figure 4: Evolution of $P(t)$ for N=10^8 agents and σ=0.2 near equilibrium

For $a<0.1$, due to the strong variations of the price process, it is more reasonable to plot the quantity $R(t)=\log(P(t)/P(0))$. We perform 10 independent simulations for every value of a, especially for values $a=O(10^{-2})$ in order to trace possible qualitative differences in the behavior of the price process which may hint on a 'catastrophic' behavior. Figure 5 shows that the statistics of the paths are basically the same for $N=10^8$ and $N=10^9$ agents, even for this small value of a. We may therefore assume that the properties of the price process are mainly influenced by a and that a value of $N=10^8$ may be considered as sufficiently large in order to perform such an investigation.

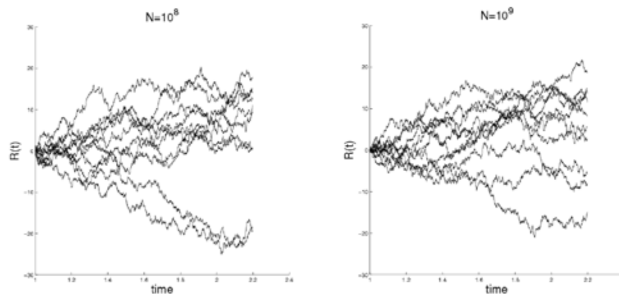

Figure 5: 10 realizations of $R(t)$ for a=0.03, σ=0.2

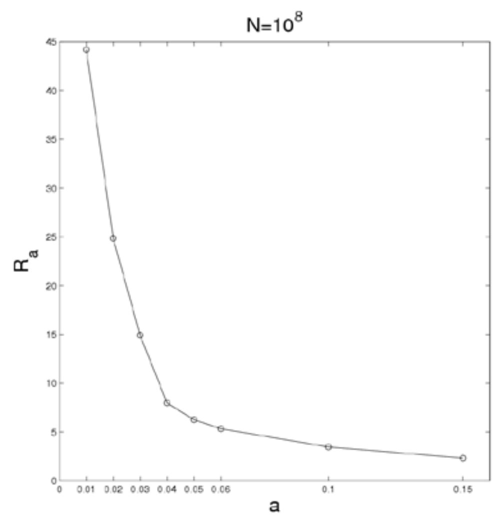

Figure 6: R_a=Ê[$\max_{t\leq T}$ $|R(t)|$]

In Figure 6 we plot the quantity $R_a=\hat{E}[\max_{t\leq T}|R(t)|]$ in dependance of a, which is the empirical mean of the maximum of the log of the price process. High values of this parameter will imply explosions or crash of the price process (as exponential function of $R(t)$). We observe clearly that since $R_{0.01}\approx 10R_{0.1}$, the values of $a=0.1$ and $a=0.01$ belong to different regimes, the same fact being pointed out also in [5].

V. The Case of Extreme Herding Behavior

In this section we analyze the model in the case of an extreme aggregation behavior for $a=a(N)\to 0$ as $N\to\infty$. As pointed out in Section II., in order that the fluctuations of the u_1-component of the stochastic process vanish in the limit $N\to\infty$, it is necessary to have the convergence property $\frac{a}{N}\int_0^t E_x[M_3(u(s))]ds\to 0.$

Additionally to the problem of convergence to some limit, a further question is if one can pass to the limit $a \to 0$ in the functionals of the equilibrium solution of (1) computed in [4] for fixed a: $\overline{u}_1 = 1/(2-a)$, $\overline{u}_2 = (1-a)/(2(2-a)^3)$, $\overline{\mathrm{M}}_2 = 1/a$, $\overline{\mathrm{M}}_3 = (2-2a+a^2)/a^3$ (for quantities becoming infinite we look at their order of magnitude).

We analyze the model numerically for $a = N^{-\alpha}$ near the equilibrium. In Figure 7 we note a qualitatively different behavior for $\alpha = 0.5$ (left) and $\alpha = 0.6$ (right). For $\alpha = 0.5$ we observe convergence of the components u_k towards the equilibrium values $\overline{u}_k$ of (1), while for $\alpha = 0.6$ these seem to be only expected values. Moreover, for both values of α it turns out that the formula for $\overline{\mathrm{M}}_3$ computed for a independent on N does not give the correct order of magnitude in our situation. For $a = N^{-0.5}$ we notice that $a\mathrm{M}_3 / N$ (the formal limit being 2), while for $a = N^{-0.6}$ we obtain a stochastic process with values of magnitude $O(1)$, the formal limit being ∞.

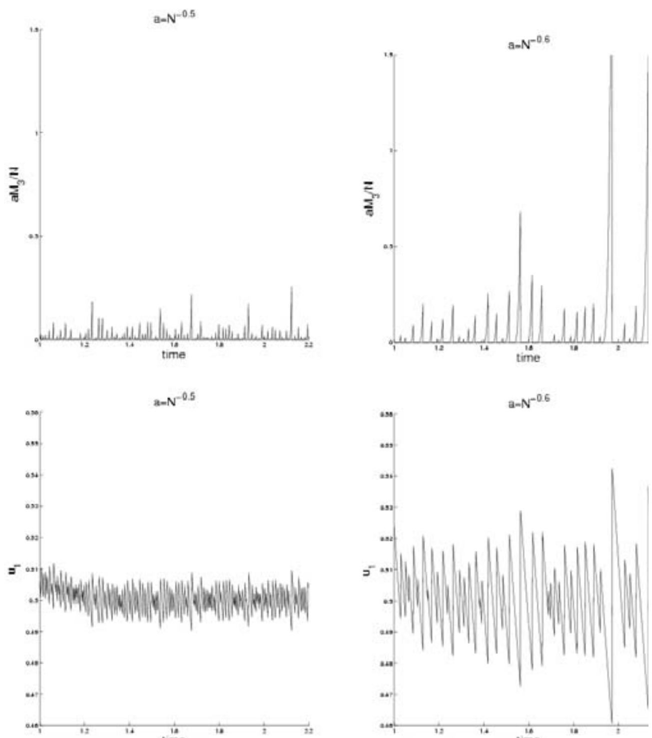

Figure 7: One realization of $a\mathrm{M}_3/N$ (top) and u_1 (bottom) for $N=10^9$ agents and α=$N^{-0.5}$ (left) and α=$N^{-0.6}$ (right)

Further experiments with different values of α show that $\alpha = 0.5$ is a treshold value. We compare therefore the results of 10 independent simulations for $\alpha = 0.4$ and $\alpha = 0.6$ near equilibrium for different values of N. We note that for $\alpha = 0.4$ we have convergence towards $\overline{u}_1$ and $\overline{u}_2$ and that $a\mathrm{M}_3 / N \to 0$, while for both values of α we note that at equilibrium we have only $a\hat{E}[\mathrm{M}_2(u(t))] \approx 1$, but not a convergence of $a\mathrm{M}_2(u(t))$ towards 1. Moreover, for $\alpha = 0.6$ we note that the equilibrium values are also only expected values and that the limit process may not be deterministic.

Figure 8 illustrates these remarks. For $\alpha = 0.4$ and $\alpha = 0.6$ we compare the results of 10 independent simulations on the following quantities: u_1, u_2, $a\mathrm{M}_2$ and $a\mathrm{M}_3 / N$.

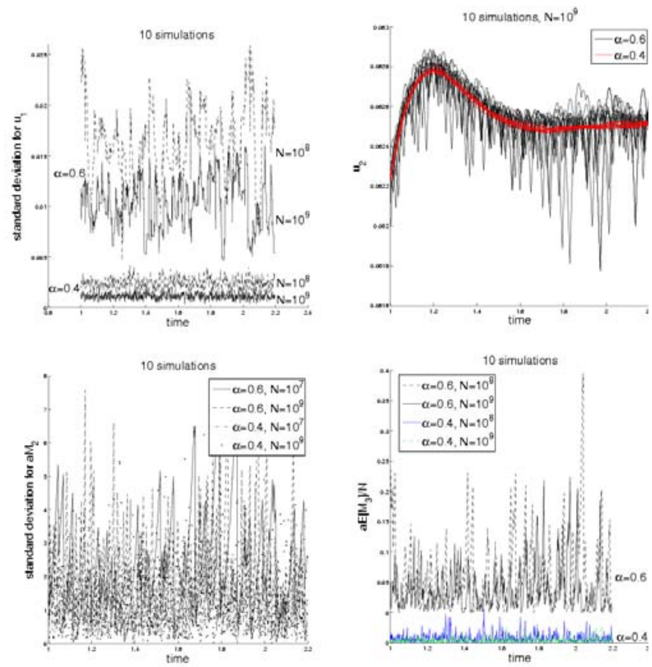

Figure 8: Different behavior for α=0.4 and α=0.6

More precisely, in the four graphics we compare in the same coordinate system:

- the standard deviation of the stochastic processes from the equilibrium value $\overline{u}_1$ for different values of α and N.
- 10 realization paths of u_2 for different α's and $N = 10^9$.
- the standard deviation of $a\mathrm{M}_2$ from the mean 1 for different values of α and N.
- $a\hat{E}[\mathrm{M}_3]/N$ for different values of α and N (empirical mean based on 10 simulations).

The conclusions are respectively:

- for $\alpha = 0.4$ we notice convergence towards $\bar{u}_1$ while for $\alpha = 0.6$ this value is only an expectation and the fluctuations do not vanish.
- the same remark as above can be made also for u_2 (see also Figure 9 for $\alpha = 0.6$ and $\alpha = 0.8$.
- for all values of α and N we note that the fluctuations of $a\mathrm{M}_2$ do not vanish and are of about the same order of magnitude.
- for $\alpha = 0.4$ we have that $a\hat{E}[\mathrm{M}_3]/N \to 0$ and the fluctuations decrease with increasing N. For $\alpha = 0.6$ the quantity $a\hat{E}[\mathrm{M}_3]/N$ is of magnitude $O(1)$ and the fluctuations for $N = 10^8$ and $N = 10^9$ are practically the same. Further simulations with $\alpha = 0.8$ shown in Figure 9 sustain the hypothesis that $a\hat{E}[\mathrm{M}_3]/N = O(1)$ for $\alpha > 0.5$ and $o(1)$ for $\alpha \le 0.5$.

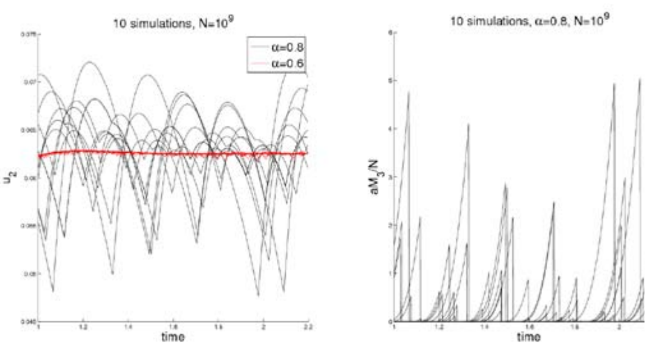

Figure 9: u_2 and $a\mathrm{M}_3/N$ for $a=N^{-0.5}$ and $N=10^9$

In order to construct a price process we assume a more general scaling of the dynamics than in (2), (3), namely the transitions and the corresponding rates are given by:

$$(u,R) \to \left(u - \frac{1}{N}e_k + \frac{k}{N}e_1, R + k\sigma\eta(N)\right) \quad \text{at rate} \quad \frac{1}{2}Naku_k \qquad (12)$$

$$(u,R) \to \left(u - \frac{1}{N}e_k + \frac{k}{N}e_1, R + k\sigma\eta(N)\right) \quad \text{at rate} \quad \frac{1}{2}Naku_k \qquad (13)$$

while (4) remains unchanged. The liquidity of the market ist therefore taken as $\lambda = (\sigma\eta(N))^{-1}$. By the tools in the Appendix we compute

$$Var_x(R(t)) = (\sigma N\eta(N))^2 \cdot \frac{a}{N}\int_0^t E_x[\mathrm{M}_3(u(s))]ds. \qquad (14)$$

If the convergence $\frac{a}{N}\int_0^t E_x[\mathrm{M}_3(u(s))]ds \to 0$ holds, in order to obtain a nontrivial process $R(t)$ with finite variance, one has therefore to choose the scaling $\eta(N)$ adapted to the convergence speed of the mentioned term. For $a = N^{-0.5}$ numerical simulations indicate a slow, logarithmic order if convergence. Otherwise, if we have $\frac{a}{N}\int_0^t E_x[\mathrm{M}_3(u(s))]ds = O(1)$ as in the numerical simulations for $a = N^{-\alpha}$ with $\alpha > 0.5$, we have to choose $\eta(N) = N^{-1}$.

Summarizing the results of our numerical simulations, we notice that if $a = a(N)$ goes to 0 with a convergence order of at most $O(N^{-1/2})$, the stochastic processes still converge to constant equilibrium values, while if the convergence is faster than $O(N^{-1/2})$, we do not have a deterministic limit. In all analyzed cases with a of the form $a = N^{-\alpha}$, the quantity $a\mathrm{M}_2$ has the expected value 1, but the fluctuations around this mean do not vanish. An interesting mathematical problem is therefore to formulate and to prove a convergence result to some limiting dynamics if $N \to \infty$ and $a = a(N) \to 0$ slower than $N^{-1/2}$. The numerical simulations indicate deterministic values for u_k but a stochastic behavior for $a\mathrm{M}_2$, which is a term appearing in the first equation of (1).

VI. Conclusions

Our theoretical and numerical analysis of the model for herd behavior of agents in financial markets confirms on the one hand the known properties already reported in the existing literature. Moreover, we additionally point out some (for the moment) unsolved mathematical problems, such as the rigorous investigation of the approximation properties of the stochastic processes by the mean-field model for a large number of agents, convergence towards equilibrium in the mean field equations, the behavior of the model and the scaling of the price dynamics in the case of an extreme herding behavior $a = a(N) \to 0$. Based on numerical simulations we are able to give some hints on the validity of these conjectures, which turn out to be challenging research topics for the future.

APPENDIX

Basics of Markov Jump Processes in Continuous Time

For our purpose it is sufficient to assume here a discrete state space $E \subseteq \mathbb{R}^n$. Let $X(t), t \geq 0$ be a continous-time Markov jump process with state space E and let x, x' be two possible states. Let $q_{x \to x'}(t)$ be the corresponding transition kernel, that is the probability of being in state x' after t time units, conditioned by the current state x. The *infinitesimal transition rate* $R_{x \to x'}$ is then defined by $R_{x \to x'} := \lim_{t \to 0} \frac{q_{x \to x'}(t)}{t}$.

The *infinitesimal generator* Λ of the Markov jump process $X(t), t \geq 0$ is an operator acting on the bounded continuous test functions $\varphi \in C_b(E)$ by

$$\Lambda\varphi(x) = \sum_{x \to x'} (\varphi(x') - \varphi(x)) R_{x \to x'},$$

where the sum is taken over all possible transitions $x \to x'$.

Useful in the calculus with Markov jump processes turns out to be their martingale characterization given by the *Dynkin-formula*:

$$\varphi(X(t)) = \varphi(X(0)) + \int_0^t \Lambda\varphi(X(s))ds + M_\varphi(t) \qquad (A.1)$$

Where $M_\varphi(t)$ is a martingale with respect to the filtration associated to the Markov process $X(t)$.

As a special case we obtain the following: define $F : E \to \mathbb{R}^n$ by $F(x) = \sum_{x \to x'} (x' - x) R_{x \to x'}$ and assume that $\sum_{x \to x'} | x' - x | R_{x \to x'} < \infty$ holds. We then have that for $i = 1, \dots, N$ the processes

$$M_i(t) = X_i(t) - X_i(0) - \int_0^t F_i(X(s))ds \qquad (A.2)$$

are martingales and that the pair $(X(t), M(t))_{t \geq 0}$ is a Markov process. The bounded smooth functions depending only on the second component: $f(z, m) \equiv f(m) \in C_b^1(\mathbb{R}^n)$ are in the domain of its infinitesimal generator A and we have:

$$\mathrm{A}f(x, m) = \sum_{x \to x'} (f(x' - x + m) - f(m) - \\ - (x - x') f'(z) R_{x \to x'}.$$

We further have:

Proposition 1.
If $f \in C^1(\mathbb{R}^n)$ is nonnegative with absolutely continuous first derivative and positively (semi-)definite second derivative, then the conditional expectation $E_x[f(M(t))]$ given $X(0) = x$ can be computed by

$$E_x[f(M(t)] = f(0) + \int_0^t E_x[\mathrm{A}f(X(s), M(s))]ds$$

□

Of interest in our case is the choice $f(m) = m_i^2$. We have in this case

$$E_x[M_i^2(t)] = \int_0^t E_x[\sum_{x \to x'} (x_i' - x_i)^2 R_{x \to x'}]ds. \qquad (A.3)$$

For further details on this subject see [11].

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