Discrete Mathematical Model of Diffraction on Pre-Cantor Set of Slits in Impedance Plane and Numerical Experiment

Kateryna V. Nesvit

Abstract **—** The significance of this work is directed towards research and mathematical modeling of TE wave diffraction on pre-Cantor structures. The total electromagnetic (EM) field is represented as a superposition of two independent fields in the case of E and H polarization for the solution of two-dimensional problems in the mathematical theory of diffraction. In our case the stationary Maxwell equations are reduced to the solution of two independent boundary-value problems of Dirichlet and Neumann for Helmholtz equation. In this paper the diffraction problem of E-polarized plane EM wave on pre-Cantor set of slits in the impedance plane has been investigated. The boundary-value problem was reduced to a system of boundary singular integral equations (SIEs) with supplementary conditions and a Fredholm equation of the 2-nd kind. It was done using the method of parametric representations of integral operators. A discrete mathematical model of this boundary SIE with the help of an efficient Discrete Singularities Method (DSM) has been implemented. Several numerical experiments have been carried out to investigate the performance of the developed technique for pre-Cantor grating of different degree and variable impedance of the metallic strips.

Keywords **—** Diffraction problem, integral equation, numerical experiment, pre-Cantor grating.

I. INTRODUCTION

Wave diffraction on the pre-Cantor sets of slits is a good model of a wide range of classic diffraction problems. The considered diffraction structure consists of $2^{(N-1)}$ slits, where N is the order of pre-Cantor grating. Use of the pre-fractal structures in wave diffraction problems is a recent direction of research. In papers [1], [2] and [3], the different, not pre-fractal gratings are studied. The pre-Cantor set is a particular case of the pre-fractal set. From the application viewpoint, we may state that the model which is considered in this paper is an approximation of a real fractal antenna in 2D. Fractal antennas are used in a variety of modern mobile devices due to their compact size and broadband properties, which have made them essential in wireless communication, Bluetooth, Wi-Fi and GSM standards.

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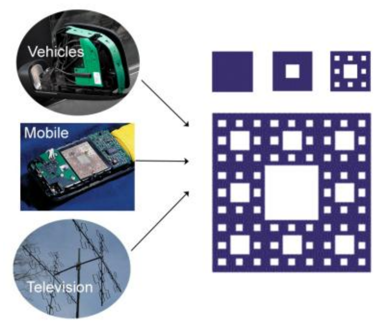


Fig. 1. Using of the fractal antennas.

From the mathematical viewpoint, the boundary-value problem of boundary integral equations (IEs) for the stationary wave equation was reduced SIEs with supplementary conditions and the Fredholm equation of the 2-nd kind.. These singularities have been resolved with the help of special quadrature formulas with nodes in the nulls of Chebychev polynomials. We have successfully built a discrete mathematical model based on the SIEs and carried out several numerical experiments with the help of an efficient DSM.

II. PROBLEM FORMULATION

In the 2D case the total EM field is represented as a superposition of two fields: the E-polarized $\overline{E}(E_{x'},\,0,\,0)$, $\overline{H}(0,H_{y'},H_{z'})$, and the H-polarized $\overline{E}(0,E_{y'},E_{z'})$, $\overline{H}(H_{x'},0,0)$. Here the time dependence is given by the factor $e^{-i\omega t}$). In this case a unique independent component (out of six) of either the electric $E_{x'}=u(y',z')$ or the magnetic $H_{x'}=u(y',z')$ field satisfies the 2D Helmholtz equation without of the strips [1-3], [6]:

$$\Delta u(y',z')+k^2u(y',z')=0,\quad k=\frac{\omega}{c}.$$

Besides, as will be demonstrated later, the total field $u(y',z')$ should satisfy the boundary conditions on the strips, the Sommerfeld radiation conditions and the Meixner edge condition.

The non-zero electric (TM case) or magnetic (TE case) field components are expressed by Maxwell's equations:

TE case:

$$H_{y'} = \frac{1}{i\omega\mu}\frac{\partial}{\partial z'}E_{x'}, \quad H_{z'} = -\frac{1}{i\omega\mu}\frac{\partial}{\partial y'}E_{x'},$$

TM case:

$$E_{y'} = -\frac{1}{i\omega\varepsilon}\frac{\partial}{\partial z'}H_{x'}, \quad E_{z'} = \frac{1}{i\omega\varepsilon}\frac{\partial}{\partial y'}H_{x'}.$$

The considered grating is based on a pre-Cantor set. Let us define the sets of pre-Cantor intervals, which are obtained by constructing a Cantor set on the *N*-th step (see Fig.1).

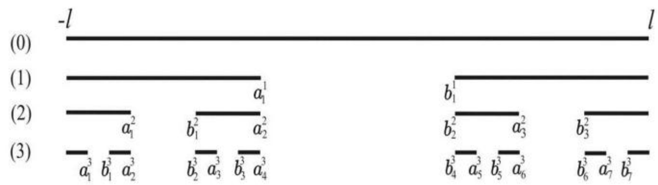


Fig. 2. Pre-Cantor sets for *N*=0,1,2,3.

Consider an E-polarized plane EM wave with a unit amplitude and an angle α to fall from infinity onto the diffraction structure (see Fig. 3):

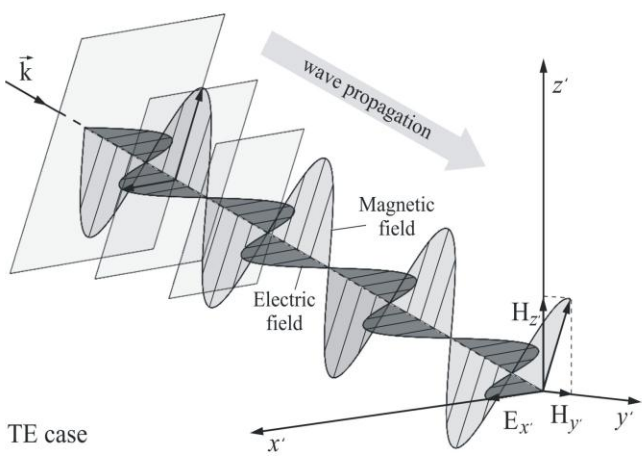


Fig. 3. Propagation of the plane electromagnetic wave in TE case.

$$u_{inc}^N(y', z') = E_{x'}(y', z') = e^{ik(y'\sin\alpha - z'\cos\alpha)}. \tag{1}$$

In the case of E-polarization the propagation direction of a plane wave is given by direction of the wave vector k. Cartesian coordinate system is chosen so that the set of strips is located in the X′Y′ plane, and the X′ axis is parallel to the strips' edges (see Fig. 4):

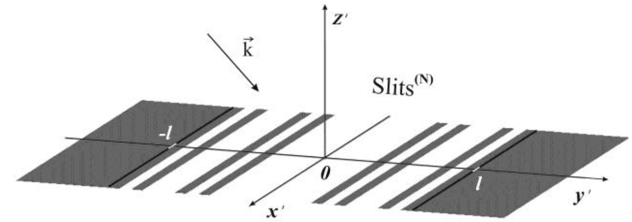


Fig. 4. Schematic of the considered diffraction structure.

$$Slits^{(N)} = \left\{(x', y', z') \in \Re^3, y' \in S_l^{(N)}, z' = 0\right\},$$
$$Sl_l^{(N)} = \bigcup_{q=1}^{2^N-1}\left(a_q^N, b_q^N\right), \tag{2}$$

where

$$a_q^N = \left(\frac{P_q - 1}{2\cdot 3^N} + \frac{1}{3^N}\right)\cdot 2l - 1, \quad b_q^N = \left(\frac{P_{q+1} - 1}{2\cdot 3^N}\right)\cdot 2l - 1,$$

$$P_{s+2^{k-1}} = 2\cdot 3^k - P_{2^{k-1}-s+1}, \; P_1 = 1, \; q = \overline{1, 2^N - 1}.$$

It is convenient to introduce to dimensionless coordinates:

$$x = \frac{x'}{l}, y = \frac{y'}{l}, z = \frac{z'}{l}, \alpha_q^N = \frac{a_q^N}{l}, \beta_q^N = \frac{b_q^N}{l},$$
$$\kappa = kl, \quad Sl^{(N)} = \bigcup_{q=1}^{2^N-1}(\alpha_q^N, \beta_q^N),$$

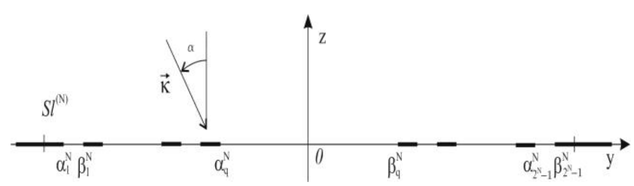


Fig. 5. Cross section of the diffraction structure in YZ plane.

We seek the total field $u^{(N)}(y, z)$, which results from the scattering of E-polarized monochromatic plane wave on the considered diffraction structure (see Fig. 3). The total field is considered to have the following form:

$$u^{(N)}(y, z) = \begin{cases} u_0^N(y, z) + u_+^N(y, z), & z > 0, \\ u_-^N(y, z), & z < 0, \end{cases} \tag{3}$$

where $u_0^N(y,z)$ is a known solution of the Helmholtz equation representing the sum of the incident and reflected waves in half-space $z \geq 0$ when the slits are closed. The functions $u_+^N(y,z),\ u_-^N(y,z)$ will be determined, $u_+^N(y,z), u_-^N(y,z) \in C^2$.

The total field (3) must satisfy the following conditions [1-3]:

- the Helmholtz equation off of the impedance strips:

$$\Delta u^{(N)}(y,z) + \kappa^2 u^{(N)}(y,z) = 0,\ \kappa = \frac{\omega l}{c}; \tag{4}$$

- the boundary conditions on the strips:

$$\frac{\partial u^{(N)}}{\partial z}(y,+0) - A u^{(N)}(y,+0) = 0,\ y \in CSl^{(N)} = \Re \setminus Sl^{(N)}, \tag{5}$$

$$\frac{\partial u^{(N)}}{\partial z}(y,-0) + A u^{(N)}(y,-0) = 0,\ y \in CSl^{(N)}, \tag{6}$$

where

$$\gamma(\lambda) = \begin{cases} \sqrt{\lambda^2 - \kappa^2}, & |\lambda| > \kappa, \\ -i\sqrt{\kappa^2 - \lambda^2}, & |\lambda| < \kappa, \end{cases}$$

$$A = i\kappa \frac{Z_0}{Z_c}, \quad Z_c = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega \mu_c}{\sigma_c}}, \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}},$$

Z_c is impedance of metal, Z_0 is impedance of free space, μ_c is magnetic permittivity of metal, σ_c is magnetic conductivity of metal, ε_0 is dielectric constant, μ_0 is magnetic constant, ω is angular frequency;

- the conditions of conjugation in the slits:

$$u^{(N)}(y,+0) = u^{(N)}(y,-0), \quad y \in Sl^{(N)}, \tag{7}$$

$$\frac{\partial u^{(N)}}{\partial z}(y,+0) = \frac{\partial u^{(N)}}{\partial z}(y,-0), \quad y \in Sl^{(N)}; \tag{8}$$

- the Sommerfeld radiation condition at infinity:

$$\frac{\partial u_\pm^N}{\partial r}(y,z) - i\kappa u_\pm^N(y,z) = \bar{o}\left(\frac{1}{\sqrt{r}}\right), \quad r = |\,y,z\,| \to \infty; \tag{9}$$

- the Meixner edge condition in an integral form (the condition of local energy limited):

$$\int_\Omega \left(\kappa^2 |u_\pm^N(y,z)|^2 + |\nabla u_\pm^N(y,z)|^2\right) d\sigma < \infty, \tag{10}$$

where Ω is any boundary field in $\Re^2$.

The initial field $u_0^N(y,z)$ can be written in the form:

$$u_0^N(y,z) = e^{i\kappa(y\sin\alpha - z\cos\alpha)} + \left(\frac{i\kappa\cos\alpha + A}{i\kappa\cos\alpha - A}\right) e^{i\kappa(y\sin\alpha + z\cos\alpha)}, \tag{11}$$

and the sought functions are considered as Fourier series in the integral form:

$$u_+^N(y,z) = \int_{-\infty}^{+\infty} C_+^N(\lambda) e^{i\lambda y - \gamma(\lambda) z} d\lambda,\ z > 0,$$
$$u_-^N(y,z) = \int_{-\infty}^{+\infty} C_-^N(\lambda) e^{i\lambda y + \gamma(\lambda) z} d\lambda,\ z < 0, \tag{12}$$

These series satisfy all the aforementioned conditions (4)-(8). The radiation condition will be satisfied if $\gamma(\lambda) = \sqrt{\lambda^2 - \kappa^2}$ is given by $\operatorname{Re}\gamma(\lambda) \geq 0$, $\operatorname{Im}\gamma(\lambda) \leq 0$, $\lambda \in \Re$.

As shown in [1], [2] we have obtained two coupled integral equations using the definitions (11), (12) and the conditions (4) – (8):

$$\begin{cases} \int_{-\infty}^{+\infty} \left[C_+^N(\lambda) - C_-^N(\lambda)\right](\gamma(\lambda) + A) e^{i\lambda y} d\lambda = 0, & y \in CSl^{(N)}, \\ \int_{-\infty}^{+\infty} \left[C_+^N(\lambda) - C_-^N(\lambda)\right] e^{i\lambda y} d\lambda = -u_0^N(y,+0), & y \in Sl^{(N)}, \end{cases} \tag{13}$$

$$\begin{cases} \int_{-\infty}^{+\infty} \left[C_+^N(\lambda) + C_-^N(\lambda)\right](\gamma(\lambda) + A) e^{i\lambda y} d\lambda = 0, & y \in CSl^{(N)}, \\ \int_{-\infty}^{+\infty} \left[C_+^N(\lambda) + C_-^N(\lambda)\right] \gamma(\lambda) e^{i\lambda y} d\lambda = \frac{\partial u_0^N}{\partial z}(y,+0), & y \in Sl^{(N)}. \end{cases} \tag{14}$$

III. SIE with supplementary conditions and Fredholm equation of 2-nd kind

Let us define

$$B_1^N(\lambda) = \left[C_+^N(\lambda) - C_-^N(\lambda)\right](\gamma(\lambda) + A), \tag{15}$$

and write down the coupled integral equation (13) in the following form:

$$\begin{cases} \int_{-\infty}^{+\infty} B_1^N(\lambda) e^{i\lambda y} d\lambda = 0,\ y \in CSl^{(N)}, \\ \int_{-\infty}^{+\infty} B_1^N(\lambda) e^{i\lambda y} \frac{d\lambda}{\gamma(\lambda) + A} = -u_0^N(y,+0),\ y \in Sl^{(N)}. \end{cases} \tag{16}$$

Next, let us introduce the new unknown function:

$$G_1^N(y) = \int_{-\infty}^{+\infty} B_1^N(\lambda) e^{i\lambda y} d\lambda,\ y \in \Re. \tag{17}$$

The first equation in (16) has the following property:

$$G_1^N(y) = 0, \quad y \in CSl^{(N)}. \tag{18}$$

The Fourier representation for the function (17):

$$B_1^N(\lambda) = \frac{1}{2\pi} \int_{Sl^{(N)}} G_1^N(\xi) e^{-i\lambda\xi} d\xi. \tag{19}$$

By differentiating the second equation in (16) we can write down:

$$\int_{-\infty}^{+\infty} B_1^N(\lambda) i \frac{\lambda}{|\lambda|} e^{i\lambda y} d\lambda +$$

$$+\int_{-\infty}^{+\infty} B_1^N(\lambda)\left[\frac{i\lambda}{\gamma(\lambda)+A} - \frac{i\lambda}{|\lambda|}\right] e^{i\lambda y} d\lambda = \tag{20}$$

$$= -\frac{\partial u_0^N}{\partial y}(y,+0), \quad y \in Sl^{(N)}.$$

Using a parametric representation of the Hilbert operator [1], [2]:

$$\begin{cases} F(\xi) = \int_{-\infty}^{+\infty} C(\lambda) e^{i\lambda\xi} d\lambda, \\ \frac{1}{\pi}\int_{Sl^{(N)}} \frac{F(\xi)d\xi}{\xi - y} = \int_{-\infty}^{+\infty} C(\lambda) i \frac{\lambda}{|\lambda|} e^{i\lambda y} d\lambda. \end{cases} \tag{21}$$

For the equation (20), using an explicit expression (19) for $B_1^{(N)}(\lambda)$ and after simple transformations (17) for the unknown function $G_1^{(N)}(y)$, and finally from (20), we obtain a singular integral equation:

$$\int_{-\infty}^{+\infty} B_1^N(\lambda) \frac{e^{i\lambda\tilde{y}}}{\gamma(\lambda)} d\lambda +$$

$$+\int_{-\infty}^{+\infty} B_1^N(\lambda) e^{i\lambda\tilde{y}} \left(\frac{1}{\gamma(\lambda)+A} - \frac{1}{\gamma(\lambda)}\right) d\lambda = \tag{22}$$

$$= -u_0^N(\tilde{y},+0), \quad \tilde{y} \in Sl^{(N)}.$$

where $K_1^N(y,\xi)$, $f_1^N(y)$ are known functions.

Further, we can write down a supplementary condition for (22):

$$\int_{-\infty}^{+\infty} B_1^N(\lambda) \frac{e^{i\lambda\tilde{y}}}{\gamma(\lambda)} d\lambda +$$

$$+\int_{-\infty}^{+\infty} B_1^N(\lambda) e^{i\lambda\tilde{y}} \left(\frac{1}{\gamma(\lambda)+A} - \frac{1}{\gamma(\lambda)}\right) d\lambda = \tag{23}$$

$$= -u_0^N(\tilde{y},+0), \quad \tilde{y} \in Sl^{(N)}.$$

where $\tilde{y}$ are centers of the slits.

Using the parametric representation [1], [2]:

$$\begin{cases} U(\xi) = \int_{-\infty}^{+\infty} C(\lambda) e^{i\lambda\xi} d\lambda, \\ \frac{i}{2}\int_{-\infty}^{+\infty} H_0^{(1)}(\kappa|\xi - y|) U(\xi) d\xi = \int_{-\infty}^{+\infty} \frac{C(\lambda)}{\sqrt{\lambda^2 - \kappa^2}} e^{i\lambda y} d\lambda. \end{cases} \tag{24}$$

and with the help of (23), (24) we have obtained a supplementary conditions for SIE (22):

$$\frac{i}{2}\int_{-\infty}^{+\infty} H_0^{(1)}(\kappa|\xi - \tilde{y}|) G_1^N(\xi) d\xi +$$

$$+\frac{1}{\pi}\int_{Sl^{(N)}} G_1^N(\xi)\left[\frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{1}{\gamma(\lambda)+A} - \frac{1}{\gamma(\lambda)}\right) e^{i\lambda(\tilde{y}-\xi)} d\lambda\right] d\xi, \tag{25}$$

$$\tilde{y} \in Sl^{(N)}.$$

By virtue of the fact that IEs (25) have a singularity only if $\xi = \tilde{y}$, using the Hankel series expansions [8] and with the help of Bessel function of 1-st kind, we can separate the singularity from (25). All the remaining equations can be then written in the form of remainder series which do not include a singularity. Thus, the supplementary conditions have the form:

$$\frac{1}{\pi}\int_{S^{(N)}} \ln|\xi - \tilde{y}| G_1^N(\xi) d\xi +$$

$$+\int_{S^{(N)}} G_1^N(\xi) K_2^N(\xi,\tilde{y}) d\xi = f_2^N(\tilde{y}), \tag{26}$$

$$\tilde{y} \in Sl^{(N)},$$

where $K_2^N(\tilde{y},\xi)$, $f_2^N(\tilde{y})$ are known functions.

By denoting

$$B_2^N(\lambda) = \left[C_+^N(\lambda) + C_-^N(\lambda)\right](\gamma(\lambda)+A), \tag{27}$$

We can write down a coupled integral equation (14) in the form:

$$\begin{cases} \int_{-\infty}^{+\infty} B_2^N(\lambda) e^{i\lambda y} d\lambda = 0, \ y \in CSl^{(N)}, \\ \int_{-\infty}^{+\infty} B_2^N(\lambda) e^{i\lambda y} d\lambda - A\int_{-\infty}^{+\infty} \frac{B_2^N(\lambda)}{\gamma(\lambda)+A} e^{i\lambda y} d\lambda = \\ = \frac{\partial u_0^N}{\partial z}(y,+0), \ y \in Sl^{(N)}. \end{cases} \tag{28}$$

A new unknown function for (28) is introduced:

$$G_2^N(y) = \int_{-\infty}^{+\infty} B_2^N(\lambda) e^{i\lambda y} d\lambda, \ y \in \Re, \tag{29}$$

and suggests the following property of the first equation in (28): $G_2^N(y) = 0, \ y \in CS^{(N)}$.

The Fourier representation for function (29) is:

$$B_2^N(\lambda) = \frac{1}{2\pi}\int_{Sl^{(N)}} G_2^N(\xi) e^{-i\lambda\xi} d\xi. \tag{30}$$

Using (30) and the parametric representation (24) we obtain the Fredholm equation of the 2-nd kind from the second integral equation in (28):

$$G_2^N(y)-\frac{Ai}{2}\int_{Sl^{(N)}}H_0^{(1)}\left(\kappa|y-\xi|\right)G_2^N(\xi)d\xi-$$

$$-\frac{A}{\pi}\int_{Sl^{(N)}}K_{22}^N(y,\xi)G_2^N(\xi)d\xi= \tag{31}$$

$$=f_3^N(y),\quad y\in Sl^{(N)},$$

where $K_{22}^N(y,\xi)$, $f_3^N(y)$ are known functions.

Considering the asymptotic expansion of the Hankel function and after several simple transformations the formula (31) can be written in the form:

$$G_2^N(y)-\frac{A}{\pi}\int_{Sl^{(N)}}\ln|y-\xi|G_2^N(\xi)d\xi+$$

$$+\frac{A}{\pi}\int_{Sl^{(N)}}K_2^N(y,\xi)G_2^N(\xi)d\xi= \tag{32}$$

$$=f_3^N(y),\quad y\in Sl^{(N)},$$

where $K_2^N(y,\xi)$, $f_3^N(y)$ are known functions.

Finally, we obtain a system of SIEs with supplementary conditions and the Fredholm equation of the 2-nd kind:

$$\begin{cases}\frac{1}{\pi}\int_{Sl^{(N)}}\frac{G_1^N(\xi)}{\xi-y}d\xi+\frac{1}{\pi}\int_{Sl^{(N)}}G_1^N(\xi)K_1^N(y,\xi)d\xi=\\ =f_1^N(y),\quad y\in Sl^{(N)},\\ \frac{1}{\pi}\int_{Sl^{(N)}}\ln|\tilde{y}-\xi|G_1^N(\xi)d\xi+\\ +\frac{1}{\pi}\int_{Sl^{(N)}}G_1^N(\xi)K_2^N(\tilde{y},\xi)d\xi=\\ =f_2^N(\tilde{y}),\quad \tilde{y}\in Sl^{(N)},\\ G_2^N(y)-\frac{A}{\pi}\int_{Sl^{(N)}}\ln|y-\xi|G_2^N(\xi)d\xi+\\ +\frac{A}{\pi}\int_{Sl^{(N)}}G_2^N(\xi)K_2^N(y,\xi)d\xi=\\ =f_2^N(y),\quad y\in Sl^{(N)},\end{cases} \tag{33}$$

where

$$f_1^N(y)=-\frac{\partial u_0^N}{\partial y}(y,+0),\ f_2^N(\tilde{y})=u_0^N(\tilde{y},+0),$$

$$f_3^N(y)=\frac{\partial u_0^N}{\partial z}(y,+0),$$

$$K_1^N(y,\xi)=\frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{i\lambda}{\gamma(\lambda)+A}-\frac{i\lambda}{|\lambda|}\right)e^{i\lambda(y-\xi)}d\lambda,$$

$$K_{22}^N(\tilde{y},\xi)=\frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{1}{\gamma(\lambda)+A}-\frac{1}{\gamma(\lambda)}\right)e^{i\lambda(\tilde{y}-\xi)}d\lambda.$$

$$K_2^N(\tilde{y},\xi)=-\frac{\pi i}{2}H_0^{(1)}\left(\kappa|\xi-\tilde{y}|\right)-\ln|\xi-\tilde{y}|-K_{22}^N(\tilde{y},\xi).$$

Let us introduce the restrictions of functions

$$f_i^N(y), i=1,3,\ G_i^N(\xi), i=1,2,\ f_2^N(\tilde{y}),\ G_2^N(y),$$

on the intervals $S_q^{(N)}=\left(\alpha_q^N,\beta_q^N\right)$, $q=\overline{1,2^N-1}$,

$$f_i^N(y)\Big|_{y\in Sl_p^{(N)}}=f_{i,p}^N(y),\quad G_i^N(\xi)\Big|_{\xi\in Sl_q^{(N)}}=G_{i,q}^N(\xi),$$

$$f_2^N(\tilde{y})\Big|_{\tilde{y}\in Sl_p^{(N)}}=f_{2,p}^N(\tilde{y}),\quad G_2^N(y)\Big|_{y\in Sl_p^{(N)}}=G_{2,p}^N(y). \tag{34}$$

The Meixner edge condition will be satisfied if functions

$$G_{i,q}^N(\xi), i=1,2,\ q=\overline{1,2^N-1},\ G_{2,p}^N(y)$$

are represented as:

$$G_{i,q}^N(\xi)=\frac{v_{i,q}^N(\xi)}{\sqrt{\left(\beta_q^N-\xi\right)\left(\xi-\alpha_q^N\right)}},\ \xi\in Sl_q^{(N)},$$

$$G_{2,p}^N(y)=\frac{v_{2,p}^N(y)}{\sqrt{\left(\beta_p^N-y\right)\left(y-\alpha_p^N\right)}},\ y\in Sl_p^{(N)}. \tag{35}$$

Thus (33) can be written in the form of $p=\overline{1,2^N-1}$:

$$\begin{cases}\frac{1}{\pi}\int_{\alpha_p^N}^{\beta_p^N}\frac{v_{1,p}^N(\xi)}{\xi-y}\frac{d\xi}{\sqrt{\left(\beta_p^N-\xi\right)\left(\xi-\alpha_p^N\right)}}+\\ +\frac{1}{\pi}\sum_{q=1}^{2^N-1}\int_{\alpha_q^N}^{\beta_q^N}\frac{M_1^N(y,\xi)v_{1,q}^N(\xi)}{\sqrt{\left(\beta_q^N-\xi\right)\left(\xi-\alpha_q^N\right)}}d\xi=\\ =f_{1,p}^N(y),\quad y\in Sl_p^{(N)},\\ \frac{1}{\pi}\sum_{q=1}^{2^N-1}\int_{\alpha_q^N}^{\beta_q^N}\frac{\ln|\xi-\tilde{y}|v_{1,q}^N(\xi)}{\sqrt{\left(\beta_q^N-\xi\right)\left(\xi-\alpha_q^N\right)}}d\xi+\\ +\frac{1}{\pi}\sum_{q=1}^{2^N-1}\int_{\alpha_q^N}^{\beta_q^N}\frac{M_{2,p}^N(\tilde{y},\xi)v_{1,q}^N(\xi)}{\sqrt{\left(\beta_q^N-\xi\right)\left(\xi-\alpha_q^N\right)}}d\xi=\\ =f_{2,p}^N(\tilde{y}),\quad \tilde{y}\in Sl_p^{(N)},\\ \frac{v_{2,p}^N(y)}{\sqrt{\left(\beta_p^N-y\right)\left(y-\alpha_p^N\right)}}-\frac{A}{\pi}\int_{\alpha_p^N}^{\beta_p^N}\frac{\ln|y-\xi|v_{2,p}^N(\xi)}{\sqrt{\left(\beta_p^N-\xi\right)\left(\xi-\alpha_p^N\right)}}d\xi+\\ +\frac{1}{\pi}\sum_{q=1}^{2^N-1}\int_{\alpha_q^N}^{\beta_q^N}\frac{M_3^N(y,\xi)v_{2,q}^N(\xi)}{\sqrt{\left(\beta_q^N-\xi\right)\left(\xi-\alpha_q^N\right)}}d\xi=\\ =f_{3,p}^N(y),\ y\in Sl_p^{(N)},\end{cases} \tag{36}$$

where

$$M_1^N(y,\xi)=\begin{cases} K_1^N(y,\xi), \quad y\in Sl_p^{(N)}, \\ \xi\in Sl_p^{(N)},\ p=q, \\ \\ \frac{1}{\xi-y}+K_1^N(y,\xi), \quad y\in Sl_p^{(N)}, \\ \xi\in Sl_q^{(N)},\ p\neq q, \end{cases}$$

$$M_2^N(\tilde{y},\xi)=\begin{cases} K_2^N(\tilde{y},\xi), \quad \tilde{y}\in S_p^{(N)}, \\ \xi\in Sl_p^{(N)},\ p=q, \\ \\ \ln\left|\tilde{y}-\xi\right|+K_2^N(\tilde{y},\xi), \\ \tilde{y}\in Sl_p^{(N)},\ \xi\in Sl_q^{(N)},\ p\neq q. \end{cases}$$

$$M_3^N(y,\xi)=\begin{cases} AK_2^N(y,\xi), \quad y\in Sl_p^{(N)}, \\ \xi\in Sl_p^{(N)},\ p=q, \\ \\ -A\ln\left|y-\xi\right|+AK_2^N(y,\xi), \\ y\in Sl_p^{(N)},\ \xi\in Sl_q^{(N)},\ p\neq q. \end{cases}$$

Consider a standard interval of (-1,1) and use it to represent the intervals $S_q^{(N)}=\left(\alpha_q^N,\beta_q^N\right),\ q=1,\ldots,2^N-1,$ such that

$$g_q^{(N)}\ :\ (-1,1)\ \to\left(\alpha_q^N,\beta_q^N\right)\ :$$

$$t\ \to\ g_q^{(N)}(t)=\frac{\beta_q^N-\alpha_q^N}{2}t+\frac{\beta_q^N+\alpha_q^N}{2}.$$

By making a substitution for $q=1,\ldots,2^N-1,$

$$\begin{gathered} \xi=g_q^{(N)}(t),\quad y=g_p^{(N)}(t_0),\quad \tilde{y}=g_p^{(N)}(\tilde{t}_0), \\ t,t_0,\tilde{t}_0\in[-1,1],\quad \xi\in\left(\alpha_q^N,\beta_q^N\right), \\ y,\tilde{y}\in\left(\alpha_p^N,\beta_p^N\right) \end{gathered} \tag{38}$$

and then for $i=1,2,\ |t|<1,\ |t_0|<1$:

$$\begin{gathered} G_{i,q}^N\left(g_q^{(N)}(t)\right)=\frac{2v_{i,q}^N\left(g_q^{(N)}(t)\right)}{\left(\beta_q^N-\alpha_q^N\right)\sqrt{1-t^2}}, \\ G_{2,p}^N\left(g_p^{(N)}(t_0)\right)=\frac{2v_{2,p}^N\left(g_p^{(N)}(t_0)\right)}{\left(\beta_p^N-\alpha_p^N\right)\sqrt{1-t_0^2}}, \end{gathered} \tag{39}$$

and finally denoting

$$\begin{gathered} w_{i,q}^N(t)=v_{i,q}^N\left(g_q^{(N)}(t)\right), i=1,2, \\ w_{2,p}^N(t_0)=v_{2,q}^N\left(g_p^{(N)}(t_0)\right), \\ \hat{f}_{i,p}^N(t_0)=f_{i,p}^N\left(g_p^{(N)}(t_0)\right), i=1,3, \\ \\ \hat{f}_{2,p}^N(\tilde{t}_0)=f_{2,p}^N\left(g_p^{(N)}(\tilde{t}_0)\right), \\ Q_1^N(t_0,t)=M_1^N\left(g_p^{(N)}(t_0),g_q^{(N)}(t)\right), \end{gathered}$$

$$Q_2^N(\tilde{t}_0,t)=\begin{cases} \ln\left|\frac{\beta_p^N-\alpha_p^N}{2}\right|+M_2^N\left(g_p^{(N)}(\tilde{t}_0),g_q^{(N)}(t)\right), \\ p=q, \\ \\ M_2^N\left(g_p^{(N)}(\tilde{t}_0),g_q^{(N)}(t)\right), \quad p\neq q, \end{cases}$$

$$Q_3^N(t_0,t)=\begin{cases} \ln\left|\frac{\beta_p^N-\alpha_p^N}{2}\right|+M_3^N\left(g_p^{(N)}(t_0),g_q^{(N)}(t)\right), \\ p=q, \\ \\ M_3^N\left(g_p^{(N)}(t_0),g_q^{(N)}(t)\right), \quad p\neq q. \end{cases}$$

We have obtained a system of SIEs with supplementary conditions and the Fredholm equation of the 2-nd kind on the standard interval for $|t_0|<1, |\tilde{t}_0|<1, p=\overline{1,2^N-1}$:

$$\begin{cases} \frac{1}{\pi}\frac{2}{\beta_p^N-\alpha_p^N}\int\limits_{-1}^{1}\frac{w_{1,p}^N(t)}{t-t_0}\frac{dt}{\sqrt{1-t^2}}+ \\ +\frac{1}{\pi}\sum\limits_{q=1}^{2^N-1}\int\limits_{-1}^{1}Q_1^N(t_0,t)w_{1,q}^N(t)\frac{dt}{\sqrt{1-t^2}}= \\ =\hat{f}_{1,p}^N(t_0), \\ \frac{1}{\pi}\int\limits_{-1}^{1}\ln\left|t-\tilde{t}_0\right|w_{1,p}^N(t)\frac{dt}{\sqrt{1-t^2}}+ \\ +\frac{1}{\pi}\sum\limits_{q=1}^{2^N-1}\int\limits_{-1}^{1}Q_2^N(\tilde{t}_0,t)w_{1,q}^N(t)\frac{dt}{\sqrt{1-t^2}}= \\ =\hat{f}_{2,p}^N(\tilde{t}_0), \\ \frac{2}{\beta_p^N-\alpha_p^N}\frac{w_{2,p}^N(t_0)}{\sqrt{1-t_0^2}}-\frac{A}{\pi}\int\limits_{-1}^{1}\ln\left|t-t_0\right|\frac{w_{2,p}^N(t)dt}{\sqrt{1-t^2}}+ \\ +\frac{1}{\pi}\sum\limits_{q=1}^{2^N-1}\int\limits_{-1}^{1}Q_3^N(t_0,t)\frac{w_{2,q}^N(t)dt}{\sqrt{1-t^2}}=\hat{f}_{3,p}^N(t_0), \end{cases} \tag{40}$$

where $Q_i^N(t_0,t)$, $\hat{f}_{i,p}^N(t_0), i=1,3,$ $Q_2^N(\tilde{t}_0,t)$, $\hat{f}_{2,p}^N(\tilde{t}_0)$ are known functions.

IV. Discrete mathematical model

We have built a discrete mathematical model of a system of SIEs with the supplementary conditions and Fredholm equation of the 2-nd kind based on a mathematical model (40), and discretized the boundary IEs (40) with the help of specific quadrature formulas [4]. Then we have interpolated the unknown functions

$w_{i,q,(n-1)}^N(t),\ q=\overline{1,2^N-1},$

by Lagrange polynomials

$w_{i,q,(n-1)}^N(t),\ q=\overline{1,2^N-1},$

in the nodes, which are the nulls of Chebyshev polynomials of the 1-st kind. Finally, we have obtained a system for approximate solutions and a system of linear algebraic equations (SLAE) for

$$\begin{cases}
\frac{1}{n}\frac{2}{\beta_p^N-\alpha_p^N}\sum\limits_{k=1}^{n}\frac{w_{1,p,(n-1)}^N(t_k^n)}{t_k^n-t_{0j}^n}+ \\
+\frac{1}{n}\sum\limits_{q=1}^{2^N-1}\sum\limits_{k=1}^{n}Q_1^N(t_{0j}^n,t_k^n)w_{1,q,(n-1)}^N(t_k^n)= \\
=\hat{f}_{1,p}^N(t_{0j}^n),\quad j=\overline{1,n-1}, \\
-\frac{1}{n}\sum\limits_{k=1}^{n}w_{1,p,(n-1)}^N(t_k^n)\left[\ln 2+2\sum\limits_{r=1}^{n-1}T_r\left(\tilde{t}_0\right)\frac{T_r\left(t_k^n\right)}{r}\right]+ \\
+\frac{1}{n}\sum\limits_{q=1}^{2^N-1}\sum\limits_{k=1}^{n}Q_2^N(\tilde{t}_0,t_k^n)w_{1,q,(n-1)}^N(t_k^n)= \\
=\hat{f}_{2,p}^N(\tilde{t}_0),\ \ j=n, \\
\frac{2}{\beta_p^N-\alpha_p^N}\frac{w_{2,p}^N\left(t_j^N\right)}{\sqrt{1-(t_j^n)^2}}+\frac{A}{n}\sum\limits_{k=1}^{n}w_{2,p,(n-1)}^N(t_k^n)\times \\
\times\left[\ln 2+2\sum\limits_{r=1}^{n-1}T_r\left(t_j^n\right)\frac{T_r\left(t_k^n\right)}{r}\right]+ \\
+\frac{1}{n}\sum\limits_{q=1}^{2^N-1}\sum\limits_{k=1}^{n}w_{2,q,(n-1)}^N(t_k^n)Q_3^N(t_j^n,t_k^n)= \\
=\hat{f}_{3,p}^N(t_j^n),\quad j=\overline{1,n},
\end{cases} \tag{41}$$

where

$t_k^n=\cos(\frac{2k-1}{2n}\pi), k=1,\dots,n$,

are the nulls of Chebyshev polynomial of the 1-st kind of the n-th degree,

$t_{0j}^n=\cos(\frac{j}{n}\pi),\ j=1,\dots,n-1,$

are the nulls of Chebyshev polynomial of the 2-nd kind of the (n-1) degree.

By solving the SLAE (41) we find the values of unknown functions in the node points and calculate the unknown coefficients (19), (30) to obtain the scattered and diffracted fields. The discretization of these coefficients has been done:

$$C_-^N(\lambda)=\frac{1}{4n}\sum_{q=1}^{2^N-1}\sum_{k=1}^{n}\left[w_{1,q,(n-1)}^N\left(g_q^{(N)}\left(t_k^n\right)\right)+w_{2,q,(n-1)}^N\left(g_q^{(N)}\left(t_k^n\right)\right)\right]\times \tag{42}$$
$$\times\left(\frac{e^{-i\lambda g_q^{(N)}\left(t_k^n\right)}-1}{\gamma(\lambda)+A}\right),$$

$$C_+^N(\lambda)=\frac{1}{4n}\sum_{q=1}^{2^N-1}\sum_{k=1}^{n}\left[w_{2,q,(n-1)}^N\left(g_q^{(N)}\left(t_k^n\right)\right)-w_{1,q,(n-1)}^N\left(g_q^{(N)}\left(t_k^n\right)\right)\right]\times \tag{43}$$
$$\times\left(\frac{e^{-i\lambda g_q^{(N)}\left(t_k^n\right)}-1}{\gamma(\lambda)+A}\right).$$

Thus, using the asymptotic representation of the Hankel function we derive the expressions for the radiation patterns of scattered and diffracted fields in the far zone:

$$D_\pm(\varphi)=\lim_{r\to\infty}\frac{u_\pm^N(r,\varphi)}{\sqrt{\frac{2}{\pi\kappa r}}e^{i\left(\kappa r-\frac{\pi}{4}\right)}},\ r=\sqrt{y^2+z^2},$$
$$H_0^{(1)}(\kappa r)\sim\sqrt{\frac{2}{\pi\kappa r}}e^{i\left(\kappa r-\frac{\pi}{4}\right)},$$
$$u_+^N(r,\varphi)\sim\int_{-\kappa}^{\kappa}C_+^N(\lambda)e^{ir\left(\lambda\cos\varphi+\sqrt{\kappa^2-\lambda^2}\sin\varphi\right)}d\lambda,$$
$$u_-^N(r,\varphi)\sim\int_{-\kappa}^{\kappa}C_-^N(\lambda)e^{ir\left(\lambda\cos\varphi-\sqrt{\kappa^2-\lambda^2}\sin\varphi\right)}d\lambda. \tag{44}$$

Using the results of [1], [2], [8], we can evaluate a convergence rate of the approximate solutions to the exact ones in the Hilbert metric and in the uniform metric for physical quantities.

V. NUMERICAL EXPERIMENTS

A numerical experiment with the help of an efficient DSM [1], [5], [9] has been performed. A few results shows in [7]. Figs. 4 show radiation patterns (RPs) of the scattered field in the far zone. The RPs are obtained from the numerical solution of SLAE (41) and the calculated coefficients (42), (43). Both plots in Fig. 4 show the dependence of the RPs on the *N*-th order of pre-Cantor grating and the impedance of strips material: Niobium (Ni), Stannum (Sn) and Plumbum (Pb).

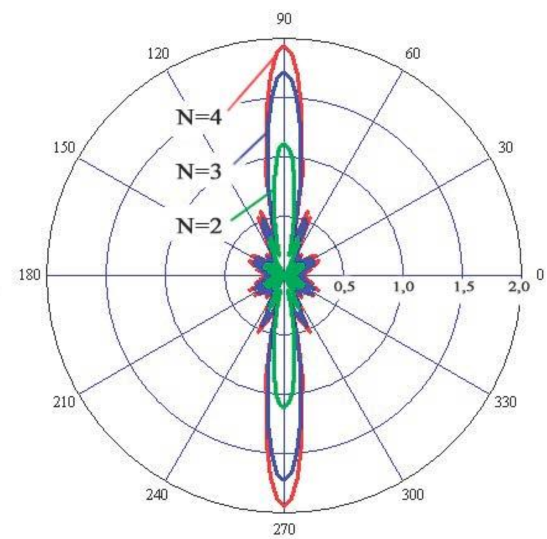


Fig. 6. Dependence of RP $D_{+}(\varphi)$ on N in far field where material of grating is Ni, T/T_{cr}= 0.7 (T_{cr}=9.25^0 K), f=11.2GHz, $\alpha=0^0$, $Zc=1.99122\cdot10^{-4}+4.52763\cdot10^{-3}i$, l=0.05m.

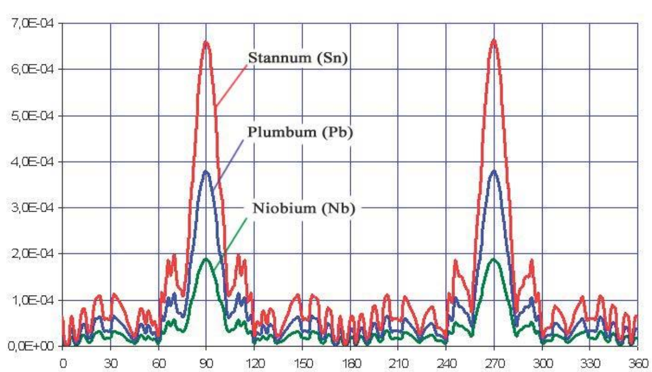


Fig. 7. Dependence of RP $D_{+}(\varphi)$ on impedance in far field of material Zc (Pb, Sn, Ni) where, T/T_{cr}= 0.7 (T_{cr}=9.25^0 K), f=11.2GHz, $\alpha=0^0$, l=0.05m, N=3.

As for the radiation properties, Fig. 6 and Fig. 7 give an overview of the diffraction structure behavior. Note here, that these radiation patterns are omnidirectional.

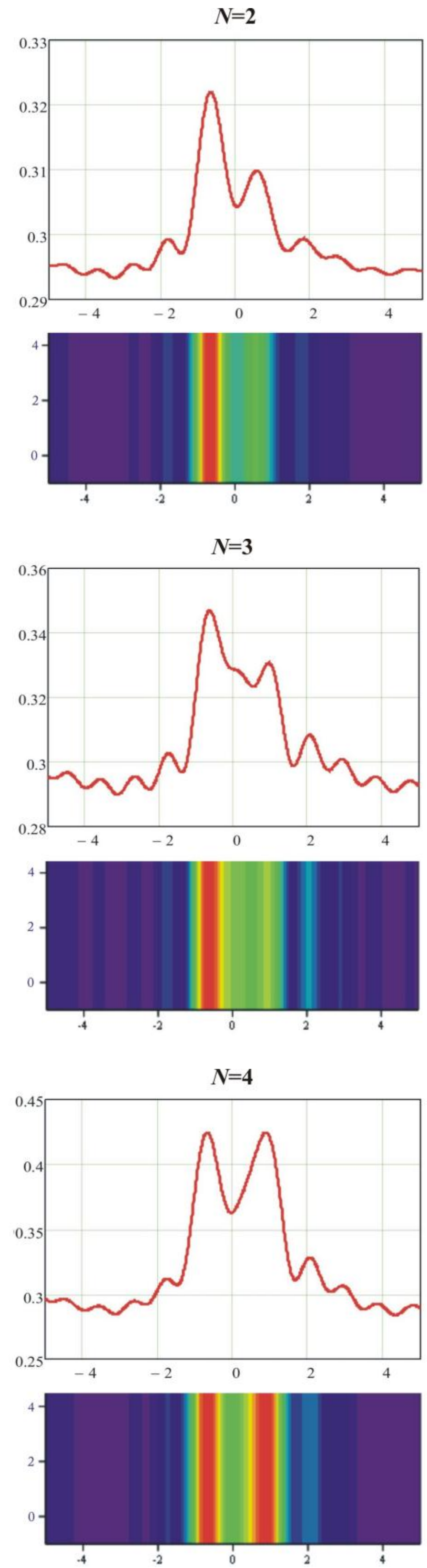
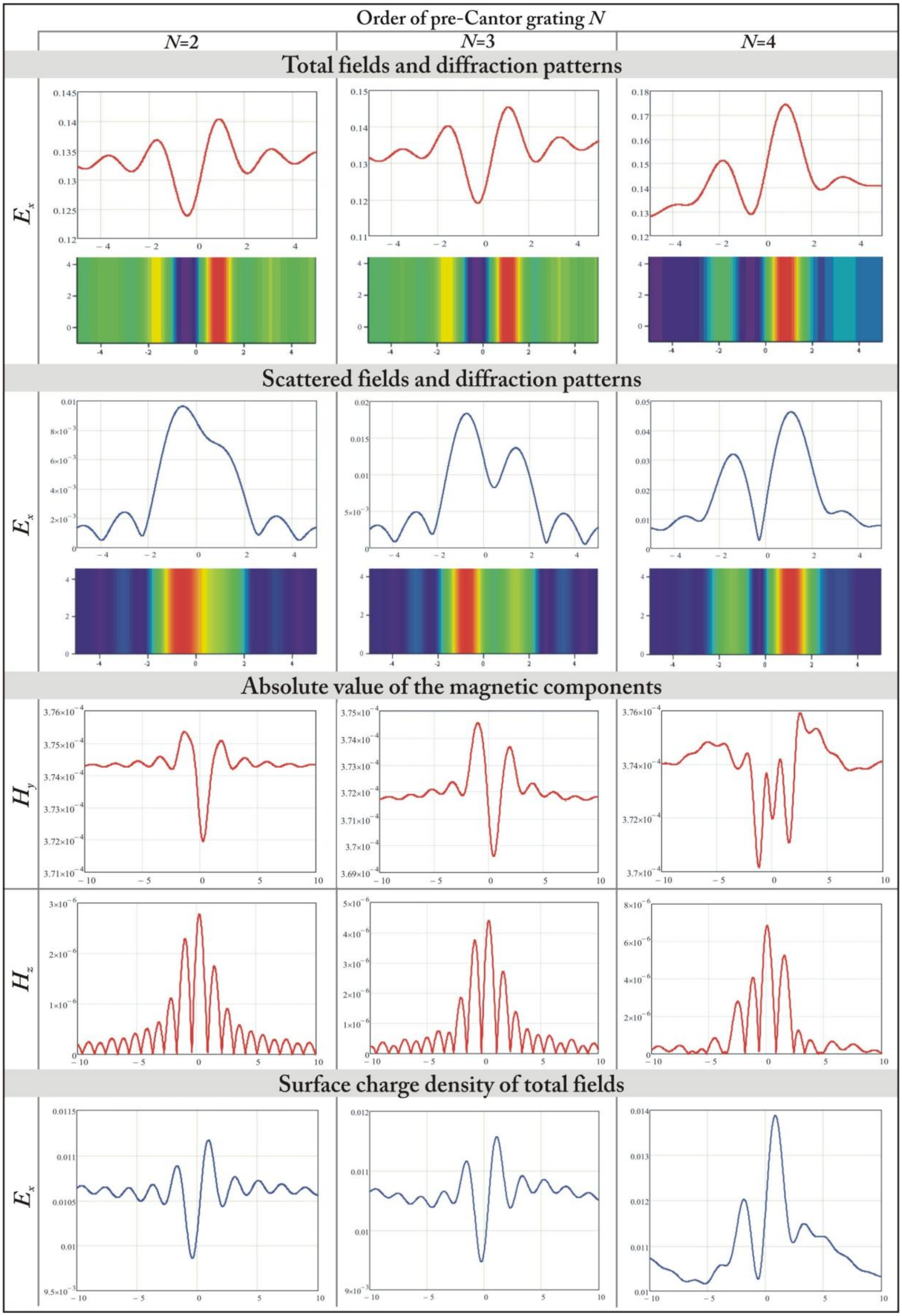


Fig. 8. Near zone of the total fields and diffraction patterns for strips from Constantan where N=2,3,4, f=2GHz, l=0,1 m, $\alpha=45^0$.

TABLE NEAR FIELDS FOR CONSIDERED DIFFRACTION STRUCTURE FOR CONSTANTAN STRIPS.



The main results of this work are the investigation of diffraction problem TE wave in near and far fields. If we look at the Fig. 8 we can see that the maximum evaluate of the total fields have increases with *N*. These results are calculated for frequency 2 GHz and all next results for 0.9 GHz, *l*=0,1 m, $\alpha=45^0$. Some interesting phenomena are depicted in Table: the total, scattered fields and its diffraction patterns of electric component E_x , the absolute value of the magnetic components H_y , H_z . In additions to the aforesaid has been calculated a surface charge density of total fields for different values of the order of pre-Cantor grating at three last figures in Table.

VI. Conclusions

The overall aim of this paper was to build a discrete mathematical model of the diffraction problem on pre-Cantor grating and perform a broad numerical experiment. In the previous works on this topic, this problem had not been carried out to a numerical solution and thus it is new and actual. From the mathematical point of view the problem has been reduced to SIEs with supplementary conditions and the Fredholm IE of the 2-nd kind. The singularities in the kernels of considered IEs have been avoided with the help of specific quadrature formulas and DSM. To summarize, we can conclude that the aim of this work has been achieved.

As a straightforward follow-up of this work we consider the development of a discrete mathematical model of the diffraction problem on pre-Cantor grating with a flat screen reflector or a screened dielectric layer, supported by a broad range of numerical experiments.

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