Spectral properties of internal waves in rotating stratified fluid for various boundary value problems

A. Giniatoulline, and T. Castro

***Abstract*—** We investigate the mathematical properties of rotating exponentially stratified fluid for bounded three-dimensional domains, such as existence and uniqueness of the solutions, the structure of the spectrum of normal vibrations, the relationship between the essential spectrum and non-uniqueness of the solutions, etc. We consider the both cases of viscous and inviscid fluid, as well as compressible and uncompressible cases. For various boundary problems of viscous barotropic fluid, we prove that the essential spectrum consists of isolated real points. For inviscid fluid, we prove that the essential spectrum is an interval of the imaginary axis which is symmetrical with respect to zero. Since the considered model of the stratified fluid corresponds to a distribution of the initial density in a homogeneous gravitational field, the obtained results may find their application in the models of the Atmosphere and the Ocean which consider the rotation of the Earth over the vertical axis. The novelty of this research is to consider simultaneously the effects of rotation and stratification, which has been studied separately in previous works. Three different techniques are used to localize and investigate the spectrum: the theory of the operator bundles, the verification of ellipticity in sense of Douglis-Nirenberg and the Lopatinski conditions, and the construction of the explicit Weyl sequence for the essential spectrum, which is our main result.

***Keywords*—** Essential spectrum, partial differential equations, Sobolev spaces, stratified fluid, rotational fluid.

I. INTRODUCTION

LET us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^1 and the following system of fluid dynamics

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$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 - \nu\Delta u_1 - \nu\beta \dfrac{\partial}{\partial x_1}\left(\operatorname{div}\vec{u}\right) + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 - \nu\Delta u_2 - \nu\beta \dfrac{\partial}{\partial x_2}\left(\operatorname{div}\vec{u}\right) + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} - \nu\Delta u_3 - \nu\beta \dfrac{\partial}{\partial x_3}\left(\operatorname{div}\vec{u}\right) + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2 \dfrac{\partial p}{\partial t} + \operatorname{div}\vec{u} = 0 \qquad x\in\Omega, \quad t\geq 0. \end{cases} \tag{1}$$

Here $\vec{u} = \left(u_1, u_2, u_3\right)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure and $\rho(x,t)$ is the dynamic density. In this model, the stationary distribution of density is described by the function e^{-Nx_3}, so N is a positive constant. For the compressibility coefficient α, the kinematic viscosity coefficient ν, and the volume (bulk) viscosity coefficient β we assume $\alpha > 0$, $\nu > 0$, $\beta \geq 0$.

We also suppose that ω is a positive constant so that the system (1) describes linear motions of compressible stratified barotropic viscous fluid which is rotating over the vertical axis with a constant angular velocity $\vec{\omega} = \left(0, 0, \omega\right)$.

We consider as well the inviscid case of the model described by (1):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2 \dfrac{\partial p}{\partial t} + \operatorname{div}\vec{u} = 0 \qquad x\in\Omega, \quad t\geq 0. \end{cases} \tag{2}$$

For inviscid case, the equations (1) are deduced in [1]-[3]. For viscous compressible fluid, the system (1) is deduced, for example, in [4].

May we observe that, despite an extensive study of stratified flows from the physical point of view (see, for example, [5]-

[10]), there have been relatively few works considering the mathematical aspect of the problem.

The fundamental solution of internal waves in incompressible stratified flows was first constructed in [11].

The mathematical properties of rotational inviscid fluid were studied in various works of S. Sobolev, starting from the famous paper [12].

The studying of qualitative properties of solutions of PDE systems modeling rotational compressible flows was started by V. Maslennikova in [13] and was developed later in her future works.

The solutions for a Cauchy problem for the homogeneous system (1) with $\omega = 0$, for the viscous case of intrusion for homogeneous system were constructed in [14], and the uniqueness of the homogeneous Cauchy problem for the viscous case was studied in [15]. For the inviscid incompressible case, the spectral properties of the differential operator of (1) with $\omega = 0$ were considered in [16]-[19], and the case of inviscid compressible fluid was first considered in [20]. Particularly, for $\nu = 0$ and $\beta = 0$, for the case of non-rotational compressible fluid ($\alpha > 0$), in [20] it was proved that the essential spectrum of operator of normal vibrations is the interval of the imaginary axis $[-iN, iN]$.

For non-compressible inviscid stratified fluid ($\alpha = \beta = \nu = 0$), in [16], [17], [19] it was proved that the essential spectrum is the same interval of the imaginary axis $[-iN, iN]$, outside of which there can be only the eigenvalues of finite multiplicity.

For rotational inviscid fluid, the corresponding result ($[-i\omega, i\omega]$) was proved in [21], [27].

However, the case of the viscous compressible barotropic fluid has not been considered previously. Neither has been considered the case of rotating stratified (either inviscid or viscous, compressible or non-compressible) fluid. The novelty of these problems, the explicit relationship between the parameters of rotation and stratification in the description of the spectral properties and their possible applications to the dynamics of the Atmosphere and the Ocean was the motivation of this paper, some partial results of which were reported in [28], [29].

II. Statement of the Problem and Preliminaries

Let us consider first the system (2) with $\alpha = 0$ and the boundary condition

$$\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0. \tag{3}$$

Let us define the following functional spaces:

$$G_2(\Omega) = \left\{ \vec{u}(x) \in L_2(\Omega) : \vec{u}(x) = \nabla\varphi,\ \varphi \in W_2^1(\Omega) \right\},$$

$$J_0(\Omega) = \left\{ \vec{u}(x) : \vec{u} \in C^1(\Omega),\ \operatorname{div} \vec{u} = 0,\ \vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0 \right\},$$

and let $J_2(\Omega)$ be a closure of $J_0(\Omega)$ in the norm of $L_2(\Omega)$.

In what follows in this section, we will use the techniques and ideas which were first introduced by S. Sobolev in [12].

It is proved in [30] that there is valid the decomposition

$$L_2(\Omega) = J_2(\Omega) \oplus G_2(\Omega).$$

Now, if we denote $q = \frac{\partial p}{\partial t}$, $\vec{e}_3 = (0,0,1)$ and use the above decomposition for $L_2(\Omega)$, we can exclude the unknown function ρ from (2) and thus write the system (2) with $\alpha = 0$ in the following way.

$$\begin{cases} \dfrac{\partial^2 \vec{u}}{\partial t^2} + \omega \dfrac{\partial}{\partial t}[\vec{e}_3, \vec{u}] + N^2 u_3 \vec{e}_3 + \nabla q = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases} \tag{4}$$

Now, let us denote as P the operator of orthogonal projection of $L_2(\Omega)$ onto $J_2(\Omega)$ and let us consider the following operators:

$$K\vec{u} = P[\vec{e}_3, \vec{u}], \quad L\vec{u} = P\{u_3 \vec{e}_3\}.$$

In this way, if we consider the initial conditions

$$\vec{u}(x,0) = \vec{u}_0(x), \quad \frac{\partial \vec{u}}{\partial t}(x,0) = \vec{u}_1(x), \tag{5}$$

then we can reduce the problem (4), (5) to the following abstract differential equation for the linear operators *K*, *L*:

$$\begin{aligned} &\frac{\partial^2 \vec{u}}{\partial t^2} + \omega K \frac{\partial \vec{u}}{\partial t} + N^2 L \vec{u} = 0 \\ &\vec{u}, \vec{u}_0, \vec{u}_1 \in J_2(\Omega) \end{aligned} . \tag{6}$$

It is easy to see that $\|K\| = \|L\| = 1$.

In addition, we observe that for $\vec{u}, \vec{v} \in J_2(\Omega)$ the relations hold:

$$(\vec{u}, iK\vec{v})_{L_2(\Omega)} = (iK\vec{u}, \vec{v})_{L_2(\Omega)}, \quad (\vec{u}, L\vec{v})_{L_2(\Omega)} = (L\vec{u}, \vec{v})_{L_2(\Omega)},$$

from which we can easily conclude that the operators iK and L are self-adjoint operators in $J_2(\Omega)$. Thus, from [31], [32] we have that the following Theorem is valid:

<u>Theorem 1</u>.

If $\vec{u}_0, \vec{u}_1 \in J_2(\Omega)$ then there exists a unique solution of the problem (3)-(5) such that $\vec{u}(t,x) \in C^\infty[0,\infty) \times J_2(\Omega)$.

Let us consider the problem of what is called *normal oscillations* of considered fluid in the domain Ω, in other terms, we would like to consider the particular solutions of (6) which are represented by the form

$$\vec{u} = \vec{v}(x) \exp\{i\lambda t\},$$

where $\vec{v}(x) \in J_2(\Omega)$ and he value $\vec{v}(x)$ is defined as a solution of the equation

$$\lambda^2 \vec{v} = \lambda i\omega K \vec{v} + N^2 L \vec{v}.$$

If we denote as $B(\lambda)$ the following quadratic bundle of operators

$$B(\lambda) = \lambda^2 I - i\lambda\omega K - N^2 L, \tag{7}$$

then the problem of the spectrum of normal oscillations in $L_2(\Omega)$ can be considered as the problem of the spectrum for the quadratic bundle of operators

$$B(\lambda)\vec{v}=0 \qquad (8)$$

in the functional space $J_2(\Omega)$.

(We recall that the spectrum of the bundle of operators $B(\lambda)v=0$, $v\in H$, is a set of all λ such that for the operator $B(\lambda)$ there is no bounded inverse operator in *H*).

Property 1.

The point $\lambda=0$ is an eigenvalue of infinite multiplicity of the bundle (7). The corresponding eigenfunctions are the elements $\vec{v}(x)\in J_2(\Omega)$ such that $\vec{v}=(v_1,v_2,0)$.

Proof.

For $\lambda=0$ we have that the solutions of the equation $B(0)\vec{v}=0$ coincide with the kernel of the operator L in $J_2(\Omega)$. From the definition of the operator L it follows immediately that its kernel is the subspace of $J_2(\Omega)$ with trivial third component and thus the Property is proved.

Now, for $\lambda\neq 0$ we can write the bundle (7) as follows.

$$\vec{v}=\frac{1}{\lambda}i\omega K\vec{v}+\frac{N^2}{\lambda^2}L\vec{v}. \qquad (9)$$

To localize the spectrum of (9), we consider the elements

$$z=\{\vec{v},\vec{w}\}\in H=J_2(\Omega)\oplus J_2(\Omega),$$

where $\vec{w}=(N/\lambda)L^{1/2}\vec{v}$, being $L^{1/2}$ a square root of L and thus a bounded and self-adjoint operator in $J_2(\Omega)$. Now, if we introduce the matrix operator

$$T=\begin{bmatrix} i\omega K & NL^{1/2} \\ NL^{1/2} & 0\end{bmatrix},$$

then we obtain that the bundle (9) acting in $J_2(\Omega)$ has its equivalent matrix form

$$\lambda z=Tz \qquad (10)$$

for the matrix operator *T* acting in $H=J_2(\Omega)\oplus J_2(\Omega)$.

It is easy to see that, for $\lambda\neq 0$, the spectrum of the bundle (7) coincides with the spectrum of the operator *T*.

Let us observe that the spectrum of the bundle (7) is real and symmetrical with respect to the origin. Since the operator of complex conjugation (we denote it as$(^*)$), is an involution operator in $J_2(\Omega)$ (see[33]), then, from the evident identity

$$B^{-1}(\lambda_0)f=\left(B^{-1}(-\lambda_0)f^*\right)^*,\ f\in J_2(\Omega)$$

we obtain that the existence of a bounded inverse operator for $B(\lambda_0)$ implies the existence of a bounded inverse operator for $B(-\lambda_0)$, and vice versa.

The fact that the spectrum is real is a consequence of the self-adjointness of the operators *K*, *L*.

Evidently, the upper limit of the spectrum of a bounded self-adjoint operator *T* is:

$$M=\sup_{\|z\|_H}(z,Tz)_H.$$

Let us estimate the value M. We have

$$(z,Tz)_H=\omega(\vec{v},iK\vec{v})_{L_2}+N\left(\vec{v},L^{1/2}\vec{w}\right)_{L_2}+N\left(\vec{w},L^{1/2}\vec{v}\right)_{L_2},$$

where

$$\left|(\vec{v},iK\vec{v})_{L_2}\right|=\left|\int_\Omega(v_1\bar{v}_2-\bar{v}_1v_2)\,dx\right|\le\int_\Omega\left(v_1^2+v_2^2\right)dx=\|\vec{v}\|_{L_2}^2-\|v_3\|_{L_2}^2,$$

$$\left|\left(\vec{v},L^{1/2}\vec{w}\right)_{L_2}+\left(\vec{w},L^{1/2}\vec{v}\right)_{L_2}\right|=2\left|\mathrm{Re}\left(L^{1/2}\vec{v},\vec{w}\right)_{L_2}\right|\le$$
$$\le\left\|L^{1/2}\vec{v}\right\|_{L_2}^2+\|\vec{w}\|_{L_2}^2\le\|v_3\|_{L_2}^2+\|\vec{w}\|_{L_2}^2,$$

from which we finally obtain

$$(z,Tz)_H\le\omega\|\vec{v}\|_{L_2}^2+N\|\vec{w}\|_{L_2}^2+(N-\omega)\|v_3\|_{L_2}^2.$$

Since $\|z\|_H^2=\|\vec{v}\|_{L_2}^2+\|\vec{w}\|_{L_2}^2$ and $\|v_3\|_{L_2}^2\le\|\vec{v}\|_{L_2}^2$, then we finally have

$$M\le\max\{N,\omega\}=A. \qquad (11)$$

Let us resume the obtained results as the following

Property 2.

The spectrum of normal oscillations for inviscid uncompressible rotating stratified fluid, defined as the spectrum of quadratic bundle (7) in $J_2(\Omega)$, belongs to the interval of the real axis $[-A,A]$. Moreover, it is symmetrical with respect to the point $\lambda=0$, which is an eigenvalue of infinite multiplicity.

Remark 1.

In the proof of Property 2, we used the separation of varianbles $\vec{u}=\vec{v}(x)\exp\{i\lambda t\}$. If, instead of that, we use the separation of variables in the form $\vec{u}=\vec{v}(x)\exp\{\lambda t\}$, $\lambda\in C$, then the result of Property 2 will be the interval of the imaginary axis $[-iA,iA]$.

Now, let us consider the system (2) with $\alpha\neq 0$ and the boundary condition

$$\vec{u}\cdot\vec{n}\big|_{\partial\Omega}=0\ . \qquad (12)$$

We consider the following problem of normal oscillations

$$\begin{aligned}\vec{u}(x,t)&=\vec{v}(x)e^{-\lambda t}\\ \rho(x,t)&=Nv_4(x)e^{-\lambda t}\\ p(x,t)&=\frac{1}{\alpha}v_5(x)e^{-\lambda t}\quad,\ \lambda\in C.\end{aligned} \qquad (13)$$

We denote $\tilde{v}=(\vec{v},v_4,v_5)$ and write (2) as

$$L\tilde{v}=0 \qquad (14)$$

where $L=M-\lambda I$ and

$$M=\begin{pmatrix} 0 & -\omega & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ \omega & 0 & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}. \qquad (15)$$

Let us denote as M^1 the differential operator (15) corresponding to the boundary conditions (12).

We define the domain of the differential operator M^1 as follows.

$$D(M^1)=\left\{\begin{matrix} \vec{v}\in\left(L_2(\Omega)\right)^3 \ \mid \exists f\in L_2(\Omega): \\ (\vec{v},\nabla\varphi)=(f,\varphi)\ \forall\varphi\in W_2^1(\Omega) \end{matrix}\right\}\times W_2^1(\Omega)\times W_2^1(\Omega).$$

On the other hand, we will consider the system (1) with the boundary conditions

$$\vec{u}\big|_{\partial\Omega}=0 \qquad (16)$$

or

$$\sum_{j=1}^{3}T_{ij}n_j\Big|_{\partial\Omega}=0\ ,\quad i=1,2,3, \qquad (17)$$

where the components of the stress tensor T_{ij} are expressed by

$$T_{ij}=\delta_{ij}\nu(\beta-1)\operatorname{div}\vec{u}+\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-p\delta_{ij}$$

and n_j are the components of the exterior normal $\vec{n}$ to the surface $\partial\Omega$.

For the system (1) we also apply the separation of variables (13), (14) , and thus the matrix M will take the form

$$M=\begin{pmatrix} -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_1^2} & -\nu\beta\frac{\partial^2}{\partial x_1\partial x_2}-\omega & -\nu\beta\frac{\partial^2}{\partial x_1\partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ -\nu\beta\frac{\partial^2}{\partial x_1\partial x_2}+\omega & -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_2^2} & -\nu\beta\frac{\partial^2}{\partial x_2\partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ -\nu\beta\frac{\partial^2}{\partial x_1\partial x_3} & -\nu\beta\frac{\partial^2}{\partial x_2\partial x_3} & -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_3^2} & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix} \qquad (18)$$

Let us denote as M^2 the differential operator (18) with the boundary conditions (16), and, respectively, let M^3 be the operator (18) associated with the boundary conditions (17). We define the domains of these differential operators as follows.

$$D(M^2)=\left\{\begin{matrix} \vec{v}\in\left(\overset{0}{W}{}_2^1(\Omega)\right)^3, v_4\in L_2(\Omega), v_5\in L_2(\Omega): \\ M\tilde{v}\in\left(L_2(\Omega)\right)^5 \end{matrix}\right\},$$

$$D(M^3)=\left\{\begin{matrix} \vec{v}\in\left(W_2^1(\Omega)\right)^3, v_4\in L_2(\Omega), v_5\in L_2(\Omega): \\ M\tilde{v}\in\left(L_2(\Omega)\right)^5,\ \sum_{j=1}^{3}T_{ij}n_j\big|_{\partial\Omega}=0 \end{matrix}\right\},$$

where $\overset{0}{W}{}_2^1(\Omega)$ is a closure of the functional space $C_0^\infty(\Omega)$ in the norm of $W_2^1(\Omega)$.

From the physical point of view, the separation of variables (13) serves as a tool to establish the possibility to represent every non-stationary process described by (1) or (2), as a linear superposition of the normal vibrations. The knowledge of the spectrum of normal vibrations may be very useful for studying the stability of the flows. Also, the spectrum of operators M^1, M^2, M^3 is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operators M^1, M^2, M^3.

In this paper we study first the spectrum of operators M^1, M^2, M^3.

As we have seen in Theorem 1 and Property 2, for the spectral study of the incompressible case of the operator M^1, the theory of operator bundles could be used. For compressible case of operator M^1, as well as for the operators M^2, M^3, we will use a different technique, which is based on the concepts of ellipticity in sense of Douglis-Nirenberg and Lopatinski conditions.

For the operator M^1, additionally, we will prove the property of skew-selfadjointness and construct the explicit Weyl sequence for the essential spectrum. We will find the essential spectrum of the operators M^2, M^3 and localize the sector of the complex plane to which all the eigenvalues belong.

Finally, we will compare the obtained spectral results for stratified rotating fluid with the previous analogous results considering separately the cases of rotation and stratification, either for viscous or for inviscid fluid.

We observe first that the operators M^1, M^2, M^3 are closed operators, and their domains are dense in $\left(L_2(\Omega)\right)^5$.

Let us denote by $\sigma_{ess}(M)$ the essential spectrum of a closed linear operator M. We recall that the essential spectrum

$$\sigma_{ess}(M)=\{\lambda\in C:(M-\lambda I)\ \text{is not of Fredholm type}\},$$

is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity (see [22] ,[23]).

In this way, every spectral point which does not belong to the essential spectrum, is an eigenvalue of finite multiplicity.

To find the essential spectrum of the operator M , we will use the following property (see [24]):

$$\sigma_{ess}(M)=Q\cup S,$$

where

$$Q=\left\{\begin{matrix}\lambda\in C:(M-\lambda I)\text{is not elliptic}\\ \text{in sense of Douglis-Nirenberg}\end{matrix}\right\}$$

and

$$S=\left\{\begin{matrix}\lambda\in C\setminus Q:\text{the boundary conditions of }(M-\lambda I)\\ \text{do not satisfy Lopatinski conditions}\end{matrix}\right\}.$$

For the ellipticity in sense of Douglis-Nirenberg, we will use the definition from [25]. For the Lopatinski conditions, we will use the definition from [24]. Since it is less widely known, we recall it hereafter.

Definition 1.

Let us consider $\xi=(\xi_1,\xi_2,\xi_3)$, $\tilde{\xi}=(\xi_1,\xi_2)$, $L^*(\xi)$ - the matrix of the algebraic complements of the main symbol matrix $\tilde{L}(\xi)$, $G(\xi)$ is the main symbol of the matrix *G(D)* which defines the boundary conditions, $M^+(\tilde{\xi},\tau)=\prod\left(\tau-\tau_j(\tilde{\xi})\right)$, $\tau_j(\tilde{\xi})$ are the roots of the equation $\det\tilde{L}(\tilde{\xi},\tau)=0$ with positive imaginary part.

If the rows of the matrix $G(\tilde{\xi},\tau)\hat{L}(\tilde{\xi},\tau)$ are linearly independent with respect to the module $M^+(\tilde{\xi},\tau)$ for $|\tilde{\xi}|\neq 0$, then we will say that the *conditions of Lopatinski are satisfied* (see [24]).

We also will use the following criterion which is attributed to Weyl ([22],[23]): a necessary and sufficient condition that a real finite value λ be a point of the essential spectrum of a self-adjoint operator M is that there exist a sequence of elements $v_n\in D(M)$ such that

$$\|v_n\|=1\,,\quad v_n\rightharpoonup 0\,,\quad \|(M-\lambda I)v_n\|\to 0\,. \tag{19}$$

III. The Solution of the Problem for the Case of Inviscid Compressible Rotating Stratified Fluid

Theorem 2.

The operator M^1 is skew-selfadjoint.

Proof.

We observe that M^1 can be represented as

$$M_1=M_0+B_\omega+B_N, \tag{20}$$

where

$$B_\omega=\begin{pmatrix}0&-\omega&0&0&0\\ \omega&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix},\quad B_N=\begin{pmatrix}0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&N&0\\ 0&0&-N&0&0\\ 0&0&0&0&0\end{pmatrix}.$$

Since B_ω, B_N are anti-symmetric bounded operators, then it is sufficient to prove the skew-selfadjointness for the operator M_0 with the domain

$$D(M_0)=D(M^1).$$

Let $\tilde{u},\tilde{v}\in D(M_0)$. Integrating by parts, we obtain

$$(M_0\tilde{u},\tilde{v})=-(\tilde{u},M_0\tilde{v})\,.$$

Now, let $\tilde{v}\in D(M_0^*)$. Thus, $\tilde{v}\in L_2(\Omega)$ and there exists $\tilde{f}\in L_2(\Omega)$ such that

$$(M_0\tilde{u},\tilde{v})=(\tilde{u},\tilde{f})\ \text{ for all }\tilde{u}\in D(M_0)\,.$$

Take $\tilde{u}=(0,0,0,0,u_5)$, $u_5\in W_2^1(\Omega)$. Then, we will have

$$(\nabla u_5,\vec{v})=(u_5,f_5)\,.$$

For $\tilde{u}=(u_1,u_2,u_3,0,0)$ we obtain

$$(div\ \vec{u},v_5)=(\vec{u},\vec{f})\,.$$

From the last two relations we conclude that v_5 has a weak gradient from $L_2(\Omega)$ and $v_5\in W_2^1(\Omega)$.

Since M_0 is not acting on the fourth component of the vector $\tilde{u}$, we may consider $u_4=v_4=f_4=0$.

In this way, we have verified that

$$D(M_0^*)\subset D(M_0).$$

The reciprocal inclusion can be proved analogously and thus the theorem is proved.

Theorem 3.

Let $a=\min\{\omega,N\}$, $A=\max\{\omega,N\}$. Then, the essential spectrum of M^1 is the following symmetrical set of the imaginary axis:

$$\{0\}\cup[-iA,-ia]\cup[ia,iA]\,.$$

Moreover, the points $\{0\},\pm\{ia\},\pm\{iA\}$ are eigenvalues of infinite multiplicity.

Proof 1.

According to [25], [27], for operator M in (15), we can choose the numbers $s_i=t_j=0$ for $i,j=1,2,3,4$ and $s_5=t_5=1$. In this way, the main symbol $\tilde{L}(\xi)$ takes the following form:

$$\tilde{L}(\xi)=\begin{pmatrix}-\lambda&-\omega&0&0&\frac{1}{\alpha}\xi_1\\ -\omega&-\lambda&0&0&\frac{1}{\alpha}\xi_2\\ 0&0&-\lambda&N-\lambda&\frac{1}{\alpha}\xi_3\\ 0&0&-N&0&0\\ \frac{1}{\alpha}\xi_1&\frac{1}{\alpha}\xi_2&\frac{1}{\alpha}\xi_3&0&0\end{pmatrix}$$

and thus

$$\det\tilde{L}(\xi)=\frac{\lambda}{\alpha^2}\left[\left(\lambda^2+N^2\right)\left(\xi_1^2+\xi_2^2\right)+\left(\lambda^2+\omega^2\right)\xi_3^2\right]. \tag{21}$$

We can see from (21) that if

$$\lambda \notin \left[\{0\} \cup (-iA, -ia) \cup (ia, iA) \right],$$

then the operator L is elliptic in sense of Douglis-Nirenberg. Now, let us prove that the boundary condition (12) satisfies Lopatinski conditions.

Indeed, if we write the conditions (12) in form

$$G\tilde{u}\big|_{\partial\Omega} = 0,$$

we obtain immediately that

$$G = (n_1, n_2, n_3, 0, 0)$$

and G is a vector row. It can be easily seen that $\hat{L}\left(\tilde{\xi}, \tau\right)$ is a matrix whose size is 5x5, and that $G\hat{L}$ is a non-zero row with five components. In other terms, the Lopatinski condition is satisfied, which completes the Proof 1.

Proof 2. (construction of an explicit Weyl sequence)

From Theorem 2 we know that the spectrum of the operator M^1 belongs to the imaginary axis. Taking into account (21), we consider $\lambda_0 \in \pm(ia, iA)\setminus\{0\}$ and choose a vector $\xi \neq 0$ such that

$$\left(\lambda_0^{\,2} + N^2\right)\left(\xi_1^2 + \xi_2^2\right) + \left(\lambda_0^{\,2} + \omega^2\right)\xi_3^2 = 0.$$

Therefore, there exist the vector η such that

$$\tilde{L}(\xi)\eta = 0. \tag{22}$$

Solving (22) with respect to η, we obtain one of possible solutions:

$$\begin{cases} \eta_1 = \dfrac{\lambda_0\xi_1 - \omega\xi_2}{\alpha\left(\lambda_0^2 + \omega^2\right)}, \quad \eta_2 = \dfrac{\lambda_0\xi_2 + \omega\xi_1}{\alpha\left(\lambda_0^2 + \omega^2\right)}, \\ \eta_3 = \dfrac{\lambda_0\xi_3}{\lambda_0^2 + N^2}, \quad \eta_4 = \dfrac{-N\xi_3}{\alpha\left(\lambda_0^2 + N^2\right)}, \quad \eta_5 = 1. \end{cases}$$

We observe that $\eta_i \neq 0$, $i = 1,2,3,4,5$. Now, let us choose a function $\psi_0 \in C_0^\infty(\Omega)$, $\int\limits_{\|x\|\le 1} \psi_0^2(x)dx = 1$. We fix $x_0 \in \Omega$ and put

$$\psi_k(x) = k^{\frac{3}{2}}\psi_0\left(k(x - x_0)\right), k = 1,2,\ldots$$

We define the Weyl sequence $\tilde{v}^k$ as follows:

$$\begin{cases} v_j^k(x) = \eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k + \dfrac{i}{k^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right), \ j = 1,2,3 \\ v_4^k(x) = \eta_4\psi_k e^{ik^3\langle x,\xi\rangle} \\ v_5^k(x) = -\dfrac{i}{k^3}\psi_k e^{ik^3\langle x,\xi\rangle} \\ \langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2 + x_3\xi_3, \qquad k = 1,2,\ldots \end{cases} \tag{23}$$

Now we will verify that the sequence (23) satisfies all the Weyl conditions (19).

We observe first that, for the functions (23), the weak convergence to zero is evident, due to the weak convergence to zero for the functions $e^{ik^3\langle x,\xi\rangle}$, $k \to \infty$.

To verify that the norms $\tilde{v}^k$ are separated from zero, it is sufficient to prove that at least the norms of one component of the field $\tilde{v}^k$ are separated from zero as $k \to \infty$.

Let us consider the two summands $v_1^k = v_{11}^k + v_{12}^k$, where

$$\begin{cases} v_{11}^k(x) = \eta_1\psi_k e^{ik^3\langle x,\xi\rangle} \\ v_{12}^k(x) = \eta_1\dfrac{i}{k^3\xi_1}\dfrac{\partial\psi_k}{\partial x_1}e^{ik^3\langle x,\xi\rangle} \end{cases}.$$

From (23) we can easily see that

$$\|\psi_k\|_{L_2} = 1, \quad \left\|\frac{\partial\psi_k}{\partial x_j}\right\|_{L_2} = C_j^1 k, \quad \left\|\frac{\partial^2\psi_k}{\partial x_j^2}\right\|_{L_2} = C_j^2 k^2,$$

where the constants $C_j^i \neq 0$ do not depend on k.

Therefore, we obtain that

$$\lim_{k\to\infty}\left\|v_{12}^k\right\|_{L_2} = \lim_{k\to\infty}\frac{|\eta_1|}{k^3|\xi_1|}\left\|\frac{\partial\psi_k}{\partial x_1}\right\|_{L_2} = 0.$$

However, for the first summand v_{11}^k we have

$$\left\|v_{11}^k\right\|_{L_2} = \left\|\eta_1\psi_k e^{ik^3\langle x,\xi\rangle}\right\|_{L_2} = |\eta_1|\|\psi_k\|_{L_2} = |\eta_1| \neq 0.$$

Now we will prove the property $\left\|(M - \lambda_0 I)\tilde{v}^k\right\| \to 0$.

For example, for the first component we will have

$$-\lambda_0 v_1^k - \omega v_2^k + \frac{1}{\alpha}\frac{\partial v_5^k}{\partial x_1} =$$

$$= \left(-\lambda_0\eta_1 - \omega\eta_2 + \frac{1}{\alpha}\xi_1\right)\psi_k e^{ik^3\langle x,\xi\rangle} -$$

$$-\frac{i}{k^3}e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_1}\left[\frac{\lambda_0\eta_1}{\xi_1} + \frac{\omega\eta_2}{\xi_2} + \frac{1}{\alpha}\right].$$

Since

$$-\lambda_0\eta_1 - \omega\eta_2 + \frac{1}{\alpha}\xi_1 = 0,$$

then,

$$\left\|-\lambda_0 v_1^k - \omega v_2^k + \frac{1}{\alpha}\frac{\partial v_5^k}{\partial x_1}\right\|_{L_2} \le \frac{Const}{k^2} \to 0 \text{ as } k \to \infty.$$

The proof for the other components is analogous.

For $\lambda = 0$, the system (14) transforms into

$$\begin{cases} -\omega v_2 + \dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_1} = 0 \\ \omega v_1 + \dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_2} = 0 \\ N v_4 + \dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_3} = 0 \\ -N v_3 = 0 \\ \dfrac{1}{\alpha}\operatorname{div}\vec{v} = 0 \end{cases}.$$

It can be easily seen that every vector-function of the form

$$v=\left(\frac{-1}{\alpha\omega}\frac{\partial\varphi}{\partial x_2},\frac{1}{\alpha\omega}\frac{\partial\varphi}{\partial x_1},0,\frac{-1}{\alpha N}\frac{\partial\varphi}{\partial x_3},\varphi\right),\quad \varphi\in C_0^\infty(\Omega),$$

satisfies the last system and thus the value $\lambda=0$ is an eigenvalue of infinite multiplicity.

Since the essential spectrum of a linear operator is always a closed set, the points $\pm\{ia\},\pm\{iA\}$, belong to it. These limit points are also eigenvalues of infinite multiplicity, for example, let $\lambda=iN$, $N>\omega$. Then, the system (14) takes the form

$$\begin{cases} -iNv_1-\omega v_2+\dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_1}=0 \\ \omega v_1-iNv_2+\dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_2}=0 \\ -iNv_3+Nv_4+\dfrac{1}{\alpha}\dfrac{\partial v_5}{\partial x_3}=0 \\ -Nv_3-iNv_4=0 \\ \dfrac{1}{\alpha}\operatorname{div}\vec{v}-iNv_5=0 \end{cases}.$$

Evidently, any function of the type

$$\left(0,0,\varphi(x_1,x_2),i\varphi(x_1,x_2),0\right),\quad \varphi\in C_0^\infty,$$

satisfies the last system and thus the Proof 2 is concluded.

We would like to observe as well that the sequence (23), being an explicit solution of the system (14), (15) for λ belonging to the essential spectrum, serves as an example of non-uniqueness of the solution, due to the arbitrary election of the function ψ_0.

<u>Remark 2.</u>

Let us denote as M_q^1 the differential operator in (14), (15) associated to the boundary condition

$$p|_{\partial\Omega}=0,$$

and let us consider the following domain of the operator M_q^1:

$$D(M_q^1)=\left\{\begin{matrix}\vec{v}\in\left(L_2(\Omega)\right)^3 \mid \exists f\in L_2(\Omega): \\ \left(\vec{v},\nabla\varphi\right)=\left(f,\varphi\right)\ \forall\varphi\in \overset{0}{W}{}_2^1(\Omega)\end{matrix}\right\}\times W_2^1(\Omega)\times \overset{0}{W}{}_2^1(\Omega).$$

It can be easily seen that the statement of the Theorem 3 is valid for the operator M_q^1, either Proof 1 or Proof 2 can be applied. Particularly, the Weyl sequence (23) is also valid for the operator M_q^1.

IV. The Solution of the Problem for the Case of Compressible Viscous Barotropic Rotating Stratified Fluid

<u>Theorem 4.</u>

The essential spectrum of the operator M^2 is composed of three real isolated points

$$\sigma_{ess}\left(M^2\right)=\left\{0,\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2(\beta+2)}\right\}.$$

<u>Proof.</u>

We observe that, according to [25], [27], we can choose

$$s_1=s_2=s_3=0,\ s_4=s_5=-1,$$
$$t_1=t_2=t_3=2,\ t_4=t_5=1,$$

so that the main symbol of the operator $L=M^2-\lambda I$ will be expressed as:

$$\tilde{L}(\xi)=\begin{pmatrix} -\nu|\xi|^2-\nu\beta\xi_1^2 & -\nu\beta\xi_1\xi_2 & -\nu\beta\xi_1\xi_3 & 0 & \frac{1}{\alpha}\xi_1 \\ -\nu\beta\xi_1\xi_2 & -\nu|\xi|^2-\nu\beta\xi_2^2 & -\nu\beta\xi_2\xi_3 & 0 & \frac{1}{\alpha}\xi_2 \\ -\nu\beta\xi_1\xi_3 & -\nu\beta\xi_2\xi_3 & -\nu|\xi|^2-\nu\beta\xi_3^2 & 0 & \frac{1}{\alpha}\xi_3 \\ 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & 0 & -\lambda \end{pmatrix}$$

We calculate the determinant of the last matrix:

$$\det\left(\widetilde{M^2-\lambda I}\right)(\xi)=\frac{\lambda\nu^2}{\alpha^2}|\xi|^6\left(\nu\lambda\alpha^2(\beta+1)-1\right),$$

and thus we can see that for two points

$$\lambda=0\quad\text{and}\quad\lambda=\frac{1}{\nu\alpha^2(\beta+1)}$$

the operator $L=M^2-\lambda I$ is not elliptic in sense of Douglis-Nirenberg. Now we will show, additionally, that for the point $\lambda=1/\nu\alpha^2(\beta+2)$ the condition of Lopatinski is not satisfied.

The Dirichlet boundary condition can be written in a matrix form

$$Gv^*|_{\partial\Omega}=0\quad,\qquad G=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix}.$$

If we denote $\tilde{\xi}=(\xi_1,\xi_2)$, $\xi_3=\tau$, then

$$\det\left(\widetilde{M^2-\lambda I}\right)\left(\tilde{\xi},\tau\right)=$$
$$=\frac{\lambda\nu^2}{\alpha^2}\left(\left|\tilde{\xi}\right|^2+\tau^2\right)^3\left(\nu\lambda\alpha^2(\beta+1)-1\right)$$

and thus the equation $\det\left(\widetilde{M^2-\lambda I}\right)\left(\tilde{\xi},\tau\right)=0$ for $\lambda\neq 0,\dfrac{1}{\nu\alpha^2(\beta+1)}$ has one root $\tau=i\left|\tilde{\xi}\right|$ of triple multiplicity in the upper half of the complex plane.

In this way, $M^+\left(\tilde{\xi},\tau\right)=\left(\tau-i\left|\tilde{\xi}\right|\right)^3$. Since the elements of the matrices $\widetilde{M^2-\lambda I}$ and G are homogeneous functions with respect to $\tilde{\xi},\tau$, then it is sufficient to verify the Lopatinski condition for unitary vectors $\tilde{\xi}$. Let us choose a local system of coordinates so that $\xi_1=1$, $\xi_2=0$. Then, we have $M^+\left(\tilde{\xi},\tau\right)=\left(\tau-i\right)^3$ and $\left(\widetilde{M^2-\lambda I}\right)(\tau)=$

$$=\begin{pmatrix} -\nu(\beta+1)-\nu\tau^2 & 0 & -\nu\beta\tau & 0 & \frac{1}{\alpha} \\ 0 & -\nu\left(1+\tau^2\right) & 0 & 0 & 0 \\ -\nu\beta\tau & 0 & -\nu-\nu(\beta+1)\tau^2 & 0 & \frac{\tau}{\alpha} \\ 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{\alpha} & 0 & \frac{\tau}{\alpha} & 0 & -\lambda \end{pmatrix}$$

For the matrix $\left(\widetilde{M^2-\lambda I}\right)$ we construct first the adjoint matrix $\left(\widehat{M^2-\lambda I}\right)$ (which is composed of algebraic complements of the original matrix), then we multiply $\left(\widehat{M^2-\lambda I}\right)$ by the boundary conditions matrix G, after which we divide $G\left(\widehat{M^2-\lambda I}\right)$ by the polynomial $\left(\tau-i\right)^3$, and, finally, we consider the matrix M_1 which is composed of the residues of that division. After some elementary transformations of the rows of M_1 and making the notation $\eta=\beta-1/\nu\lambda\alpha^2$, we obtain the matrix M_2 from which we withdraw the two last zero columns and thus obtain the square 3x3 matrix which we denote as M_3:

$$M_3=-\nu^2\lambda(\tau-i)\begin{pmatrix} (4+3\eta)i-(5\eta+4)\tau & 0 & -\eta(1+3i\tau) \\ 0 & -4(1+\eta)(\tau-i) & 0 \\ -\eta(1+3i\tau) & 0 & (4+\eta)i+(\eta-4)\tau \end{pmatrix}.$$

Evidently, the linear independence of the rows of the matrices M_2, M_1 can be verified by calculating the determinant of the matrix M_3. We observe that

$$\det M_3=16(\eta+1)(\eta+2)^2\left(\nu^2\lambda\right)^3(\tau-i)^6=$$
$$=16\left((\beta+1)-\frac{1}{\nu\alpha^2\lambda}\right)\left((\beta+2)-\frac{1}{\nu\alpha^2\lambda}\right)^2\left(\nu^2\lambda\right)^3(\tau-i)^6.$$

As we can see, for $\lambda\neq 0,\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2(\beta+2)}$ the rows of the matrix M_1 are linearly independent. We have already proved that the first two points belong to the essential spectrum of the operator M. In this way, due to the verification of Lopatinski condition, we have that the point $\lambda=\frac{1}{\nu\alpha^2(\beta+2)}$, also belongs to the essential spectrum of the operator M^2 and thus

$$\sigma_{ess}\left(M^2\right)=\left\{0,\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2(\beta+2)}\right\}$$

which concludes the proof of the Theorem 4.

Theorem 5.

The essential spectrum of the operator M^3 is composed of three real isolated points

$$\sigma_{ess}\left(M^3\right)=\left\{0,\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2\beta}\right\}.$$

Proof.

For the points $\lambda=0$, $\lambda=\frac{1}{\nu\alpha^2(\beta+1)}$, just like in the case of the operator M^2, we have that the operator $L=M^3-\lambda I$ is not elliptic in sense of Douglis-Nirenberg and thus these two points belong to the essential spectrum. We shall prove now that for the point $\lambda=\frac{1}{\nu\alpha^2\beta}$ the Lopatinski condition is not satisfied. Let $x\in\partial\Omega$. Since the system (14), (18) and the boundary condition (17) are invariants with respect to translations of origin and rotations of the coordinate axis, we may consider them, without loss of generality, in a local system of coordinates with the origin in x. In this way, we have

$$\eta_1=\eta_2=0,\ \eta_3=1,$$

and the boundary condition (17) takes the following form in a local system of coordinates on a rectilinear portion of the boundary $\partial\Omega'$:

$$\begin{aligned} &\nu\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right)\Bigg|_{\partial\Omega'}=0 \\ &\nu\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right)\Bigg|_{\partial\Omega'}=0 \\ &\nu(\beta-1)\operatorname{div}\vec{v}+2\nu\frac{\partial v_3}{\partial x_3}-\frac{1}{\alpha}v_5\Bigg|_{\partial\Omega'}=0\ . \end{aligned}\tag{24}$$

For the boundary condition (17) we have the following characteristic matrix $G\left(\tilde{\xi},\tau\right)$:

$$\begin{pmatrix} \nu\tau & 0 & \nu\xi_1 & 0 & 0 \\ 0 & \nu\tau & \nu\xi_2 & 0 & 0 \\ \nu(\beta-1)\xi_1 & \nu(\beta-1)\xi_2 & \nu(\beta+1)\xi_3 & 0 & \frac{-1}{\alpha} \end{pmatrix}.$$

As we have shown in the proof of Theorem 4, it is sufficient to verify the Lopatinski condition for unitary vectors $\tilde{\xi}$, i.e. $\xi_1 = 1,\ \xi_2 = 0$. In this way,

$$G(\tau) = \begin{pmatrix} \tau & 0 & 1 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 \\ \beta - 1 & 0 & (\beta+1)\tau & 0 & \frac{-1}{\nu\alpha} \end{pmatrix}.$$

We proceed in an analogous way as we did in the proof of the Theorem 4.

For the matrix $\left(\widetilde{M^3 - \lambda I}\right)$ we construct first the adjoint matrix $\left(\widehat{M^3 - \lambda I}\right)$ (which is composed of algebraic complements of the original matrix), then we multiply $\left(\widehat{M^3 - \lambda I}\right)$ by the boundary conditions matrix *G*, after which we divide $G\left(\widehat{M^3 - \lambda I}\right)$ by the polynomial $(\tau - i)^3$, and, finally, we consider the matrix M_4 which is composed of the residues of that division. After some elementary transformations of the rows of M_4 and making the notation $\eta = \beta - 1/\nu\lambda\alpha^2$, we obtain the matrix M_5 from which we withdraw the two last zero columns and thus obtain the square 3x3 matrix which we denote as M_6:

$$M_6 = = -2\nu^2\lambda(\tau-i)\begin{pmatrix} 2\eta - i(5\eta+2)(\tau-i) & 0 & 2i\eta + (3\eta-2)(\tau-i) \\ 0 & -2i(1+\eta)(\tau-i) & 0 \\ 2i\eta + (3\eta+2)(\tau-i) & 0 & -2\eta + i(\eta-2)(\tau-i) \end{pmatrix}$$

.

We observe that

$$\det M_6 = -64i\left(\nu^2\lambda\right)^3 \eta(\eta+1)(\eta+2)(\tau-i)^6,$$

from which it follows that, for $\eta \neq 0,\ -1,\ -2$, the Lopatinski condition is fulfilled.

For $\eta = 0$ we have $\lambda = \dfrac{1}{\nu\alpha^2\beta}$ and thus

$$\lambda = \frac{1}{\nu\alpha^2\beta} \in \sigma_{ess}\left(M^3\right).$$

For $\eta = -1$ we have $\lambda = \dfrac{1}{\nu\alpha^2(\beta+1)}$ for which the operator $M^3 - \lambda I$ is not elliptic in sense of Douglis-Nirenberg and thus

$$\lambda = \frac{1}{\nu\alpha^2(\beta+1)} \in \sigma_{ess}\left(M^3\right).$$

Finally, for $\eta = -2$ we have $\lambda = \dfrac{1}{\nu\alpha^2(\beta+2)}$. For that value of λ, we calculate the matrix M_4:

$$M_4 = -\frac{2\nu(\tau-i)}{\alpha^2(\beta+2)} \times$$

$$\times \begin{pmatrix} -4+8i(\tau-i) & 0 & -4i-8(\tau-i) & \frac{-4}{\lambda\alpha}(\tau-i) \\ 0 & 2i(\tau-i) & 0 & 0 \\ -4i-4(\tau-i) & 0 & 4-4i(\tau-i) & \frac{4}{\lambda\alpha}(\tau-i) \end{pmatrix}.$$

Now, we multiply the third column by *i*, then sum it with the first column, withdraw the first zero column an denote the resulting 3x3 square matrix as M_7. We observe that

$$\det M_7 = \frac{-64\nu^4 i(\tau-i)^6}{\alpha^5(\beta+2)^2},$$

and thus the Lopatinski condition is fulfilled for

$$\lambda = \frac{1}{\nu\alpha^2(\beta+2)}.$$

Summing up the results of all the calculations for matrices M_4 - M_7, and recalling that for $\lambda = 0$ the operator $M^3 - \lambda I$ is not elliptic in sense of Douglis-Nirenberg, we have that

$$\sigma_{ess}\left(M^3\right) = \left\{0, \frac{1}{\nu\alpha^2(\beta+1)}, \frac{1}{\nu\alpha^2\beta}\right\},$$

and thus the Theorem 5 is proved.

<u>Theorem 6</u>.

Let $A = \omega + N$. Then, the spectrum of operator M^2 is symmetrical with respect to the real axis, and all the eigenvalues of operator M^2 are in the following sector of the complex plane:

$$Z = \left\{\lambda \in C:\ \operatorname{Re}\lambda \geq \nu\mu_1,\ |\operatorname{Im}\lambda| \leq A + \frac{(\operatorname{Re}\lambda)}{\nu\alpha^2\beta A}\right\},$$

where μ_1 is the smallest eigenvalue of the operator $(-\Delta)$ in Ω for zero boundary Dirichlet condition.

<u>Remark 3.</u>

The value μ_1 is estimated as follows: $\mu_1 \geq d^{-2}$, where d is the width of the 3-dimensional strip region which includes the domain Ω. In that case, all the eigenvalues of the operator M^2 are situated at the right side of the straight line $\operatorname{Re}\lambda = \nu d^{-2}$.

<u>Proof.</u>

Let us denote $\tilde{v} = (v_1, v_2, v_3, v_4)$ and take notations for the matrices B_ω, B_N from (20).

Then, the system $\left(M^2-\lambda I\right)\{\tilde{v},v_5\}=0$ can be written in the form

$$\begin{cases} -\lambda\tilde{v}+B_\omega\tilde{v}+B_N\tilde{v}-\nu\Delta\vec{v}-\nu\beta\operatorname{div}\vec{v}+\dfrac{1}{\alpha}\nabla v_5=0 \\ -\lambda v_5+\dfrac{1}{\alpha}\operatorname{div}\vec{v}=0 \end{cases}. \tag{25}$$

We apply the complex-conjugation to the original system of $M^2-\lambda I=0$:

$$\begin{cases} -\overline{\lambda}\overline{\tilde{v}}+B_\omega\overline{\tilde{v}}+B_N\overline{\tilde{v}}-\nu\Delta\overline{\vec{v}}-\nu\beta\operatorname{div}\overline{\vec{v}}+\dfrac{1}{\alpha}\nabla\overline{v}_5=0 \\ -\overline{\lambda}\overline{v}_5+\dfrac{1}{\alpha}\operatorname{div}\overline{\vec{v}}=0 \end{cases}$$

from which we can see that, if λ is an eigenvalue of M^2, then $\overline{\lambda}$ is also an eigenvalue of operator M^2, and thus the spectrum is symmetrical with respect to the real axis.

Now we multiply the system (25) by $\overline{\{\tilde{v},v_5\}}$ and then integrate by parts in Ω. In this way, we obtain the following equations:

$$-\lambda\|\tilde{v}\|^2+\left(B_\omega\tilde{v},\tilde{v}\right)+\left(B_N\tilde{v},\tilde{v}\right)+\nu\sum_{k=1}^{3}\|\nabla v_k\|^2+$$

$$+\nu\beta\|\operatorname{div}\vec{v}\|^2-\frac{1}{\alpha}\left(v_5,\operatorname{div}\vec{v}\right)=0$$

$$-\lambda\|v_5\|^2+\frac{1}{\alpha}\left(\operatorname{div}\vec{v},v_5\right)=0$$

We sum up these two equations

$$-\lambda\left(\|\tilde{v}\|^2+\|v_5\|^2\right)+\left(B_\omega\tilde{v},\tilde{v}\right)+\left(B_N\tilde{v},\tilde{v}\right)+\nu\sum_{k=1}^{3}\|\nabla v_k\|^2+$$

$$+\nu\beta\|\operatorname{div}\vec{v}\|^2+\frac{1}{\alpha}\left[\left(\operatorname{div}\vec{v},v_5\right)-\left(v_5,\operatorname{div}\vec{v}\right)\right]=0$$

and then separate the real and the imaginary parts, keeping in mind the fact that for a skew-symmetric matrix B the expression $\left(B\tilde{v},\tilde{v}\right)$ is imaginary.

$$\operatorname{Re}\lambda=\frac{\nu\sum_{k=1}^{3}\|\nabla v_k\|^2+\nu\beta\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2+\|v_5\|^2}>0,$$

$$\left|\operatorname{Im}\lambda\right|=-i\frac{\left(B_\omega\tilde{v},\tilde{v}\right)+\left(B_N\tilde{v},\tilde{v}\right)+\frac{1}{\alpha}\left[\left(\operatorname{div}\vec{v},v_5\right)-\left(v_5,\operatorname{div}\vec{v}\right)\right]}{\|\tilde{v}\|^2+\|v_5\|^2}.$$

We estimate

$$\left|\operatorname{Im}\lambda\right|\le\frac{A\|\tilde{v}\|^2+\frac{2}{\alpha}\|\operatorname{div}\vec{v}\|\|v_5\|}{\|\tilde{v}\|^2+\|v_5\|^2}\le$$

$$\le\frac{A\|\tilde{v}\|^2+A\|v_5\|^2+\frac{\|\operatorname{div}\vec{v}\|^2}{\alpha^2A}}{\|\tilde{v}\|^2+\|v_5\|^2}.$$

Here we used the inequalities

$$\left(f,g\right)_{L_2}\le\|f\|_{L_2}\|g\|_{L_2},$$

$$2\frac{a}{\sqrt{A}}b\sqrt{A}\le\frac{a^2}{A}+b^2A.$$

Since

$$\frac{\operatorname{Re}\lambda}{\nu\alpha^2\beta A}=\frac{\frac{1}{\alpha^2\beta A}\sum_{k=1}^{3}\|\nabla v_k\|^2}{\|\tilde{v}\|^2+\|v_5\|^2}+\frac{\frac{1}{\alpha^2A}\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2+\|v_5\|^2}$$

and

$$\left|\operatorname{Im}\lambda\right|\le A+\frac{\frac{1}{\alpha^2A}\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2+\|v_5\|^2},$$

then we finally have

$$\left|\operatorname{Im}\lambda\right|\le A+\frac{\left(\operatorname{Re}\lambda\right)}{\nu\alpha^2\beta A}.$$

Let us prove now that $\operatorname{Re}\lambda\ge\nu\mu_1$. We shall suppose the contrary. In other terms, we suppose that there exists λ such that $\operatorname{Re}\lambda<\nu\mu_1$ and there exists the vector $v^1=\left(\tilde{v}^1,v_5^1\right)$ such that

$$-\lambda\tilde{v}^1+B_\omega\tilde{v}^1+B_N\tilde{v}^1-\nu\Delta\vec{v}^1-\nu\beta\operatorname{div}\vec{v}^1+\frac{1}{\alpha}\nabla v_5^1=0$$

$$-\lambda v_5^1+\frac{1}{\alpha}\operatorname{div}\vec{v}^1=0.$$

$$\tilde{v}^1\Big|_{\partial\Omega}=0$$

Let us consider the following auxiliary problem;

$$\Delta\varphi=-\lambda\alpha v_5^1\quad,\quad x\in\Omega$$

$$\left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{\partial\Omega}=0. \tag{26}$$

Since $\operatorname{div}\vec{v}^1=\lambda\alpha v_5^1$ and $\vec{v}^1\big|_{\partial\Omega}=0$, then we have

$$\int_\Omega\lambda\alpha v_5^1(x)\,dx=\int_\Omega\operatorname{div}\vec{v}^1dx=\int_{\partial\Omega}\vec{v}^1\cdot\vec{n}\,ds=0.$$

In that way, the necessary condition for the solvability of the problem (26) is satisfied and there exists a solution $\varphi(x)$ which is defined within an additive constant.

Now, we consider one more auxiliary problem;

$$-\lambda\tilde{v}^2 + B_\omega\tilde{v}^2 + B_N\tilde{v}^2 - \nu\Delta\vec{v}^2 + \nabla v_5^2 = B_\omega\nabla\varphi + B_N\nabla\varphi$$
$$\operatorname{div}\vec{v}^2 = 0 \ . \qquad (27)$$
$$\tilde{v}^2\big|_{\partial\Omega} = 0$$

For the problem (27) it is known (see [26]) that the spectrum of the homogeneous problem is situated to the right of the straight line $\operatorname{Re}\lambda = \nu\mu_1$, therefore there exists a unique solution of the problem (27). Now, let us put

$$\tilde{u} = \tilde{v}^1 + \tilde{v}^2 + \nabla\varphi \ ,$$
$$u_5 = \frac{1}{\alpha}v_5^1 + v_5^2 + \lambda\varphi + \nu\Delta\varphi - \nu\beta\operatorname{div}\vec{v}^1 \ .$$

Then, we shall have

$$-\lambda\tilde{u} + S\tilde{u} - \nu\Delta\vec{u} + \nabla u_5 = 0$$
$$\operatorname{div}\vec{u} = 0 \ . \qquad (28)$$
$$\tilde{u}\big|_{\partial\Omega} = 0$$

After multiplying the first line in (28) by $\bar{\tilde{u}}$ and integrating by parts in Ω, we obtain

$$-\lambda\|\tilde{u}\|^2 + (S\tilde{u},\tilde{u}) + \nu\sum_{k=1}^{3}\|\nabla u_k\|^2 = 0 \ .$$

From the last equation we have

$$\operatorname{Re}\lambda = \frac{\nu\left(\sum_{k=1}^{3}\|\nabla u_k\|^2\right)}{\|\tilde{u}\|^2} \ .$$

Finally, using the Friedrichs inequality, we obtain

$$\|\tilde{u}\|^2 \le \frac{1}{\mu_1}\left(\sum_{k=1}^{3}\|\nabla u_k\|^2\right).$$

In this way, $\operatorname{Re}\lambda \ge \nu\mu_1$, which contradicts the assumption of $\operatorname{Re}\lambda < \nu\mu_1$. We conclude therefore that for all the eigenvalues of the operator M^2 the following property is valid:

$$\operatorname{Re}\lambda \ge \nu\mu_1 \ ,$$

and thus the Theorem 6 is proved.

Theorem 7.

The spectrum of operator M^3 is symmetrical with respect to the real axis, and all the eigenvalues of operator M^3 are in the following sector of the complex plane:

$$Z = \left\{\lambda \in C: \ \operatorname{Re}\lambda \ge 0, \ \left|\operatorname{Im}\lambda\right| \le A + \frac{(\operatorname{Re}\lambda)}{\nu\alpha^2\beta A}\right\}.$$

The proof of the Theorem 7 is analogous to the proof of the Theorem 6. Particularly, just like in the proof of the previous theorem, we obtain that

$$\operatorname{Re}\lambda = \frac{\nu\sum_{k=1}^{3}\|\nabla v_k\|^2 + \nu\beta\|\operatorname{div}\vec{v}\|^2}{\|\tilde{v}\|^2 + \|v_5\|^2} \ge 0 \ .$$

We only would like to observe that, we cannot exclude the case of equality in the last relation, since $\lambda = 0$ is an eigenvalue of the operator M^3 with the corresponding eigenfunction $\tilde{v} = (v_1, v_2, 0, 0)$, where v_1, v_2 are arbitrary constants.

V. Conclusion

For the inviscid case of compressible rotating stratified fluid, as we have seen, the essential spectrum of inner oscillations is the symmetrical bounded set of the imaginary axis

$$\{0\} \cup [-iA, -ia] \cup [ia, iA] .$$

Comparing these results with the compressible viscous case, we can conclude that, just as in case of the explicit representations of Cauchy problems, where, in most cases, the inviscid solution cannot be obtained from viscous solutions as a limit for vanishing viscosity parameter; the essential spectrum of normal oscillations for inviscid stratified fluid cannot be obtained from the essential spectrum for viscous stratified fluid by putting the viscosity parameters equal to zero. Therefore, the considered problems and the results of Theorems 3, 4 and 5, are remarkable and interesting due to the special property that, for the viscous fluid, (for example, for the operator M^2), the two points of the essential spectrum

$$\frac{1}{\nu\alpha^2(\beta+1)}, \frac{1}{\nu\alpha^2(\beta+2)}$$

move to infinity for $\nu, \beta \to 0$; while the essential spectrum of the inviscid fluid contains an interval of the imaginary axis.

Additionally, as we can see, the results of Theorem 3 obtained for the inviscid fluid , correspond to the statement of Theorem 7 if we put $\operatorname{Re}\lambda = 0$: $\left(\operatorname{Re}\lambda = 0, \ \left|\operatorname{Im}\lambda\right| \le A\right)$.

Finally, we would like to observe that, if we put, for example, $N = 0$ in (2), then, according to theorem 3, the essential spectrum will be the interval of the imaginary axis $[-i\omega, i\omega]$, the result which was proved for rotating (non-stratified) compressible fluid in [27].

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