Option pricing under stochastic environment of volatility and market price of risk

Nattakorn Phewchean, Yong Hong Wu, and Yongwimon Lenbury

***Abstract*—**Since Black-Scholes model was proposed in 1973, it has been applied widely for option pricing. The aim of this paper is to develop European option pricing model taking into account stochastic volatility and stochastic market price of risk (MPR) under the framework of Black-Scholes. Both volatility and market price of risk are assumed to be stochastic and assumed to follow Ornstein-Uhlenbeck process. By using an analytical approach of Abraham Loui, explicit formulas are derived for European call and put option prices. Sensitivity of option price to model parameters are tested and the simulation results show the strong characteristic of stochastic model.

***Keywords*—**European option pricing model, stochastic volatility, stochastic market price of risk, Ornstein-Uhlenbeck process, Black-Scholes model.

I. Introduction

In 1973, Fischer Black and Myron Scholes proposed a mathematical model for option pricing [1]. Since then option has been playing a very significant role for many investors in financial market as a derivative financial instrument for traders who need to secure their investments in order to gain confidence of making a profit from the stock market. Technically, the mathematical model of Black-Scholes is based on the financial parameters of stock price *S(t)*, strike price of an option *K(t)*, interest rate *r*, time *t*, maturity date *T* and constant volatility σ. Many interesting mathematical models for option pricing has been developed and proposed by many researchers. The purpose of their studies is to estimate the price of an option by applying their mathematical models or approaches [2]-[12].

Under the Black-Scholes setting, many strong assumptions have been assumed such as stock prices are followed the normal distribution with known mean and constant volatility. In order to make the option pricing model more applicable to the real world, some assumptions of the Black-Scholes model are relaxed and some parameters of the model are assumed to follow the real-world situation.

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The constant volatility assumption has been changed and generalized by considering deviation of volatility. Many researchers have developed the option pricing model taking into account stochastic volatility. Scott [2], Hull and White [3], Wiggins [4] and Heston [5]-[7] have generalized the option pricing model by allowing the stochastic characteristic of volatility. The study of option pricing model has been extended and the stochastic volatility has been analyzed for more properties on parameters by Jacquire, Polson and Rossi [8], Carr, German, Madan and Yor [9] and Barndorff-Nielsen and Shephard [13].

In this paper, we extend the Black-Scholes model by considering the stochastic volatility to follow Ornstein-Unlenbeck process [14] and allowing the existence of stochastic market price of risk to follow the mean-reverting process. The stochastic volatility and stochastic market price of risk allow the random deviation which is the same as the real-world situation of financial market.

II. The Models

The financial market is assumed to be complete with no arbitrage opportunity as the important assumptions to construct the option pricing models. There exists a fixed martingale measure *Q* which is equivalent to the probability measure *P* such that the asset price that is discounted at the risk-free interest rate is martingale under the probability space (Ω*, P, F*) where Ω is the pricing outcomes space, *F* is the σ-algebra denoting measurable events, and *P* is the probability measure. This mathematical assumption assures that the financial market has no arbitrage opportunity [15]. All stochastic processes in any such pricing environment are adapted to the filtration $\{F_t\}$, generated by the Wiener processes.

According to Girsanov's theorem with multiple Brownian motions, there exists F_t-adapted process $k_1(t)$ and $k_2(t)$. The equivalent martingale measure *Q* and the measurable probability *P* are related by the following Radon-Nikodym derivative equation [16]-[17].

$$\left.\frac{dQ}{dP}\right|_{F_t} = \exp\left(-\int_0^t k_1(s)dW_1(s) - \int_0^t k_2(s)dW_2(s)r_2 - \frac{1}{2}\int_0^t \left(k_1(s)^2 + k_2(s)^2\right)ds\right). \quad (1)$$

The pricing movement of Brownian motions, $dW_1(s)$ and $dW_2(s)$ are one-dimensional uncorrelated Weiner Processes which are defined and assumed to follow a complete probability space (Ω, P, F). The infinitesimal amount of stock price $dS(t)$ depends on the risky asset price corresponds to a stock process $S(t)$ under the mean $\mu_S(t, S(t))$ in an infinitesimal amount of time dt. With the stochastic movement of the stock, the rate of change of stock price can be modeled by the following stochastic differential equation [19]

$$\frac{dS(t)}{S(t)} = \mu_S(t, S(t))dt + \sigma(t)dW_1(t) \qquad (2)$$

where $S(0) > 0$. $\mu_S(t, S(t))$ is stock yield and $\sigma(t)$ is stochastic volatility which is assumed to follow the Ornstein-Uhlenbeck process. Then we have the stochastic volatility process as

$$d\sigma(t) = \theta_\sigma(\mu_\sigma - \sigma(t))dt + \omega_{\sigma 1}dW_1(t) + \omega_{\sigma 2}dW_2(t) \qquad (3)$$

where $\sigma(0) = 0$, θ_σ, μ_σ, $\omega_{\sigma 1}$ and $\omega_{\sigma 2}$ are constants.

Equation (3) is correlated to (2). This correlation explicitly allows the relationship between corresponding processes $S(t)$ and $\sigma(t)$. In particular case, the absence of correlation between two processes are easily obtained by setting the parameter $\omega_{\sigma 1}$ or $\omega_{\sigma 2}$ equals to zero.

The market price of risk or MPR is considered as a risk factor k_i which follows the Ornstein-Uhlenbeck process

$$dk_i(t) = \theta_{ki}(\mu_{ki} - k_i(t))dt + \omega_{ki}dW_i(t) \qquad (4)$$

where $k(0) = 0$, θ_{ki}, μ_{ki} and ω_{ki} are constants when $i = 1, 2$.

In the real world of financial market, the stock process and volatility process are governed by the market price of risk MPR. Consequently, in the next section, the option pricing models are constructed to enclose the market price of risk factor which yield some advantages for asset pricing.

III. The Methods

The method of deriving the explicit formula for European option price is performed by following Abraham Lioui's analytic approach [20]-[21] with some extra mathematical assumptions. Some derivations of basic formulae have been presented in [22], reproduced here for completeness. Extra results are then presented in this work.

Under the assumptions of arbitrage-free, the formulas for European options are derived by encapsulating the market price of risk $k_i(t)$, the kernel of movement in stock process and volatility process are changed. Accordingly, the Wiener processes $W_i(t)$ in the processes (2)–(4) are reconstructed by changing $W_i(t)$ to $\overline{W}_i(t)$ where

$$\overline{W}_i(t) = W_i(t) + \int_0^t k_i(s)ds . \qquad (5)$$

The dynamic models (2)–(4) are changed. The equation of the stock price (2) becomes

$$\frac{dS(t)}{S(t)} = \mu_S(t, S(t))dt + \sigma(t)d\overline{W}_1(t) . \qquad (6)$$

Because of the fact that MPR has an influence on the directional movement of the underlying asset, the stochastic volatility follows the process

$$\begin{aligned} d\sigma(t) &= \theta_\sigma\left(\mu_\sigma - \sigma(t) - \frac{1}{\theta_\sigma}\left(\omega_{\sigma 1}k_1(t) + \omega_{\sigma 2}k_2(t)\right)\right)dt \\ &\quad + \omega_{\sigma 1}d\overline{W}_1(t) + \omega_{\sigma 2}d\overline{W}_2(t) \end{aligned} \qquad (7)$$

From (7), we have

$$\begin{aligned} \int_t^T \sigma(s)ds &= \mu_\sigma(T-t) - \frac{1}{\theta_\sigma}\int_t^T\left(\omega_{\sigma 1}k_1(s) + \omega_{\sigma 2}k_2(s)\right)ds \\ &\quad - \frac{1}{\theta_\sigma}(\sigma(T) - \sigma(t)) \\ &\quad + \frac{\omega_{\sigma 1}}{\theta_\sigma}\int_t^T d\overline{W}_1(t) + \frac{\omega_{\sigma 2}}{\theta_\sigma}\int_t^T d\overline{W}_2(t) \end{aligned} \qquad (8)$$

when market price of risk MPR is defined by Ornstein-Uhlenbeck process [17]-[18]

$$dk_i(t) = \theta_{ki}(\mu_{ki} - k_i(t))dt + \omega_{ki}d\overline{W}_i(t) \qquad (9)$$

where $\overline{\theta}_{ki} = \theta_{ki}(1 + \omega_{ki})$ and $\overline{\mu}_{ki} = \mu_{ki}/(1 + \omega_{ki})$.

From (9), we have

$$\int_t^T k_i(s)ds = \overline{\mu}_{ki}(T-t) - \frac{1}{\overline{\theta}_{ki}}(k_i(T) - k_i(t)) - \frac{\omega_{ki}}{\overline{\theta}_{ki}}\int_t^T d\overline{W}_i(t) \qquad (10)$$

By using integrating factor, we obtain

$$\begin{aligned} k_i(T) &= k_i(t)e^{-\overline{\theta}_{ki}(T-t)} + \overline{\mu}_{ki}(1 - e^{-\overline{\theta}_{ki}(T-t)}) \\ &\quad + \omega_{ki}\int_t^T e^{-\overline{\theta}_{ki}(T-t)}d\overline{W}_i(t) \end{aligned} \qquad (11)$$

Then we have

$$\int_t^T k_i(s)ds = \frac{1}{\overline{\theta}_{ki}}k_i(t)(1 - e^{-\overline{\theta}_{ki}(T-t)}) + \overline{\mu}_{ki}(T-t)$$

$$-\frac{1}{\bar{\theta}_{ki}}\bar{\mu}_{ki}(1-e^{-\bar{\theta}_{ki}(T-t)})-\int_t^T\left(\frac{\omega_{ki}}{\bar{\theta}_{ki}}-\frac{\omega_{ki}}{\bar{\theta}_{ki}}e^{-\bar{\theta}_{ki}(T-t)}\right)d\overline{W}_i(t) \quad (12)$$

European call option is valued first in order to obtain European put option formula by using the concept of put-call parity. The European call option *C(t)* is defined as in [20]

$$C(t)=E^Q\left[e^{-r(T-t)}\left[S(T)-K\right]^+ \mid F_t\right] \quad (13)$$
$$=e^{-r(T-t)}E^Q\left[S(T)1_{S(T)>K} \mid F_t\right]$$
$$-Ke^{-r(T-t)}E^Q\left[1_{S(T)>K} \mid F_t\right] \quad (14)$$

when $1_{S(T)>K}=\begin{cases}1 & \text{if} \quad S(T)>K\\ 0 & \text{if} \quad \text{otherwise}\end{cases}$.

We assume the distribution of stock price *S(T)* is modeled as follows.

$$S(T)=S(t)\exp\left\{\left(r-\frac{1}{2}\left[\int_t^T\sigma(s)ds\right]^2\right)(T-t)+\left[\int_t^T\sigma(s)ds\right]\cdot\int_t^T d\overline{W}_1(s)\right\}. \quad (15)$$

By using the analytic approach of Abraham Lioui [20]-[21] and applying simple integration techniques, we can derive integral term

$$\begin{aligned}\int_t^T\sigma(s)ds=&\frac{\sigma(t)}{\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)+\mu_\sigma\left[(T-t)-\frac{1}{\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\right]\\
&-\frac{\omega_{\sigma1}\bar{\mu}_{k1}}{\theta_\sigma}\left[(T-t)-\frac{1}{\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right.\\
&-\frac{\bar{\theta}_{k1}}{\theta_\sigma^2-\bar{\theta}_{k1}\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\\
&\left.+\frac{1}{\theta_\sigma-\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]\\
&-\frac{\omega_{\sigma2}\bar{\mu}_{k2}}{\theta_\sigma}\left[(T-t)-\frac{1}{\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\right.\\
&-\frac{\bar{\theta}_{k2}}{\theta_\sigma^2-\bar{\theta}_{k2}\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\\
&\left.+\frac{1}{\theta_\sigma-\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\right]\\
&+\frac{\omega_{\sigma1}}{\theta_\sigma}\left[\frac{1}{\theta_\sigma-\bar{\theta}_{k1}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k1}(T-t)}\right)\right.\\
&\left.-\frac{1}{\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]k_1(t)\\
&+\frac{\omega_{\sigma2}}{\theta_\sigma}\left[\frac{1}{\theta_\sigma-\bar{\theta}_{k2}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k2}(T-t)}\right)\right.\\
&\left.-\frac{1}{\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\right]k_2(t)\\
&+\frac{\omega_{\sigma1}}{\theta_\sigma}\int_t^T\left[1-\frac{\omega_{k1}}{\bar{\theta}_{k1}}+\frac{\omega_{k1}}{\bar{\theta}_{k1}}e^{-\bar{\theta}_{k1}(T-s)}-e^{-\theta_\sigma(T-s)}\right.\\
&\left.+\frac{\omega_{k1}}{\theta_\sigma-\bar{\theta}_{k1}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]d\overline{W}_1(s)\\
&+\frac{\omega_{\sigma2}}{\theta_\sigma}\int_t^T\left[1-\frac{\omega_{k2}}{\bar{\theta}_{k2}}+\frac{\omega_{k2}}{\bar{\theta}_{k2}}e^{-\bar{\theta}_{k2}(T-s)}-e^{-\theta_\sigma(T-s)}\right.\\
&\left.+\frac{\omega_{k2}}{\theta_\sigma-\bar{\theta}_{k2}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k2}(T-t)}\right)\right]d\overline{W}_2(s). \quad (16)\end{aligned}$$

By substituting (16) into (15), the formula for stock pricing is derived. However, since the integral in (16) is in a very complicated form. We will derive the formula for stock prices and option prices by simplification.

Now we define functions as follows.

$$M_1(t)=\frac{\sigma(t)}{\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)$$

$$M_2(t)=\mu_\sigma\left[(T-t)-\frac{1}{\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\right]$$

$$\begin{aligned}M_3(t)=&-\frac{\omega_{\sigma1}\bar{\mu}_{k1}}{\theta_\sigma}\left[(T-t)-\frac{1}{\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right.\\
&-\frac{\bar{\theta}_{k1}}{\theta_\sigma^2-\bar{\theta}_{k1}\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\\
&\left.+\frac{1}{\theta_\sigma-\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]\end{aligned}$$

$$\begin{aligned}M_4(t)=&-\frac{\omega_{\sigma2}\bar{\mu}_{k2}}{\theta_\sigma}\left[(T-t)-\frac{1}{\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\right.\\
&-\frac{\bar{\theta}_{k2}}{\theta_\sigma^2-\bar{\theta}_{k2}\theta_\sigma}\left(1-e^{-\theta_\sigma(T-t)}\right)\\
&\left.+\frac{1}{\theta_\sigma-\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\right]\end{aligned}$$

$$\begin{aligned}M_5(t)=&\frac{\omega_{\sigma1}}{\theta_\sigma}\left[\frac{1}{\theta_\sigma-\bar{\theta}_{k1}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k1}(T-t)}\right)\right.\\
&\left.-\frac{1}{\bar{\theta}_{k1}}\left(1-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]k_1(t)\end{aligned}$$

$$M_6(t)=\frac{\omega_{\sigma2}}{\theta_\sigma}\left[\frac{1}{\theta_\sigma-\bar{\theta}_{k2}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k2}(T-t)}\right)\right.$$

$$-\frac{1}{\bar{\theta}_{k2}}\left(1-e^{-\bar{\theta}_{k2}(T-t)}\right)\Bigg]k_2(t)$$

$$P_1(s)=\frac{\omega_{\sigma 1}}{\theta_\sigma}\left[1-\frac{\omega_{k1}}{\bar{\theta}_{k1}}+\frac{\omega_{k1}}{\bar{\theta}_{k1}}e^{-\bar{\theta}_{k1}(T-s)}-e^{-\theta_\sigma(T-s)}\right.$$
$$\left.+\frac{\omega_{k1}}{\theta_\sigma-\bar{\theta}_{k1}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k1}(T-t)}\right)\right]$$

$$P_2(s)=\frac{\omega_{\sigma 2}}{\theta_\sigma}\left[1-\frac{\omega_{k2}}{\bar{\theta}_{k2}}+\frac{\omega_{k2}}{\bar{\theta}_{k2}}e^{-\bar{\theta}_{k2}(T-s)}-e^{-\theta_\sigma(T-s)}\right.$$
$$\left.+\frac{\omega_{k2}}{\theta_\sigma-\bar{\theta}_{k2}}\left(e^{-\theta_\sigma(T-t)}-e^{-\bar{\theta}_{k2}(T-t)}\right)\right]$$

From (15), we consider the integral term

$$\left[\int_t^T\sigma(s)ds\right]^2=R(t)+\left[\int_t^T P_1(s)d\overline{W}_1(s)\right]^2+\left[\int_t^T P_2(s)d\overline{W}_2(s)\right]^2$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)d\overline{W}_1(s)$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_2(s)d\overline{W}_2(s)$$
$$+2\int_t^T P_1(s)d\overline{W}_1(s)\cdot\int_t^T P_2(s)d\overline{W}_2(s)\,. \tag{17}$$

where

$$R(t)=\sum_{i=1}^6 M^2{}_i(t)+2\sum_{i=2}^6 M_1(t)M_i(t)+2\sum_{i=3}^6 M_2(t)M_i(t)$$
$$+2\sum_{i=4}^6 M_3(t)M_i(t)+2\sum_{i=5}^6 M_4(t)M_i(t)++2\sum_{i=6}^6 M_5(t)M_i(t)$$

By applying Itō isometry, we have

$$\left[\int_t^T\sigma(s)ds\right]^2=R(t)+\int_t^T P_1^2(s)ds+\int_t^T P_2^2(s)ds$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)d\overline{W}_1(s)$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_2(s)d\overline{W}_2(s)$$
$$+2\int_t^T\int_t^T P_1(s)\cdot P_2(s)d\overline{W}_1(s)d\overline{W}_2(s)\,. \tag{18}$$

$$\left[\int_t^T\sigma(s)ds\right]^2=R(t)+\int_t^T P_1^2(s)ds+\int_t^T P_2^2(s)ds$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)d\overline{W}_1(s)$$
$$+2\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_2(s)d\overline{W}_2(s)$$
$$+2\int_t^T P_3(s)d\overline{W}_3(s)\,. \tag{19}$$

where $\cdot P_1(s)\cdot P_2(s)=P_3(s)$ and we assume

$$\int_t^T\int_t^T d\overline{W}_1(s)d\overline{W}_2(s)\approx\int_t^T d\overline{W}_3(s)\,.$$

Next, we consider

$$\left[\int_t^T\sigma(s)ds\right]\cdot\int_t^T d\overline{W}_1(s)\approx\int_t^T P_1^2(s)ds$$
$$+\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)d\overline{W}_1(s)$$
$$+\int_t^T P_3(s)d\overline{W}_3(s)\,. \tag{20}$$

We substitute (19) and (20) into (15), we obtain

$$S(T)=S(t)\exp\left\{r(T-t)+\frac{(t-T)}{2}R(t)\right.$$
$$+\frac{(2+t-T)}{2}\int_t^T P_1^2(s)ds+\frac{(t-T)}{2}\int_t^T P_2^2(s)ds$$
$$+(1+t-T)\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)d\overline{W}_1(s)$$
$$+(t-T)\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_2(s)d\overline{W}_2(s)$$
$$\left.+(1+t-T)\int_t^T P_3(s)d\overline{W}_3(s)\right\}. \tag{21}$$

We denote

$$\Omega(t)^2=(1+t-T)\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_1(s)^2ds$$
$$+(t-T)\left(\sum_{i=1}^6 M_i(t)\right)\int_t^T P_2(s)^2ds$$
$$+(1+t-T)\int_t^T P_3(s)^2ds\,. \tag{22}$$

and

$$\Lambda(t)=\frac{(t-T)}{2}R(t)$$
$$+\frac{(2+t-T)}{2}\int_t^T P_1^2(s)ds+\frac{(t-T)}{2}\int_t^T P_2^2(s)ds\,. \tag{23}$$

From (14), we solve $E^Q\left[1_{S(T)>K} \mid F_t\right]$ by denoting η as

$$\eta \le d_2 = \frac{1}{\Omega(t)}\left(\ln\frac{S(t)}{K} + r(T-t) - \Lambda(t)\right). \tag{24}$$

Since η follows a standard normal distribution, thus

$$E^Q\left[1_{S(T)>K} \mid F_t\right] = N(d_2) \tag{25}$$

where $d_2 = \frac{1}{\Omega(t)}\left(\ln\frac{S(t)}{K} + r(T-t) - \Lambda(t)\right)$.

Next we solve

$$E^Q\left[S(T)1_{S(T)>K} \mid F_t\right] = E^Q\left[S(T)1_{\eta>-d_2} \mid F_t\right]$$

$$= S(t)\exp\{r(T-t) - \Lambda(t)\}E^Q\left[e^{\Omega(t)\eta}1_{\eta>-d_2} \mid F_t\right]$$

$$= S(t)\exp\{r(T-t) - \Lambda(t)\}e^{(1/2)\Omega(t)^2}N(d_1) \tag{26}$$

where $d_1 = d_2 + \Omega(t)$.

By applying (21)–(26) into (14), the European option price with stochastic volatility and stochastic market price of risk can be formulated.

IV. The Formulas

The price of European call option *C(t)* with maturity *T*, strike price *K*, stochastic volatility $\sigma(t)$ and stochastic market price of risk $k(t)$ on stock *S(t)* can be evaluated by the formula

$$C(t) = S(t)e^{\left[\Lambda(t)+\frac{1}{2}\Omega(t)^2\right]}N(d_1) - Ke^{-r(T-t)}N(d_2). \tag{27}$$

where

$$d_1 = \frac{1}{\Omega(t)}\left(\ln\frac{S(t)}{K} + r(T-t) - \Lambda(t) + \Omega(t)^2\right)$$

and

$$d_2 = d_1 - \Omega(t).$$

The put-call parity formula between European call and put option when related to stochastic volatility and stochastic market price of risk can be derived as follows.

$$C(t) - P(t) = S(t)e^{\left[\Lambda(t)+\frac{1}{2}\Omega(t)^2\right]} - Ke^{-r(T-t)} \tag{28}$$

and by applying the formula of put-call parity, we can derive a formula of European put option *P(t)* with maturity *T*, strike price *K*, stochastic volatility $\sigma(t)$ and stochastic market price of risk $k(t)$ on stock *S(t)* as follows.

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S(t)e^{\left[\Lambda(t)+\frac{1}{2}\Omega(t)^2\right]}N(-d_1) \tag{29}$$

Consequently, the call option price, put-call parity relationship and put option price can be formulated under the circumstance of stochastic volatility and market price of risk (MPR).

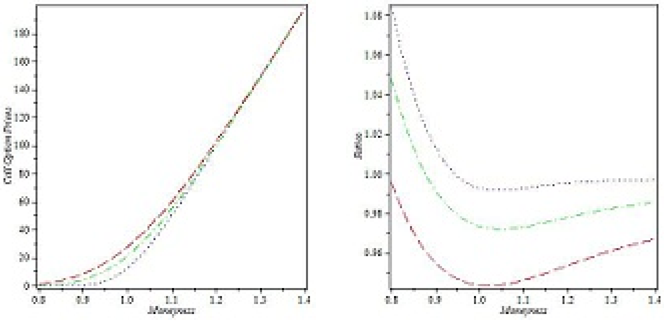
V. The Simulations

In this section, the simulations are performed to investigate our extended models. The option price sensitivity is simulated by setting the parameters as follows [23]:

$K = 500$	$\mu_\sigma = 0.10$	$\mu_{k1} = 0.20$	$\mu_{k2} = 0.20$
$r = 0.05$	$\omega_{\sigma 1} = 0.10$	$\omega_{k1} = 0.10$	$\omega_{k2} = 0.10$
	$\omega_{\sigma 2} = 0.10$		

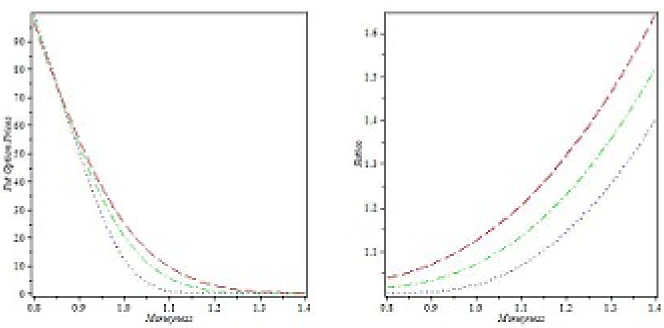
A. Sensitivity of call option prices to the maturity T

We investigate the sensitivity of the model output to the change of maturity by setting maturity *T* at three different values: *T* = 0.1, *T* = 0.3 and *T* = 0.5. We observe the change of the model results of by setting three scenarios with the same friction coefficient parameters $\theta_\sigma = 0.40$, $\theta_{k1} = 0.35$ and $\theta_{k2} = 0.35$. In the first scenario, in Fig.1 and Fig.2, we investigate the behavior of our model by setting $\sigma(0) = 0.10$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. We can see that as the maturity time increases, the call option prices decreases and the put option prices increase. In the second scenario, in Fig.3 and Fig.4, we investigate the behavior of our model by setting $\sigma(0) = 0.20$, $k_1(0) = 0.40$ and $k_2(0) = 0.40$. We can see that as the maturity time increases, the call option prices increases and the put option prices decrease. However, when it the moneyness is in-the-money, call option prices with less maturity performs higher prices compared to the other higher maturities. While the third scenario, in Fig.3 and Fig.4, when $\sigma(0) = 0.30$, $k_1(0) = 0.60$ and $k_2(0) = 0.60$, shows the similar trend for both in-the-money and out-of-the money with the different maturities.



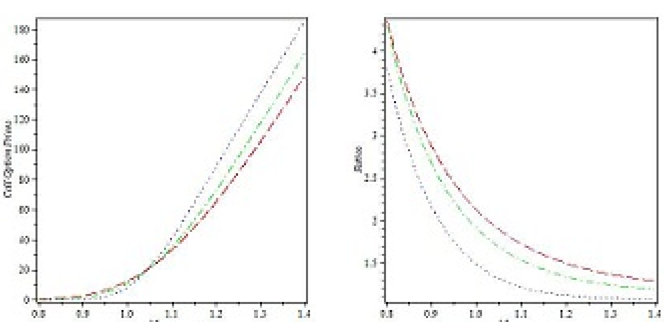
(a) (b)

Fig.1 Sensitivity of call option prices to the maturity T when $\sigma(0) = 0.10$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).



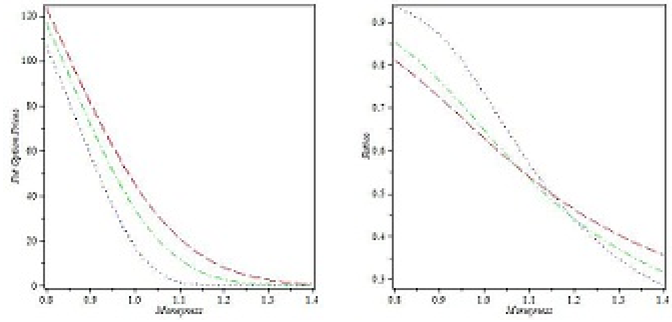
(a) (b)

Fig.2 Sensitivity of put option prices to the maturity T when $\sigma(0) = 0.10$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).



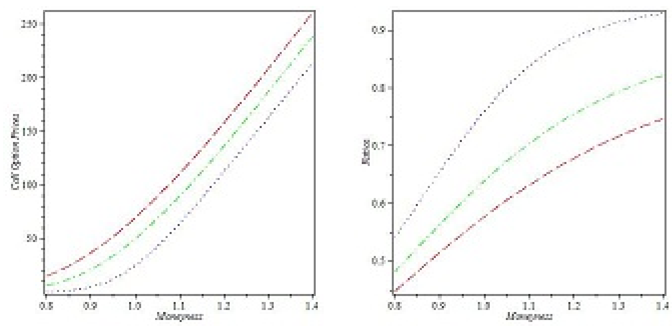
(a) (b)

Fig.3 Sensitivity of call option prices to the maturity T when $\sigma(0) = 0.20$, $k_1(0) = 0.40$ and $k_2(0) = 0.40$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).



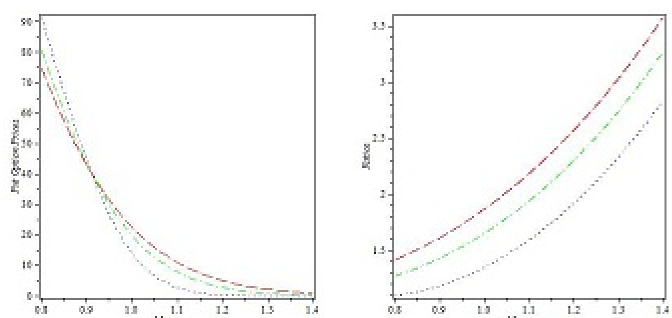
(a) (b)

Fig.4 Sensitivity of put option prices to the maturity T when $\sigma(0) = 0.20$, $k_1(0) = 0.40$ and $k_2(0) = 0.40$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).



(a) (b)

Fig.5 Sensitivity of call option prices to the maturity T when $\sigma(0) = 0.30$, $k_1(0) = 0.60$ and $k_2(0) = 0.60$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).

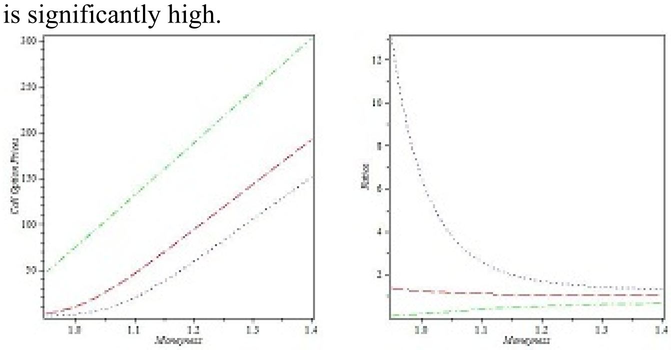


(a) (b)

Fig.6 Sensitivity of put option prices to the maturity T when $\sigma(0) = 0.30$, $k_1(0) = 0.60$ and $k_2(0) = 0.60$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).

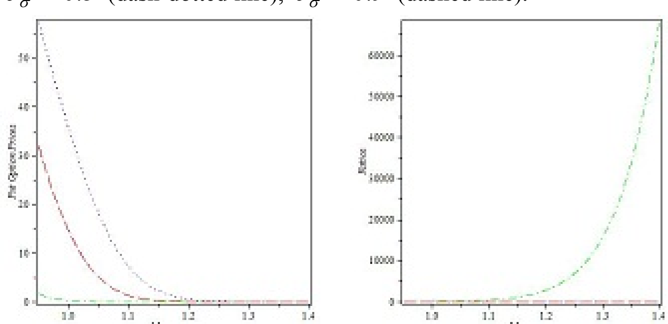
B. Sensitivity of call option prices to the volatility friction coefficient θ_σ

We investigate the sensitivity of the model output to the change of volatility friction coefficient θ_σ by setting the volatility friction coefficient θ_σ at three different values: $\theta_\sigma = 0.20$, $\theta_\sigma = 0.50$ and $\theta_\sigma = 0.90$. We observe the change of the model results of by setting three scenarios with the same market price of risk friction coefficient parameters, $\theta_{k1} = 0.35$, $\theta_{k2} = 0.35$; and the other parameters as $\sigma(0) = 0.10$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. First scenario, in Fig.7 and Fig.8, we investigate behavior of our model by setting $T = 0.1$ as the short range of maturity. Second scenario, in Fig.9 and Fig.10, we investigate behavior of our model by setting $T = 0.3$ as the middle range of maturity. And third scenario, in Fig.11 and Fig.12, we investigate behavior of our model by setting $T = 0.5$ as the long range of maturity. Obviously, the model results show that the sensitivity of the model output to the change of volatility friction coefficient θ_σ is significantly high.



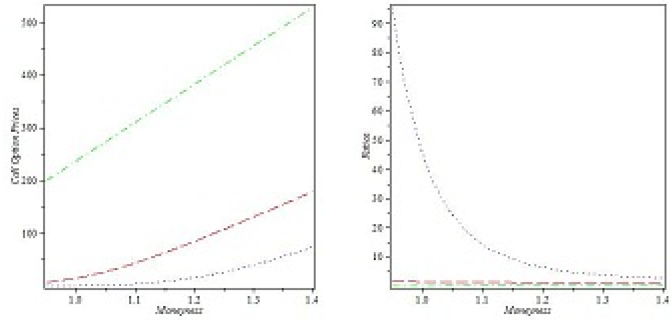
(a) (b)

Fig.7 Sensitivity of call option prices to volatility friction coefficient θ_σ when $T = 0.1$, $\sigma(0) = 0.20$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_\sigma = 0.2$ (dotted line), $\theta_\sigma = 0.5$ (dash-dotted line), $\theta_\sigma = 0.9$ (dashed line).



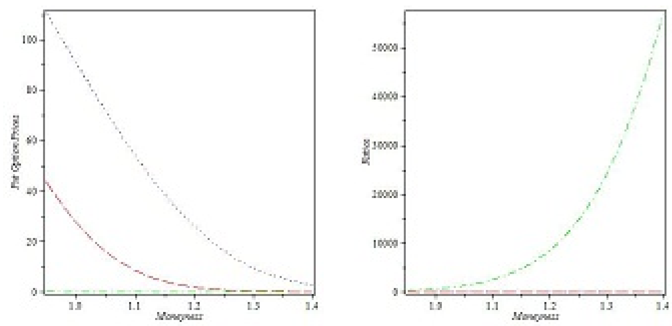
(a) (b)

Fig.8 Sensitivity of put option prices to volatility friction coefficient θ_σ when $T = 0.1$, $\sigma(0) = 0.20$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_\sigma = 0.2$ (dotted line), $\theta_\sigma = 0.5$ (dash-dotted line), $\theta_\sigma = 0.9$ (dashed line).



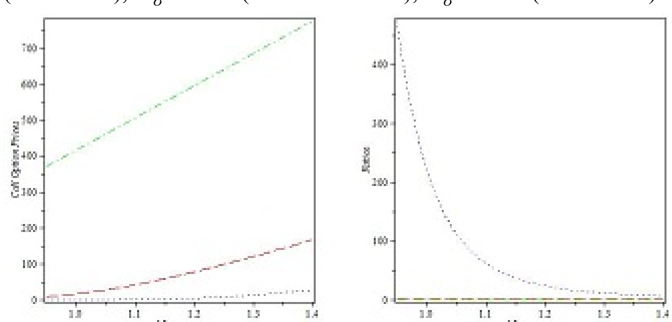
(a) (b)

Fig.9 Sensitivity of call option prices to volatility friction coefficient θ_σ when $T = 0.3$, $\sigma(0) = 0.20$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_\sigma = 0.2$ (dotted line), $\theta_\sigma = 0.5$ (dash-dotted line), $\theta_\sigma = 0.9$ (dashed line).



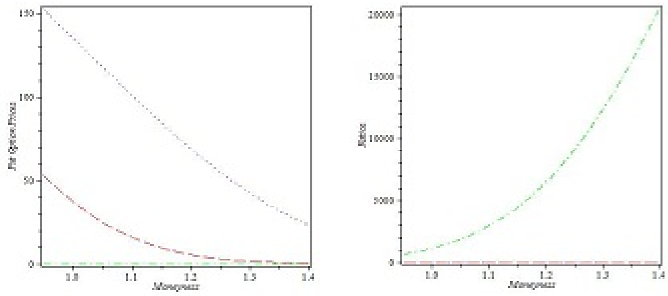
(a) (b)

Fig.10 Sensitivity of put option prices to the volatility friction coefficient θ_σ when $T = 0.3$, $\sigma(0) = 0.20$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_\sigma = 0.2$ (dotted line), $\theta_\sigma = 0.5$ (dash-dotted line), $\theta_\sigma = 0.9$ (dashed line).



(a) (b)

Fig.11 Sensitivity of call option prices to the volatility friction coefficient θ_σ when $T = 0.5$, $\sigma(0) = 0.20$, $k_1(0) = 0.20$ and $k_2(0) = 0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_\sigma = 0.2$ (dotted line), $\theta_\sigma = 0.5$ (dash-dotted line), $\theta_\sigma = 0.9$ (dashed line).

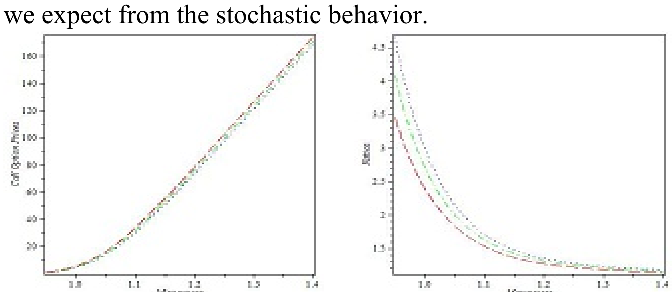


(a) (b)

Fig.12 Sensitivity of put option prices to the volatility friction coefficient θ_σ when T = 0.5 $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, T = 0.1 (dotted line), T = 0.3 (dash-dotted line), T = 0.5 (dashed line).

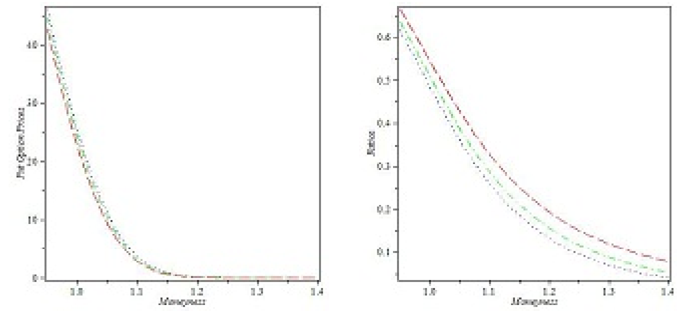
C. Sensitivity of call option prices to the market price of risk friction coefficient θ_{ki}

We investigate the sensitivity of the model output to the change of market price of risk friction coefficient θ_{ki} by setting the market price of risk friction coefficient θ_{ki} at three different values: $\theta_{ki}=0.2$, $\theta_{ki}=0.5$ and $\theta_{ki}=0.9$. We observe the change of the model results of by setting three scenarios with the same volatility friction coefficient parameters, $\theta_\sigma=0.35$; and the other parameters as $\sigma(0)=0.10$, $k_1(0)=0.20$ and $k_2(0)=0.20$. First scenario, in Fig.13 and Fig.14, we investigate behavior of our model by setting T = 0.1 as the short range of maturity. Second scenario, in Fig.15 and Fig.16, we investigate behavior of our model by setting T = 0.3 as the middle range of maturity. And third scenario, in Fig.17 and Fig.18, we investigate behavior of our model by setting T = 0.5 as the long range of maturity. Obviously, the model results show that the sensitivity of the model output to the change of volatility friction coefficient θ_σ is high as we expect. These are results of different scenarios as we expect from the stochastic behavior.



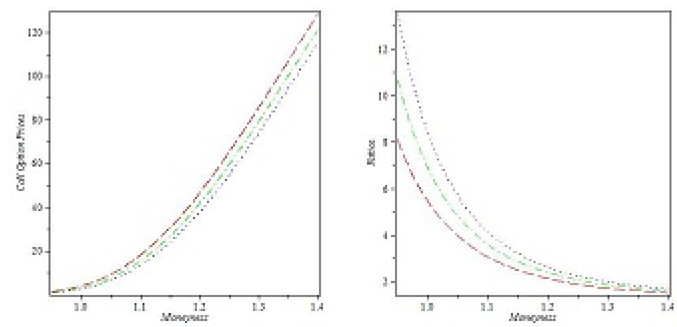
(a) (b)

Fig.13 Sensitivity of call option prices to market price of risk friction coefficient θ_{ki} when T = 0.1, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.



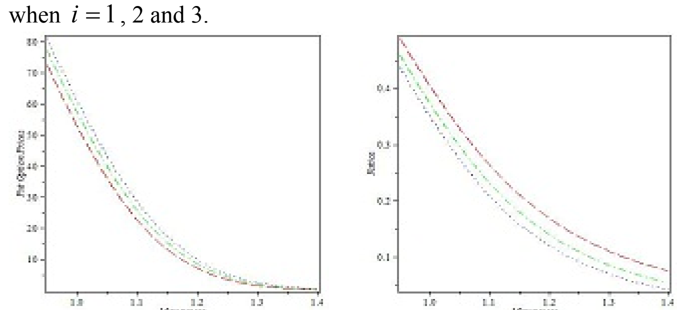
(a) (b)

Fig.14 Sensitivity of put option prices to market price of risk friction coefficient θ_{ki} when T = 0.1, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.



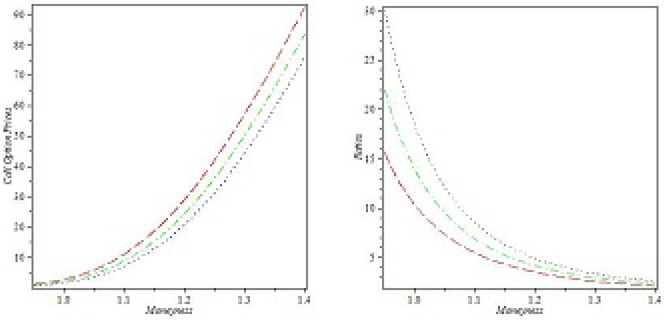
(a) (b)

Fig.15 Sensitivity of call option prices to market price of risk friction coefficient θ_{ki} when T = 0.3, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.



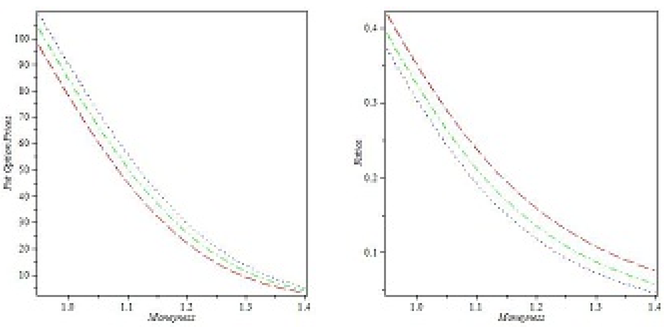
(a) (b)

Fig.16 Sensitivity of put option prices to market price of risk friction coefficient θ_{ki} when T = 0.3, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.



(a) (b)

Fig.17 Sensitivity of call option prices to market price of risk friction coefficient θ_{ki} when T = 0.3, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.



(a) (b)

Fig.18 Sensitivity of put option prices to market price of risk friction coefficient θ_{ki} when T = 0.3, $\sigma(0)=0.20$, $k_1(0)=0.20$ and $k_2(0)=0.20$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes model over the option price obtained from our stochastic model. In the figure, $\theta_{ki}=0.2$ (dotted line), $\theta_{ki}=0.5$ (dash-dotted line), $\theta_{ki}=0.9$ (dashed line) when $i=1$, 2 and 3.

VI. Conclusion

In this paper, our purpose is to develop an option pricing model under the stochastic behavior of both volatility and market price of risk. By setting up the volatility and market price of risk process to follow Ornstein-Unlenbeck processes, we can analytically derive the formula for European option price with the same approach of Abraham Loui's previous work [20]. We test the sensitivity of each parameter to the model price by setting different scenarios. Each scenario of parameter setting implies each situation of the real-world market. The simulation results of sensitivity test perform the obvious stochastic behavior as we expected.

Acknowledgement

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