Kernels For the Remainder Term of Gauss Quadrature Formulae Type

DANIEL VLADISLAV, University of Petrosani, Department of Mathematics,
University Street 20, 332006 Petrosani ROMANIA daniel.vladislav@yahoo.com

Abstract - We study the kernels in the contour integral representation of the remainder term of Gauss-Lobatto quadratures, in particular the location of their maxim on circular and elliptic contours. Quadrature rules with Chebyshev weight functions of all four kinds receive special attention. We also study a general Gauss Chebyshev-Stancu quadrature with double fixed nodes.

Key—Words - quadrature, kernels contour integral representation, remainder term, Gauss-Lobatto, Gauss-Chebyshev-Stancu

I. Introduction

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and D be its interior. Let f be analytic in D and continuous on $\bar{D}$. We consider an interpolatory quadrature rule

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{\nu=1}^{N} \lambda_\nu f(\tau_\nu) + R_N(f) \quad (1)$$

with

$$-1 \le \tau_N < \tau_{N-1} < \ldots < \tau_1 \le 1 \quad (2)$$

and

$$\omega_N(z) = \omega_N(z,w) = \prod_{\nu=1}^{N} (z-\tau_\nu),\ z \in \mathbb{C} \quad (3)$$

denote its node polynomial (which in general depends on w), and define:

$$\rho_N(z,w) = \int_{-1}^{1} \frac{\omega_N(t,w)}{z-t} w(t)\,dt,\ z \in \mathbb{C} \setminus [-1,1] \quad (4)$$

then, as is well known, the remainder term R_N in (1) admits contour integral representation

$$R_N(f) = \frac{1}{2\pi i} \oint_\Gamma K_N(z,w)\, f(z)\,dz \quad (5)$$

where the "kernel" K_N can be expressed, e.g., in the form (see [1]):

$$K_N(z,w) = \frac{\rho_N(z,w)}{\omega_N(z,w)},\ z \in \Gamma \quad (6)$$

Note that ω_N in (3) and (4) may be multiplied by any constant $C \neq 0$ without affecting the validity of (6). It is also evident from (6) that

$$K_N(\bar{z},w) = \overline{K_N(z,w)} \quad (7)$$

In order to estimate the error in (1) by means of

$$|R_N(f)| \leqslant (2\pi)^{-1} l(\Gamma) \max_{z\in\Gamma} |K_N(z,w)| \max_{z\in\Gamma} |f(z)| \quad (8)$$

where $l(\Gamma)$ is the length of the contour Γ, it becomes necessary to study the magnitude of $|K_N|$ on Γ. This has been done in a number of papers (see [1]) for Gauss-type and other quadrature formulae, and for contours Γ that are either concentric circles centered at the origin or confocal ellipses with focal points at ± 1. The thrust of this work has been directed towards upper bounds, or asymptotic estimates, for the maximum of $|K_N|$ in (8). In attempt to remove uncertainties inherent in such estimates, W. Gautschi [2] for Gauss, Gauss-Lobatto and Gauss-Radau formulae, the precise location on Γ where $|K_N|$ attains its maximum, and we suggested simple recursive techniques to evaluate $K_N(z,w)$ for any $z \in \mathbb{C} \setminus [-1,1]$. Here we investigate, in the same spirit quadrature rules of Gauss-Stancu type, especially for any of the four Chebyshev weight functions:

$$\begin{aligned} w_1(t) &= (1-t^2)^{-\frac{1}{2}} \\ w_2(t) &= (1-t^2)^{\frac{1}{2}} \\ w_3(t) &= (1-t)^{-\frac{1}{2}}(1+t)^{\frac{1}{2}} \\ w_4(t) &= (1-t)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}} \end{aligned} \quad (9)$$

II. SOME GENERAL RESULTS FOR CIRCULAR CONTOURS

In this sections, $\Gamma = C_r$, $C_r = \{z \in \mathbb{C} : |z| = r\}$, where $r > 1$. For positive weight functions w and quadrature rules of Gaussian type, with $N = n$ it is known from [2] that:

$$\max_{z\in C_r} |K_N(z,w)| = \begin{cases} K_N(r,w), \text{ if } \dfrac{w(t)}{w(-t)} \text{ is nondec on } (-1,1) \\ |K_N(-r,w)|, \text{ if } \dfrac{w(t)}{w(-t)} \text{ is nonin on } (-1,1) \end{cases} \tag{10}$$

We now explore the implications of this results to Gauss-Lobatto formulae (subsection 2.1.) and Gauss – Radau formulae (subsection 2.2.)

2.1. Gauss-Lobatto formulae

These are the quadrature rules (1) with N=n+2, $\tau_N = -1$, $\tau_1 = 1$ and $R_N(f) = 0$, whenever $f \in P_{2n+1}$ (the class of polynomials of degree 2n+1). They are clearly interpolatory. We denote $w^L(t) = (1-t^2)w(t)$ and write $\Pi_n(\cdot, w^L)$ for the polynomial of degree n (suitably normalized) orthogonal with respect to the weight function w^L. It is well known that:

$$\omega_{n+2}(z,w) = (1-z^2)\Pi_n(z;w^L) \tag{11}$$

from which then follows:

$$\rho_{n+2}(z,w) = \int_{-1}^{1} \frac{(1-t^2)\Pi_n(t;w^L)}{z-t} w(t)\,dt =$$
$$= \int_{-1}^{1} \frac{\Pi_n(t;w^L)}{z-t} w^2(t)\,dt = \rho_N(z,w)$$

and, therefore by (6)

$$K_{n+2}(z,w) = \frac{K_N(z;w^L)}{1-z^2} \tag{12}$$

Here $K_n(\cdot, w^L)$ is the Kernel for n-point Gauss formula relative to the weight function w^L. Since $|1-z^2|$ attains its minimum of C_r at $z = r$ and $z = -r$, and since $\dfrac{w^L(t)}{w^L(-t)} = \dfrac{w(t)}{w(-t)}$, we have as an immediate consequence of (10)

that: $$\max_{z\in C_r} |K_{n+2}(z,w)| = \begin{cases} \dfrac{1}{r^2-1} K_n(r;w^L) \\ \dfrac{1}{r^2-1} |K_n(-r;w^L)| \end{cases} \tag{13}$$

depending on whether $\dfrac{w(t)}{w(-t)}$, is nondecreasing or nonincreasing respectively. In particular, for the Jacobi weight functions $w(t) = (1-t)^\alpha (1+t)^\beta$, $\alpha > -1$, $\beta > -1$ the first relation in (13) holds if $\alpha \leqslant \beta$ and the second if $\alpha > \beta$.

2.2. Gauss – Radau formulae

These are pairs of such formulae namely (1) with N= n+1, τ_N = -1, and (1) with N= n+1, τ_1=1, both having $R_n(f)$=0 for f$\in$ $\mathbf{P}_{2n}$. It suffices to consider one of them, say the former since the kernels of the two formulae are simply related. If we denote w(-t)=w^*(t) and write $K_N^{(\pm1)}(.;w)$ for the Kernel of the Radau formula with τ_1= -1 and τ_1=1, respectively, a simple computation indeed will show that $K_N^{(+1)}(z,w) = -\overline{K_N^{(-1)}(-\bar{z};w^*)}$, where bars indicate complex conjugation. Therefore,

$$K_N^{(+1)}(z,w) = K_N^{(-1)}(-\bar{z};w^*) \tag{14 }$$

i.e., the modulus of $K_N^{(+1)}$ for the weight function at the point z has the same value as the modulus of $K_N^{(-1)}$ for the weight function w^* at the point $-\bar{z}$, the mirror image of z with respect to the imaginary axis.

For the Radau formula with τ_N = -1, we write $w^{\Re}_{(t)}$ = (1+t) w(t) and have, as is well known,

$$W_{(n+1)}(z;w) = (1+z)\pi_n(z;w^{\Re}) \tag{15}$$

There folows, similar to the case of Lobatto formulae,

$$K_{n+1}(z,w) = \frac{K_N(z;w^{\Re})}{1+z} \qquad (16),$$

Where $K_N(\cdot;w^{\Re})$ is the Kernel for the n- point Gauss formula relative to the weight function $w^{\Re}$. Since $|1+z|$ on $\square_r$ attains its minimum at z = -r, we can now apply the second results in (10), giving

$$\max\left|K_{n+1}(z,w)\right| = \frac{\left|K_n(-r,w^{\Re})\right|}{r-1} \quad (17),$$

provided $\frac{w^{\Re}(t)}{w^{\Re}(-t)}$ is no increasing on (-1,1). Unfortunately, this condition is not satisfied for the Chebysev weights w_1, w_2, w_3, (cf. (9)). We conjecture, in fact, that the maximum in (17) obtained at $z = r$, rather than $z = -r$, when $w = w_3$

III.REMAINDER KERNELS FOR CHEBYSHEV WEIGHT FUNCTIONS

In this section, after some preliminaries on orthogonal polynomials, we provide explicit formulae for Gauss-Lobatto, and Gauss-Stancu type rules, of $K_n(\cdot,w)$ when $w = w_i$, $(i=\overline{1,2})$ (cf (9)).

3.1 Preliminaries

We shall need some facts about Jacobi polynomials with half-integer parameters. They are given here in a form general enough to be applicable (if need be) to a Lobatto formulae with multiple fixed nodes.

Lemma 3.1. *The polynomial of degree n orthogonal on $(-1,1)$ with respect to the weight functions $(1-t^2)^{-\frac{1}{2+k}}$, $k \geqslant 0$ an integer, is given by $T_{n+k}^{(k)}(t)$ where T_m denotes the n-th degree Chebyshev polynomial of the first kind.*

Proof: See equations (6.21.7) in [4] and the paragraph following that equations.

The following lemma is also known, but are stated here in a form were suitable, for our purposes. We recall that Chebyshev polynomials U_n, V_n of the second and third kind (orthogonal relative to weight functions $(1-t^2)^{\frac{1}{2}}$ and $(1-t)^{-\frac{1}{2}}(1+t)^{\frac{1}{2}}$, respectively, are given by>

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

$$V_n(\cos\theta) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta} \qquad (18)$$

Lemma 3.2. *Let $U_{n,k}$ be a polynomial of degree n orthogonal on $(-1,1)$ with respect to the weight functions $(1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}+k}$, $k \geqslant 0$ an integer. Then>*

$$U_{n,0}(t) = U_n(t)$$

$$U_{n,k}(t) = \frac{1}{1+t}U_{n+1,k-1}(t) +$$

$$+\frac{\left(n+k+\frac{1}{2}\right)(n+k+1)}{(1+t)\left(n+\frac{1}{2}k+\frac{1}{2}\right)\left(n+\frac{1}{2}k+1\right)}U_{n,k-1}(t) \qquad (19)$$

Proof: Define

$$U_{n,k}(t) = $$

$$= \left[\frac{n!(n+k+1)!\sqrt{\pi}}{2\Gamma\left(n+\frac{k}{2}+1\right)\Gamma\left(n+\frac{k}{2}+\frac{3}{2}\right)}\right]P_n^{\left(\frac{1}{2},\frac{1}{2}+k\right)}(t)$$

and use the second relation in [4], eq (6.5.4) with $\alpha = \frac{1}{2}$,

$$\beta = -\frac{1}{2}+k.$$

Lemma 3.3.

Let $V_{n,k}$ be the polynomial of degree n orthogonal on (-1,1) with respect to the weight function $(1-t)^{-1/2}(1+t)^{1/2+k}$, $k \geqslant 0$ are integer. Then

$V_{n,0(t)=}\ V_{n(t)}$

$$V_{n,k}(t) = \frac{1}{1+t}\left\{V_{n+1,k-1}(t) + \frac{(n+k)\left(n+k+\frac{1}{2}\right)}{\left(n+\frac{1}{2}k\right)\left(n+\frac{1}{2}k+\frac{1}{2}\right)}V_{n,k-1}(t)\right\}$$

(20)

Proof: Define

$$V_{n,k}(t) = \left[\frac{n!(n+k)!\sqrt{\pi}}{\Gamma\left(n+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(n+\frac{k}{2}+1\right)}\right]P_n^{\left(-\frac{1}{2},\frac{1}{2}+k\right)}(t),$$

and use the second relation in [4] with

$\alpha = -\frac{1}{2}, \beta = -\frac{1}{2}+k$

3.2 Gauss-Chebyshev-Lobatto formulae

We begin with the weight functions w_1 and consider (1) with $w = w_1$, $N = n+2$, $\tau_N = -1$, $\tau_1 = 1$, $R_N(f) = 0$, for $f \in P_{2n+1}$. Since the nodes τ_ν, $2 \leqslant \nu \leqslant N-1$, are the zeros of $\Pi_n\left(\cdot;\left(1-t^2\right)w_1\right) = \Pi_n(\cdot;w_2)$, we may take

$$\omega_{n+2}(z,w_1) = \left(1-z^2\right)U_n(z) \tag{21}$$

giving

$$\rho_{n+2}(z,w_1) = \int_{-1}^{1}\frac{\left(1-t^2\right)U_n(t)}{z-t}w_1(t)\,dt = \int_{-1}^{1}\frac{U_n(t)}{z-t}w_2(t)\,dt \tag{22}$$

Now it is well known (cf. [2] pp. 177) that:

$$U_n(z) = \frac{u^{n+1}-u^{-(n+1)}}{u-u^{-1}},$$

$$\int_{-1}^{1}\frac{U_n(t)}{z-t}w_2(t)\,dt = \frac{\pi}{u^{n+1}} \tag{23}$$

where t and u are related by the familiar conformal maps

$$z = \frac{1}{2}\left(u+u^{-1}\right), |u| > 1 \tag{24}$$

which transforms the exterior of the unit circle $\{u \in \mathbb{C} : |u| > 1\}$, into the whole z-plane cut along $[-1,1]$. Concentric circles $|u| = \rho$, $\rho > 1$ there by are mapped into confocal ellipses

$$\varepsilon_\rho = \left\{z \in \mathbb{C} : z = \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}\right), 0 \leqslant \theta \leqslant 2\pi\right\} \tag{25}$$

with foci at ± 1 and sum of semiarces equal to ρ.

Substituting (19) in (17) and (18), and noting that $z^2 - 1 = \frac{\left(u-u^{-1}\right)^2}{4}$, one obtains

$$K_{n+2}(z,w_1) = -\frac{4\pi}{u^{n+1}\left(u-u^{-1}\right)\left(u^{n+1}-u^{-(n+1)}\right)} \tag{26}$$

Proceeding to the weight function W_2, we recall that the node $\tau_\nu, 2 \le \nu \le N-1$, are now the zeros of $\pi_n\left(\cdot;\left(1-t^2\right)w_2\right) = \pi_n\left(\cdot;\left(1-t^2\right)^{\frac{3}{2}}\right)$, hence, by Lemma 3.1. (with k=2), the zeros of T''_{n+2}. Therefore, $\omega_{n+2}(z;w_2) = \left(1-z^2\right)\mathrm{T}''_{n+2}(z)$ which, by the differential equation satisfied by T_{n+2}, becomes

$$\omega_{n+2}(z;w_2) = z\mathrm{T}'_{n+2}(z) - (n+2)^2\,\mathrm{T}_{n+2}(z)$$

With the help of

$\mathrm{T}_{n+2}(z) = \frac{1}{2}\left[U_{n+2}(z) - U_n(z)\right], \mathrm{T}'_{n+2} = (n+2)U_{n+1}(z)$

one then gets

$$\omega_{n+2}(z;w_2) = \frac{n+2}{2}\left\{-(n+2)\left[U_{n+2}(z)-U_n(z)\right] + 2zU_{n+1}(z)\right\}$$

Which can be simplified, using the recurrence relation $2zU_{n+1} = U_{n+2} + U_n$, to

$$\omega_{n+2}(z;w_2) = -\frac{(n+1)(n+2)}{2}\left\{U_{n+2}(z) - \frac{n+3}{n+1}U_n(z)\right\} \tag{27}$$

In terms of the variable u, cf.(20), using the first relation in (19), this can be written as:

$$\omega_{n+2}(z,w_2) = -\frac{(n+1)(n+2)}{2\left(u-u^{-1}\right)}\left\{u^{n+3} - u^{-(n+3)} - \frac{n+3}{n+1}\left(u^{n+1} - u^{-(n+1)}\right)\right\}$$

From (27), and the second relation in (19), we find

$$\rho_{n+2}(z,w_2) = -\frac{(n+1)(n+2)}{2}\left\{\int_{-1}^{1}\frac{U_{n+2}(t)}{z-t}w_2(t)\,dt - \frac{n+3}{n+2}\int_{-1}^{1}\frac{U_n(t)}{z-t}w_2(t)\,dt\right\} =$$
$$= -\frac{(n+1)(n+2)\pi}{2u^{n+1}}\left\{u^{-2} - \frac{n+3}{n+1}\right\}$$

Therefore, finally,

$$\mathrm{K}_{n+2}(z,w_2) = \frac{\pi}{u^{n+1}}\cdot\frac{u^{-1}-u^{-3}-\frac{n+3}{n+1}\left(u-u^{-1}\right)}{u^{n+3}-u^{-(n+3)}-\frac{n+3}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)} \quad (28)$$

* In the case $w = w_3$ we have

$$\omega_{n+2}(t,w_3) = \left(1-t^2\right)\pi_n\left(t;(1-t)^{\frac{1}{2}}(1+t)^{\frac{3}{2}}\right);$$

hence, by Lemma 3.2. (with k=1) and (16),

$$\omega_{n+2}(z,w_3) = \left(1-z^2\right)\cdot U_{n,1}(z) = (1-z)\left\{U_{n+1}(z)+\frac{n+2}{n+1}U_n(z)\right\}$$

Using (19) togheter with $1-z = -\dfrac{(u-1)^2}{2u}$ yields

$$\omega_{n+2}(z,w_3) = -\frac{1}{2}\cdot\frac{u-1}{u+2}\left\{u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)\right\}$$

Furtheremore ,

$$\rho_{n+2}(z,w_3) == \int_{-1}^{1}\frac{\omega_{n+2}(t,w_3)}{z-t}w_3(t)\,dt$$
$$= \int_{-1}^{1}\frac{U_{n+1}(t)+\frac{n+2}{n+1}U_n(t)}{z-t}w_2(t)\,dt = \frac{\pi}{u^{n+1}}\left(u^{-1}+\frac{n+2}{n+1}\right)$$

giving

$$\mathrm{K}_{n+2}(z,w_3) = -\frac{2\pi}{u^{n+1}}\cdot\frac{u+1}{u-1}\cdot\frac{u^{-1}+\frac{n+2}{n+1}}{u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)} \quad (29)$$

* In the case $w = w_4$ is easily transformed to the previous case, since $w_4(t) = w_3(-t)$ implies

$\omega_{n+2}(z,w_4) = (-1)^n\,\omega_{n+2}(-z;w_3)$ and

$$\rho_{n+2}(z;w_4) = (-1)^{n+1}\rho_{n+2}(-z;w_3)$$

Therefore, $\mathrm{K}_{n+2}(z;w_4) = -\mathrm{K}_{n+2}(-z;w_3)$ or equivalently,

$$\mathrm{K}_{n+2}(z;w_4) = -\overline{\mathrm{K}_{n+2}(-\bar{z};w_3)} \quad (30)$$

The Kernel for $w = w_4$ is thus obtained from that for $w = w_3$ essentially by reflection on the imaginary axis.

3.3 Main results. Gauss-Chebyshev-Stancu type formulae

3.3.1.

We consider a quadrature formulae with double fixed nodes ± 1 and the weight function $w = w_1$, $N = n+4$, $\tau_N = -1$, $\tau_N' = -1$, $\tau_1 = \tau_1' = 1$, and $R_N(f) = 0$ for $f \in P_{2n+3}$ [3]. The nodes fixed polynom's is the form $\left(1-t^2\right)^2$. From here the nodes τ_ν, $2 \leqslant \nu \leqslant N-1$ are the zeros of polynom

$$\Pi_n\left(\cdot,\left(1-t^2\right)^2 w_1\right) = \Pi_n\left(\cdot,\left(1-t^2\right)^{-\frac{1}{2}+2}\right) = T_{n+2}^{(2)}(\cdot)$$

In the last equality we are based on lemma 1. Results that the nodes polynom's is:

$$\omega_{n+4}(z,w_1) = \left(1-z^2\right)^2 T_{n+2}^{(2)}(z) \quad (31)$$

which, by the differential equations satisfied by T_{n+2}, becomes

$$\omega_{n+2}(z,w_2) = zT_{n+2}''(z) - (n+2)^2 T_{n+2}(z)$$

and with the help of:

$$T_{n+2}(z) = \frac{1}{2}\left[U_{n+2}(z) - U_n(z)\right]$$
$$T_{n+2}'(z) = (n+2)U_n(z)$$

one then gets:

$$\omega_{n+4}(z,w_1) = \left(1-z^2\right)^2\left[zT_{n+2}''(z) - (n+2)^2 T_{n+2}(z)\right]$$

or

$$\omega_{n+4}(z,w_1) = -\frac{(n+1)(n+2)}{2}\cdot$$
$$\cdot\left[\left(1-z^2\right)U_{n+2}(z) - \frac{n+3}{n+1}\left(1-z^2\right)U_n(z)\right] \quad (32)$$

and with the relations (6) and (32) we have

$$\rho_{n+4}(z,w_1)=\int_{-1}^{1}\frac{\left(1-t^2\right)^2 T_{n+2}^{(2)}(t)}{z-t}w_1(t)\,dt=$$
$$=-\frac{(n+1)(n+2)}{2u^{n+1}}\left[u^{-2}-\frac{n+3}{n+2}\right]$$

We recall (11) and operate $1-z^2=\frac{\left(u-u^{-1}\right)^2}{4}$ we obtain

$$\omega_{n+4}(z,w_1)=\frac{(n+1)(n+2)}{2}\frac{\left(u-u^{-1}\right)^2}{4}\cdot$$
$$\cdot\left[\frac{u^{n+3}-u^{-(n+3)}}{u-u^{-1}}-\frac{n+3}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)\right]$$

Therefore finally

$$K_{n+4}(z,w_1)=\frac{\rho_{n+4}(z,w_1)}{\omega_{n+4}(z,w_1)}=\frac{4\pi}{u^{n+1}\left(u-u^{-1}\right)^2}\cdot$$
$$\cdot\frac{u^{-1}-u^{-3}-\frac{n+3}{n+1}\left(u-u^{-1}\right)}{u^{n+3}-u^{-(n+3)}-\frac{n+3}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)} \qquad (33)$$

Remark 1 *The kernel of Gauss-Chebyshev-Stancu formulae verify:*

$$K_{n+4}(z,w_1)=\frac{4}{\left(u-u^{-1}\right)^2}K_{n+2}(z,w_2)$$

where $K_{n+2}(z,w_2)$ represented the kernel of Gauss-Lobatto quadrature formulae with simple fixed nodes to the Chebyshev function of second kind .

This last kernel it was determined in relation [28] by the W. Gautschi.

3.3.2. We also consider a quadrature formulae with fixed nodes $\tau_N=-1$ (simple) and $\tau_1=1=\tau_1{}'$(double) follow that the polynom of fixed nodes is $(1-t)(1+t)^2$ and the weight function $w=w_1$.

$\mathrm{N}=n+3$ and $\Re(f)=0$, for any polynom $f\in\mathrm{P}_{2n+2}$.

From here the nodes $\tau_{\nu,2\le\nu\le\mathrm{N}-1}$ are the zeros of polynom:

$$\Pi_n\left(\cdot;(1-t)(1+t)^2 w_1\right)=\Pi_n\left(\cdot;(1-t)^{\frac{1}{2}}(1+t)^{\frac{3}{2}}\right)=\Pi_n\left(\cdot;(1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}+1}\right)$$

and from here we have:

$$\omega_{n+3}(z,w_1)=(1-z)\left(1+z^2\right)U_{n,1}(z) \qquad (34)$$

But cf. Lemma 3.2. we have:

$$U_{n,1}(t)=\frac{1}{1+t}\left\{U_{n+1,0}(t)+\frac{\left(n+\frac{3}{2}\right)(n+2)}{(n+1)\left(n+\frac{3}{2}\right)}U_{n,0}(t)\right\}$$

or

$$U_n(t)=\frac{1}{1+t}U_{n+1}(t)+\frac{1}{1+t}\cdot\frac{n+2}{n+1}U_n(t)$$

Thus:

$$\rho_{n+3}(z,w_1)=\int_{-1}^{1}\frac{(1-t)(1+t)^2U_{n,1}(t)}{z-t}w_1(t)\,dt=$$
$$\int_{-1}^{1}\frac{(1-t)(1+t)^2}{z-t}\cdot\frac{1}{1+t}\left[U_{n+1}(t)+\frac{n+2}{n+1}U_n(t)\right]w_1(t)\,dt=$$
$$\int_{-1}^{1}\frac{1}{z-t}\left[U_{n+1}(t)+\frac{n+2}{n+1}U_n(t)\right]w_2(t)\,dt=$$
$$\int_{-1}^{1}\frac{U_{n+1}(t)}{z-t}w_2(t)\,dt+\frac{n+2}{n+1}\int_{-1}^{1}\frac{U_n(t)}{z-t}w_2(t)\,dt$$
$$=\frac{\Pi}{u^{n+2}}+\frac{n+2}{n+1}\cdot\frac{\Pi}{u^{n+1}}$$

Result that:

$$\rho_{n+3}(z,w_1)=\frac{\Pi}{u^{n+1}}\left(\frac{n+2}{n+1}+\frac{1}{u}\right) \qquad (35)$$

We recall (34) and operate $1-z^2=-\frac{\left(u-u^{-1}\right)^2}{4}$ we obtain :

$$w_{n+3}(z,w_1)=-\frac{\left(u-u^{-1}\right)^2}{4}\left[\frac{u^{n+2}-u^{-(n+2)}}{u-u^{-1}}+\frac{n+2}{n+1}\cdot\frac{u^{n+1}-u^{-(n+1)}}{u-u^{-1}}\right]=$$
$$\frac{\left(u-u^{-1}\right)}{4}\left[u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)\right]$$

For Kernel, with wellknown formulae, we have

$$\mathrm{K}_{n+3}(z;w_1)=\frac{\rho_{n+3}(z;w_1)}{\omega_{n+3}(z;w_1)}=$$

$$\frac{-\frac{\pi}{u^{n+1}}\left[\frac{n+2}{n+1}+\frac{1}{u}\right]}{\frac{u-u^{-1}}{4}\left[u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)\right]}$$

and

$$\mathrm{K}_{n+3}(z;w_1)=$$

$$-\frac{4\pi}{u^{n+1}\left(u-u^{-1}\right)}\cdot\frac{\frac{1}{u}+\frac{n+2}{n+1}}{u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)} \tag{36}$$

Remark 2: The Kernel $\mathrm{K}_{n+3}(z;w_1)$ verify:

$$\mathrm{K}_{n+3}(z;w_1)=\frac{2n}{(n+1)^2}\mathrm{K}_{n+2}(z;w_3)$$

Where $\mathrm{K}_{n+2}(z;w_3)$ represented the Kernel of Gauss-Lobatto quadrature formulae with simple fixed nodes to the Chebyshev weight function of third kind.

3.4.Chebyschev-Radau formulae

In analogy to (19) one has

$$V_n(z)=\frac{u^{n+1}+u^{-n}}{u+1},$$
$$\int_{-1}^{1}\frac{V_n(t)}{z-t}w_3(t)\,dt=\frac{2\pi}{(u-1)u^n} \tag{37}$$

The first relation follows from the second relation in (14) by writing all cosines in exponential form, using Euler's formula, and then putting $u=e^{i\theta}$. To prove the second relation, substitute $t=\cos\theta$ to obtain:

$$\int_{-1}^{1}\frac{V_n(t)}{z-t}w_3(t)\,dt=2\int_0^{\pi}\frac{\cos\left(n+\frac{1}{2}\right)\theta\cdot\cos\frac{1}{2}\theta}{z-\cos\theta}d\theta=$$
$$\int_0^{\pi}\frac{\cos(n+1)\theta\cdot\cos n\theta}{z-\cos\theta}$$

And then use Equation (5.3) in [1] and the equation immediately following it to evaluate the last integral. For reasons indicated in Subsection 2.2, we consider only Radau formulae with the fixed point at -1. Thus, $N=n+1,\tau_n=-1$ in (1), and $\Re_N(f)=0$ for $f\in\mathrm{P}_{2n}$. We treat in turn the four weight functions $w_i, i=1,2,3,4$ (cf. (9))

* For $w=w_1$, in wiew of $\Pi_n(\cdot;(1+t)w_1)=\Pi_n(\cdot;w_3)$, we can take

$\omega_{n+1}(z;w_1)=(1+z)V_n(z)$, which, by the first relation in (37) and $1+z=\frac{(u+1)^2}{2u}$, gives

$$\omega_{n+1}(z;w_1)=\frac{1}{2}(u+1)\left(u^n+u^{-(n+1)}\right)$$

And by the second relation in (37)

$$\rho_{n+1}(z;w_1)=\frac{2\pi}{(u-1)u^n},\qquad \text{hence}$$

$$\mathrm{K}_{n+1}(z;w_1)=\frac{4\pi u}{\left(u^2-1\right)\left(u^{2n+1}+1\right)} \tag{38}$$

* In the case $w=w_2$, we are led to

$$\Pi_n(\cdot;(1+t)w_2)=\Pi_n\left(\cdot;(1-t)^{\frac{1}{2}}(1+t)^{\frac{3}{2}}\right)$$

And may apply Lemma 3.2. ($\mathrm{K}=1$) to obtain

$$\omega_{n+1}(z;w_2)=(1+z)U_{n,1}(z)=$$
$$U_{n+1}(z)+\frac{n+2}{n+1}U_n(z)$$

Using (19) we find

$$\omega_{n+1}(z,w_2)=$$
$$\frac{1}{u-u^{-1}}\left\{u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)\right\}$$

And

$$\rho_{n+1}(z;w_2)=\frac{\pi}{u^{n+1}}\left(u^{-1}+\frac{n+2}{n+1}\right)$$

Giving

$$\mathrm{K}_{n+1}(z,w_2)=$$

$$\frac{\pi}{u^{n+1}}\cdot\frac{1-u^{-2}+\frac{n+2}{n+1}\left(u-u^{-1}\right)}{u^{n+2}-u^{-(n+2)}+\frac{n+2}{n+1}\left(u^{n+1}-u^{-(n+1)}\right)} \quad (39)$$

* For $w=w_3$ since $\Pi_n\left(\cdot;(1+t)w_3\right)=\Pi_n\left(\cdot;(1-t)^{\frac{1}{2}}(1+t)^{\frac{3}{2}}\right)$ we can appeal to Lemma 3.3 (with K=1) and obtain, similary as above, using (37) , that

$$\mathrm{K}_{n+1}(z,w_3)=$$

$$\frac{2\pi}{u^n}\cdot\frac{n+1}{n-1}\cdot\frac{u^{-1}+\frac{2n+3}{2n+1}}{u^{n+2}+u^{-(n+1)}+\frac{2n+3}{2n+1}\left(u^{n+1}+u^{-n}\right)} \quad (40)$$

* Finally, when $w=w_4$, we have $(1+t)w_4=w_2$ so that $\omega_{n+1}(z,w_4)=(1+z)U_n(z)$, and we find using (19) that

$$\mathrm{K}_{n+1}(z,w_4)=\frac{2\pi}{u^{n+1}}\cdot\frac{u-1}{(u+1)\left(u^{n+1}-u^{-(n+1)}\right)} \quad (41)$$

* We also consider a quadrature formulae with double fix node -1 and the weight function $w=w_1$, $\mathrm{N}=n+2, \tau_N=\tau'_N=-1$ and $\Re_N(f)=0$ for $f\in \mathrm{P}_{2n+1}$ [3].

The nodes fixed polynom's is in the form $(1+t)^2$. From here the nodes τ_ν, $2\le \nu\le N-1$ are the zeros of the polynom

$$\Pi_n\left(\cdot;(1+t)^2 w_1\right)=\Pi_n\left(\cdot;(1-t)^{-\frac{1}{2}}(1+t)^{\frac{3}{2}}\right)$$

From Lemma 3.3. (K=1) results for nodes polynom's that:

$$\omega_{n+2}(z;w_1)=(1+z)^2 V_{n,1}(z)=$$

$$(1+z)\left[V_{n+1}(z)+\frac{n+\frac{3}{2}}{n+\frac{1}{2}}V_n(z)\right]$$

Than, from relation (4) we have:

$$\rho_{n+2}(z,w_1)=\int_{-1}^{1}\frac{\omega_{n+2}(t,w_1)}{z-t}w_1(t)\,dt=$$

$$\int_{-1}^{1}\frac{(1+t)\left[V_{n+1}(t)+\frac{n+\frac{3}{2}}{n+\frac{1}{2}}V_n(t)\right]}{z-t}\cdot(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}dt=$$

$$\int_{-1}^{1}\frac{V_{n+1}(t)}{z-t}w_3(t)dt+\frac{n+\frac{3}{2}}{n+\frac{1}{2}}\int_{-1}^{1}\frac{V_n(t)}{z-t}w_3(t)\,dt=$$

$$\frac{2\pi}{(u-1)u^{n+1}}+\frac{n+\frac{3}{2}}{n+\frac{1}{2}}\cdot\frac{2\pi}{(u-1)u^n}=$$

$$\frac{2\pi}{(u-1)u^n}\left(\frac{1}{u}+\frac{n+\frac{3}{2}}{n+\frac{1}{2}}\right)$$

From here results:

$$\mathrm{K}_{n+2}(z,w_1)=$$

$$\frac{24\pi}{u^n}\cdot\frac{u}{u^2-1}\cdot\frac{u^{-1}+\frac{2n+3}{2n+1}}{u^{n+2}+u^{-(n+1)}+\frac{2n+3}{2n+1}\left(u^{n+1}+u^{-n}\right)} \quad (42)$$

IV. THE MAXIMUM OF THE KERNEL FOR CHEBYSEV WEIGHT FUNCTIONS

4.1. For $w=w_2$ we have only empirical and asymptotic results. The computation shows that $\left|K_{n+2}(z,w_2)\right|$, $z\in\varepsilon_\rho$ attains its maximum on the real axis if $n=1$ or $n=2$. If n is odd and $n\ge 3$, the maximum is attained on the real axis if $1<\rho<\rho_n$, and on the imaging axis if $\rho_n<\rho$ (at either place if $\rho=\rho_n$).

If $n\ge 4$ is even, the behavior is more complicated: we have a maximum on the real axis if $1<\rho<\rho'_n$, on the imaging axis if $\rho_n<\rho$, and in between if $\rho'_n<\rho<\rho_n$, where ρ'_n, ρ_n are certains numbers satisfying $1<\rho'_n<\rho_n$. Numerical value for $n=3(1)20$ have been determined by a bisection procedure in [5] and are shown in Table 1.

Table 1:

n	ρ'_n	ρ_n
3		1,4142
4	1,2093	1,5955
5		1,1170
6	1,0822	1,4483
7		1,0580
8	1,0451	1,3671
9		1,0350
10	1,0287	1,3138
11		1,0235
12	1,0199	1,2756
13		1,0169
14	1,0147	1,2466
15		1,0127
16	1,0113	1,2237
17		1,0099
18	1,0088	1,2051
19		1,0080
20	1,0073	1,1896

The empirical observations above can be verified asymptotically as $\rho \searrow 1$, or as $\rho \to \infty$, for any fixed n. In the just case a lengthy calculation reveals that when $\theta = 0$ (i.e., $z = \dfrac{\rho+\rho^{-1}}{2}$)

$$\left|K_{n+2}\left(\frac{1}{2}\left(\rho+\rho^{-1}\right);w_2\right)\right| \sim$$

$$\sim \frac{3\pi}{(n+1)(n+2)(n+3)}(\rho-1)^{-2},\ \rho \searrow 1 \quad (43)$$

The maximum of kernel $K_{n+4}(z;w_1)$ is asymptotically determined by the:

$$\max_{z\in\varepsilon_\rho}\left|K_{n+1}(z,w_1)\right| =$$

$$= \max_{z\in\varepsilon_\rho}\left|\frac{4}{\left(u-u^{-1}\right)^2}K_{n+2}(z,w_2)\right| \sim$$

$$\sim \frac{4}{\left(\rho-\rho^{-1}\right)^2}\frac{3\pi}{(n+1)(n+2)(n+3)} \sim$$

$$\sim \frac{12\pi}{(n+1)(n+2)(n+3)}\left(\rho-\rho^{-1}\right)^{-2}(\rho-1)^{-2}$$

$$(\rho \searrow 1)$$

whereas, for other values of θ, including $\theta = \dfrac{\pi}{2}$.

4.2. Radau formulae; circular contours. The case w_1. again, is amenable to analytic treatment. We now have

$z = re^{i\theta}$, $r > 1$, and

$$u = z+\sqrt{z^2-1} = e^{i\theta}\left(r+\sqrt{r^2-e^{-2i\theta}}\right),$$

where the branch of square root is taken that assigns positive values to positive arguments. There follows

$$\frac{u}{u^2-1} = \frac{1}{u-u^{-1}} = \left(2e^{i\theta}\sqrt{r^2-e^{-2i\theta}}\right)^{-1}$$

Hence $$\left|\frac{u}{u^2-1}\right| \le \frac{1}{2\sqrt{r^2-1}},$$

The bound being attained for $\theta = 0$ and $\theta = \pi$. Furthermore,

$$\left|u^{2n+1}+1\right| = \left|\left(r+\sqrt{r^2-e^{-2i\theta}}\right)^{2n+1}+e^{-(2n+1)i\theta}\right| \ge$$

$$\ge \left|r+\sqrt{r^2-e^{-2i\theta}}\right|^{2n+1}-1 \ge \left(r+\sqrt{r^2-1}\right)^{2n+1}-1$$

with equality holding for $\theta = \pi$. Consequently,

$$\max_{z\in C_r}\left|K_{n+1}(z;w_1)\right| = \left|K_{n+1}(-r;w_1)\right| = \qquad (44)$$

$$\frac{4\pi}{R-R^{-1}}\frac{1}{R^{2n+1}-1}$$

where $R = r + \sqrt{r^2 - 1}$. (45)

Theorema 1
The kernel of the (n+1) –point Radau formula (with fixed node at -1) for the Chebyshev weight function w_1 attains its maximum modulus on C_r on the negative real axis; the maximum is given by (44), (45).
For w=w2, we conjecture

$$\max_{z\in C_r}\left|K_{n+1}(z;w_2)\right| = \left|K_{n+1}(-r;w_2)\right| =$$

$$\frac{\pi}{R^{n+2}}\frac{\left(R-R^{n-1}\right)\left(R-\frac{n+1}{n+2}\right)}{\frac{n+1}{n+2}\left(R^{n+2}-R^{-(n+2)}\right)-\left(R^{n+1}-R^{-(n+1)}\right)}$$

Where the denominator is easily showmn to be positive for R>1, and for $w = w_3$,

$$\max_{z\in C_r}\left|K_{n+1}(z;w_3)\right| = \left|K_{n+1}(r;w_3)\right| =$$

$$\frac{2\pi}{R^{n+1}}\frac{R+1}{R-1}\frac{R+\frac{2n+1}{2n+3}}{\frac{2n+1}{2n+3}\left(R^{n+2}-R^{-(n+1)}\right)+\left(R^{n+1}-R^{-n}\right)},$$

where R is given by (45).
When $w = w_4$, the Kernel is sufficiently simple to be treated analytically. Note, first of all, we have $|u| \ge R$ (with equality for $\theta = \pi$) hence

$$\left|u^{n+1} - u^{-(n+1)}\right| \ge |u|^{n+1} - \frac{1}{|u|^{n+1}} \ge R^{n+1} - \frac{1}{R^{n+1}},$$

Again with equality holding for $\theta = \pi$. Next, there follows $\frac{z-1}{z+1} = \left(\frac{u-1}{u+1}\right)^2$, so that

$$\left|\frac{u-1}{u+1}\right|^4 = \left|\frac{z-1}{z+1}\right|^2 =$$

$$\frac{r^2-2r\cos\theta+1}{r^2+2r\cos\theta+1} \le \left(\frac{r+1}{r-1}\right)^2 = \left(\frac{R+1}{R-1}\right)^4$$

Here again, the bound is attained for $\theta = \pi$. Consequently,

$$\max_{z\in C_r}\left|K_{n+1}(z;w_4)\right| = \left|K_{n+1}(-r;w_4)\right| = = 2\pi\frac{R+1}{R-1}\frac{1}{R^{2n+2}-1} \qquad (46)$$

This proves

Theorema 2
The kernel of the (n+1) – point Radau formula (with fixed node at -1) for the Chebyshev weight function w_4 attains its maximum modulus on C_r on the negative real axis; the maximum is given by (46).

4.3. Radau formulae; eliptic contours.

Putting $u = \rho e^{i\vartheta}$ in (38), one obtains, for $w = w_1$

$$\left|K_{n+1}(z,w_1)\right| = = \frac{4\pi\rho}{\left[\left(\rho^4-2\rho^2\cos 2\vartheta+1\right)\left(\rho^{4+2}+2\rho^{2n+1}\cos(2n+1)\vartheta+1\right)\right]^{\frac{1}{2}}}$$

Which clearly takes on its maximum at $\vartheta = \pi$. Thus

$$\max_{z\in\varepsilon_\rho}\left|K_{n+1}(z;w_1)\right| = =\left|K_{n+1}\left(-\frac{1}{2}\left(\rho+\rho^{-1}\right)\right)\right| = \frac{4\pi\rho}{\left(\rho^2-1\right)\left(\rho^{2n+1}-1\right)} \qquad (47)$$

and we have

Theorema 3
The kernel of the (n+1) – point Radau formula (with fixed node at -1) for the Chebyshev weight function w_1 attains its maximum modulus on ε_ρ, on the real axis; the maximum is given by (47).

For $w = w_2$ and $w = w_3$, the kernel is found by computation to behave more curiously. In the former case, we have a situation similar to the Lobatto formula for the same weight function, namely, the maximum is attained on the negative real axis, when n= 1,2,3, and also when n≥ 4, but then only if $1 < \rho < \rho_n'$ or $\rho_n < \rho$, where ρ_n', ρ_n are shown in Table 1; otherwise, the maximum point moves on the ellipse ε_ρ from somewhere close to the imaginary axis to the negative rael axis as ρ increases.

TABLE 1

n	ρ_n'	ρ_n
4	1.2845	4.7385
5	1.1518	7.7651
6	1.0965	10.087
7	1.0681	12.267
8	1.0506	14.385
9	1.0394	16.470
10	1.0317	18.533

Assimptotically one finds, consistent with the above, that

$$\left|K_{n+1}(z;w_2)\right| \sim$$

$$\sim \frac{(n+2)\pi}{(n+1)\rho^{2n+2}}\left\{1-2\frac{2n+3}{(n+1)(n+2)}\rho^{-1}\cos\vartheta\right\}^{\frac{1}{2}} \ \rho\to\infty, \quad (48)$$

and (49)

$$\left|K_{n+1}\left(-\frac{1}{2}\left(\rho+\rho^{-1}\right),w_2\right)\right| \sim \frac{6\pi}{(n+1)(n+2)(2n+3)}(\rho-1)^{-2},$$

$\rho \downarrow 1$,

the value at the other end approaching the finite limit $\dfrac{(2n+3)\pi}{\left(2(n+1)(n+2)\right)}$ when $\rho \downarrow 1$,

and there being the familiar peaks of $O\left((\rho-1)^{-1}\right)$ when

$(n+1)\sin(n+2)\vartheta+(n+2)\sin(n+1)\vartheta=0$,

$0<\vartheta<\pi$

References

[1] W. Gautschi and R.S. Varga, Error bounds for Gaussian quadrature of analytic functions, *SIAM J.NUMER.Anal 20*, 1983, pp. 1170--1186.

[2] W. Gautschi, E. Tychopoulos and R.S. Varga, A note on the contour integral representation of the remainder term for a Gauss-Chebyshev quadrature rub, *SIAM J.NUMER.Anal 27*, 1990, pp. 219--224.

[3] D.D. Stancu, Ghe. Coman and P. Blaga, Analiza numerica si teoria aproximarii, *Presa Universitara Clujana, Cluj-Napoca*, 2002.

[4] G. Szego, Ortogonal polynomials, *AMS Colloq. Publ. 23, 4th edition*, 1975.

[5] D. Vladislav, Teza de doctorat, *Cercetari asupra cuadraturilor de tip Gauss-Cristoffel si Gauss-Kronrod*, Cluj-Napoca, 2004.

[6] Nataniel Greene, Formulas for the Fourier Series of Orthogonal Polynomials in Terms of Special Functions , International Journal of Mathematical Models and methods in applied science, Naun, ISSUE 3, vol.2,2008, pp.317-320.

[7] Dana Simian, Corina Simian, On some interpolations problems in polynomial space with generalized degree, International Journal of Mathematical Models and methods in applied science, Naun, ISSUE 3, vol.1,2007, pp.147-150.