Analysis of a Nonlinear Integral Equation with Modified Argument from Physics

MARIA DOBRIȚOIU
Department of Mathematics-Informatics, University of Petrosani, Romania
(e-mail: mariadobritoiu@yahoo.com)

Abstract— Using the Contraction Principle, Perov's theorem and the General data dependence theorem, some results of existence and uniqueness and data dependence of the solution of the integral equation with modified argument

$$x(t)=\int_{\Omega}K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds+f(t)\ ,\quad t\in\overline{\Omega}\ ,$$

where $\Omega\subset R^m$ is a bounded domain and the functions $K:\overline{\Omega}\times\overline{\Omega}\times R^m\times R^m\times C(\partial\Omega,R^m)\to R^m$, $f:\overline{\Omega}\to R^m$ $g:\overline{\Omega}\to\overline{\Omega}$, are given. Also several examples are given.

***Key–Words:* –** Nonlinear integral equation, existence, uniqueness, data dependence.

I. Introduction

In the study of some problems from turbo-reactors industry, in the '70, a Fredholm integral equation with modified argument appears, having the following form

$$x(t)=\int_a^b K(t,s,x(s),x(a),x(b))ds+f(t)\ , \tag{1}$$

where $K:[a,b]\times[a,b]\times\boldsymbol{R}^3\to\boldsymbol{R}$, $f:[a,b]\to\boldsymbol{R}$, $t\in[a,b]$.

This integral equation is a mathematical model reference with to the turbo-reactors working.

The results obtained by the author for the solution of this integral equation regarding the existence and uniqueness, the data dependence, the differentiability with respect to *a* and *b* and the approximation of the solution, were published in papers [1], [3], [4], [5], [7], [8]. These results have been obtained by applying the Contraction Principle, Perov's theorem, Schauder's theorem, the General data dependence theorem and the successive approximations method with several quadrature formula.

Starting with the Fredholm integral equation with modified argument (1), we have also considered a modification of the argument through a continuous function $g:[a,b]\to[a,b]$, thus obtaining the integral equation with modified argument

$$x(t)=\int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds+f(t)\ , \tag{2}$$

where $K:[a,b]\times[a,b]\times\boldsymbol{R}^4\to\boldsymbol{R}$, $f:[a,b]\to\boldsymbol{R}$, and $g:[a,b]\to[a,b]$.

In the papers [10], [11], [12], [15] and [18] the results of existence and uniqueness, data dependence, differentiability with respect to a parameter and approximation of the solution of integral equation (2) have been published.

Also some properties of the solution of the integral equation (2) were published in paper [16].

These results have been obtained by applying the Contraction Principle, Perov's theorem, the General data dependence theorem, the successive approximations method with several quadrature formula, the Abstract Gronwall lema and the comparison lema.

A generalization of the integral equation (2) is the following integral equation with modified argument

$$x(t)=\int_{\Omega}K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds+f(t)\ , \tag{3}$$

where $t\in\overline{\Omega}$, $\Omega\subset\boldsymbol{R^m}$ is a bounded domain, $K:\overline{\Omega}\times\overline{\Omega}\times R^m\times R^m\times C(\partial\Omega,R^m)\to R^m$, $f:\overline{\Omega}\to R^m$, $g:\overline{\Omega}\to\overline{\Omega}$.

Several results for the solution of the integral equation (3) have been published in paper [19].

The purpose of this paper is to give several results of existence and uniqueness and data dependence of the solution of integral equation (3).

In order to establish these results, the Contraction Principle, Perov's theorem and the General data dependence theorem, have been used.

Also, some results from the papers [2], [6], [9], [13], [14], [17], [20] and [21] are useful.

II. Notations and preliminaries

Let X be a nonempty set, d a metric on X and $A:X\to X$ an operator. In this paper we shall use the following notations:

$$F_A:=\{x\in X\mid A(x)=x\}\ -\ \text{the fixed points set of }A$$
$$A^{n+1}:=A\circ A^n,\quad A^0:=1_X\ ,\quad A^1:=A,\quad n\in\mathbf{N}.$$

Also, we will use the Banach space $C(\overline{\Omega},R^m)$

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M. Dobrițoiu is from the Department of Mathematics-Informatics, University of Petrosani, Romania, (e-mail: mariadobritoiu@yahoo.com).

$C(\overline{\Omega}, R^m)$ ={ f: $\overline{\Omega} \to \boldsymbol{R}^m$ | f is continuous function},

endowed with the generalized Chebyshev norm defined by the relation:

$$\|x\|_C := \begin{pmatrix} \|x_1\|_C \\ \ldots \\ \|x_m\|_C \end{pmatrix}, \quad \text{for all } x = \begin{pmatrix} x_1 \\ \ldots \\ x_m \end{pmatrix} \in C(\overline{\Omega}, R^m) \tag{4}$$

where $\| x_k \|_C = \max_{t\in[a,b]} | x_k(t) |$, $k = \overline{1,m}$.

In order to study the existence and uniqueness of the solution of integral equation (3), we will use in section III the following two theorems:

Theorem 1 (*Contraction Principle*) *Let* (X,d) *be a complete metric space and* $A : X \to X$ a *contraction* $(a < 1)$.

In these conditions we have:

(i) $F_A = \{x^*\}$;

(ii) $x^* = \lim_{n\to\infty} A^n(x_0)$, *for all* $x_0 \in X$;

(iii) $d(x^*, A^n(x_0)) \le \dfrac{a^n}{1-a} d(x_0, A(x_0))$.

Theorem 2 (*Perov*) *Let* (X,d) *be a complete generalized metric space with* $d(x,y)\in\mathbf{R}^\mathbf{m}$ *and* $A:X \to X$ *an operator.*

We suppose that there exists a matrix $Q\in M_{mm}(\mathbf{R}_+)$ *such that*

(i) $d(A(x),A(y)) \le Qd(x,y)$, *for all* $x,y \in X$;

(ii) $Q^n \to 0$ *as* $n\to\infty$.

Then

(a) $F_A =\{x^*\}$;

(b) $A^n(x) \to x^*$ *as* $n\to\infty$ *and*

$$d(A^n(x), x^*) \le (I - Q)^{-1}Q^n d(x_0,A(x_0)).$$

By definition, a matrix $Q \in M_{nn}(\boldsymbol{R})$ converges to zero if the matrix Q^k converges to the null matrix as $k \to \infty$.

The following theorem has two conditions which are equivalents with the convergence to zero of a matrix $Q\in M_{nn}(\boldsymbol{R}_+)$. This theorem is useful in the example B from section V.

Theorem 3 (see [20]) *Let* $Q \in M_{nn}(\boldsymbol{R}_+)$ *be a matrix. The following conditions are equivalents:*

(*i*) $Q^k \to 0$ *as* $k \to \infty$;

(*ii*) *The eigenvalues* λ_k, $k = \overline{1,n}$, *of the matrix* Q, *satisfies the condition* $|\lambda_k| < 1$, $k = \overline{1,n}$;

(*iii*) *The matrix* $I - Q$ *is non-singular and*

$$(I - Q)^{-1} = I + Q + \ldots + Q^n + \ldots \ .$$

In order to study the data dependence of the solution of integral equation (3), we will use in section IV the following theorem:

Theorem 4 (*General data dependence theorem*) *Let* (X,d) *be a complete metric space,* $f, g : X \to X$ *two operators and, suppose*:

(i) f is *α–contraction and* $F_f = \{x_f^*\}$;

(ii) $x_g^* \in F_g$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(f(x),g(x)) \le \eta, \ \textit{for all} \ x \in X .$$

In these conditions we have

$$d(x_f^*, x_g^*) \le \frac{\eta}{1-\alpha} .$$

III. Existence and uniqueness

In this section we will present four theorems of existence and uniqueness of the solution of integral equation with modified argument (3).

In order to obtain the existence and uniqueness theorems of the solution of integral equation (3) in $C(\overline{\Omega}, R^m)$ space we will reduce the problem of determination of the solutions of integral equation (3) to a fixed point problem. For this purpose we consider the operator A : $C(\overline{\Omega}, R^m) \to C(\overline{\Omega}, R^m)$, defined by the relation:

$$A(x)(t) = \int_\Omega K(t,s,x(s),x(g(s)),x|_{\partial\Omega})ds + f(t) . \tag{5}$$

The set of the solutions of integral equation (3) in $C(\overline{\Omega}, R^m)$ space, coincides with the set of fixed points of the operator A defined by the relation (5).

Applying the *Contraction Principle*, we obtain the following two theorems of existence and uniqueness of the solution of integral equation (3) in $C(\overline{\Omega}, R^m)$ space, respectively in $\tilde{B}(f;r)$ sphere.

Theorem 5 *Suppose that:*

(i) $K \in C(\overline{\Omega}\times\overline{\Omega}\times R^m \times R^m \times C(\partial\Omega, R^m),\ R^m)$, $f \in C(\overline{\Omega},\ R^m)$, $g \in C(\overline{\Omega},\ \overline{\Omega})$;

(ii) *there exists* $L > 0$, *such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$
$$\le L\left(\|u_1 - v_1\|_{R^m} + \|u_2 - v_2\|_{R^m} + \|u_3 - v_3\|_{C(\partial\Omega,R^m)}\right)$$

for all $t, s\in \overline{\Omega}$, $u_1,u_2,v_1,v_2\in\boldsymbol{R}^m$, $u_3,v_3\in C(\partial\Omega, R^m)$, $i = \overline{1,m}$;

(iii) $3\cdot L\cdot mes(\Omega) < 1$.

Under these conditions the integral equation (3) *has a unique solution* $x^* \in C(\overline{\Omega}, R^m)$, *which can be obtained by* *successive approximations method starting at any element from space* $C(\overline{\Omega}, R^m)$. *Moreover, if* x_0 *is the start function and* x_k *is the* k^{th} *successive approximation, then we have*:

$$\| x^* - x_k \|_{C(\overline{\Omega}, R^m)} \le \frac{[3Lmes(\Omega)]^k}{1-3Lmes(\Omega)} \cdot \| x_0 - x_1 \|_{C(\overline{\Omega}, R^m)}. \qquad (6)$$

Proof: From the condition (i) it results that the operator A is correctly defined.

Now we verify the conditions of the *Contraction Principle*. Let we prove that the operator A is an α-contraction.

From the condition (ii) we have:

$$| A(x)(t) - A(y)(t) | \le$$

$$\le \left| \int_\Omega \left[K(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K(t,s,y(s),y(g(s)),y|_{\partial\Omega}) \right] ds \right| \le$$

$$\le 3 \cdot L \cdot mes(\Omega) \cdot \| x - y \|_{C(\overline{\Omega}, R^m)}$$

and using the supremum norm we obtain

$$\|A(x) - A(y)\|_{C(\overline{\Omega}, R^m)} \le 3Lmes(\Omega)\|x - y\|_{C(\overline{\Omega}, R^m)}.$$

Therefore the operator A satisfies the Lipschitz condition with the constant $3 \cdot L \cdot mes(\Omega)$ and from the condition (iii) it results that the operator A is an α-contraction with the coefficient $\alpha = 3 \cdot L \cdot mes(\Omega)$. Now, the conclusion of this theorem results from the *Contraction Principle*. □

Theorem 6 *Suppose that:*

(i) $K \in C(\overline{\Omega} \times \overline{\Omega} \times (J_1 \times \cdots \times J_m) \times (J_1 \times \cdots \times J_m) \times C(\partial\Omega, R^m), R^m)$, $f \in C(\overline{\Omega}, R^m)$, $g \in C(\overline{\Omega}, \overline{\Omega})$, *where* $J_1, \ldots, J_m \subset \boldsymbol{R}$ *are closed and finite intervals*;

(ii) *there exists* $L > 0$, *such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$

$$\le L\left(\|u_1 - v_1\|_{R^m} + \|u_2 - v_2\|_{R^m} + \|u_3 - v_3\|_{C(\partial\Omega, R^m)} \right)$$

for all $t,s \in \overline{\Omega}$, $u_1,u_2,v_1,v_2 \in J_1 \times \ldots \times J_m$, $u_3,v_3 \in C(\partial\Omega, R^m)$, $i = \overline{1,m}$;

(iii) $3 \cdot L \cdot mes(\Omega) < 1$.

If there exists $r > 0$ *such that*

$$\left[x \in \tilde{B}(f;r) \right] \Rightarrow \left[x(t) \in J_1 \times \cdots \times J_m \right] \qquad (7)$$

and the following condition is met:

(iv) $M_K\, mes(\Omega) \le r$,

where M_K *is a positive constant, such that*

$$| K_i(t,s,u,v,w) | \le M_K, \qquad (8)$$

for all $t, s \in \overline{\Omega}$, $u, v \in J_1 \times \ldots \times J_m$, $w \in C(\partial\Omega, \boldsymbol{R}^m)$, *then the integral equation* (3) *has a unique solution* $x^* \in \tilde{B}(f;r) \subset C(\overline{\Omega}, R^m)$, *which can be obtained by the successive approximations method starting at any element from* $\tilde{B}(f;r)$ sphere. *Moreover, if* x_0 *is the start function and* x_k *is the* k^{th} *successive approximation, then the estimation* (6) *is met.*

Proof: From the conditions (ii) and (iii) it results that the operator $A: \tilde{B}(f;r) \to C(\overline{\Omega}, R^m)$, defined by the relation (5) is an α-contraction with the coefficient $\alpha = 3 \cdot L \cdot mes(\Omega)$.

From the condition (iv) it results that $A\left(\tilde{B}(f;r)\right) \subset \tilde{B}(f;r)$, i.e. $\tilde{B}(f;r) \in I(A)$.

Since $\tilde{B}(f;r)$ is a closed subset in Banach space $C(\overline{\Omega}, R^m)$, it results that the *Contraction Principle* can be applied and the proof is complete. □

Now, on the space $C(\overline{\Omega}, R^m)$ we consider the generalized Chebyshev norm, defined by the relation (4) and we obtain a complete generalized Banach space.

Applying the fixed point *Perov*'s theorem, we obtain the following two theorems:

Theorem 7 *Suppose that:*

(i) $K \in C(\overline{\Omega} \times \overline{\Omega} \times R^m \times R^m \times C(\partial\Omega, R^m), R^m)$, $f \in C(\overline{\Omega}, R^m)$, $g \in C(\overline{\Omega}, \overline{\Omega})$;

(ii) *there exists* $Q \in M_{mm}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\| \le$$

$$\le Q\left(\|u_1 - v_1\|_C + \|u_2 - v_2\|_C + \|u_3 - v_3\|_{C(\partial\Omega, r^M)} \right)$$

for all $t,s \in \overline{\Omega}$, $u_1,u_2,v_1,v_2 \in \boldsymbol{R}^m$, $u_3,v_3 \in C(\partial\Omega, \boldsymbol{R}^m)$;

(iii) $3 \cdot mes(\Omega) \cdot Q$ *is a matrix which converges to zero.*

Under these conditions the integral equation (3) *has a unique solution* $x^* \in C(\overline{\Omega}, R^m)$, *which can be obtained by successive approximations method starting at any element from space* $C(\overline{\Omega}, R^m)$. *Moreover, if* x_0 *is the start function and* x_k *is the* k^{th} *successive approximation, then we have*:

$$\|x^* - x_k\|_{C(\overline{\Omega}, R^m)} \le$$

$$\le [3mes(\Omega)Q]^k \left[I - 3mes(\Omega)Q \right]^{-1} \| x_0 - x_1 \|_{C(\overline{\Omega}, R^m)} \qquad (9)$$

Proof: We consider the operator $A: C(\overline{\Omega}, R^m) \to C(\overline{\Omega}, R^m)$, defined by the relation (5):

$$A(x)(t) = \int_\Omega K(t,s,x(s),x(g(s)),x|_{\partial\Omega}) ds + f(t).$$

From the condition (i) it results that the operator A is correctly defined.

The set of the solutions of integral equation (3) in $C(\overline{\Omega}, R^m)$ space, coincides with the set of fixed points of this operator A.

Now we verify the conditions of *Perov*'s theorem. Let we prove that the operator A is an contraction.

From the condition (ii) it results that the function K satisfies a Lipschitz condition with respect to the last three arguments, with a matrix $Q \in M_{mm}(\boldsymbol{R}_+)$. Therefore we have:

$$|A(x)(t)-A(y)(t)| = \begin{pmatrix} |A_1(x)(t)-A_1(y)(t)| \\ \cdots \\ |A_m(x)(t)-A_m(y)(t)| \end{pmatrix} =$$

$$= \begin{pmatrix} \left|\int_\Omega [K_1(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_1(t,s,y(s),y(g(s)),y|_{\partial\Omega})]ds\right| \\ \cdots \\ \left|\int_\Omega [K_m(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_m(t,s,y(s),y(g(s)),y|_{\partial\Omega})]ds\right| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_\Omega |K_1(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_1(t,s,y(s),y(g(s)),y|_{\partial\Omega})|ds \\ \cdots \\ \int_\Omega |K_m(t,s,x(s),x(g(s)),x|_{\partial\Omega}) - K_m(t,s,y(s),y(g(s)),y|_{\partial\Omega})|ds \end{pmatrix}$$

Now, using the generalized Chebyshev norm on $C(\overline{\Omega}, R^m)$ space, defined by the relation (4) we obtain:

$$\| A(x)-A(y) \|_{C(\overline{\Omega},R^m)} \le 3mes(\Omega)Q \| x-y \|_{C(\overline{\Omega},R^m)}$$

and it results that the operator A satisfies a Lipschitz condition with respect to the last three arguments, with the matrix $3\cdot mes(\Omega)\cdot Q \in M_{mm}(\boldsymbol{R}_+)$.

From the condition (iii) it results that the operator A is an α–contraction with the matrix $\alpha = 3\cdot L\cdot mes(\Omega)$.

Now, the conclusion of the theorem results from *Perov*'s theorem. □

Theorem 8 *Suppose that:*

(i) $K \in C(\overline{\Omega}\times\overline{\Omega}\times(J_1\times\cdots\times J_m)\times(J_1\times\cdots\times J_m)\times C(\partial\Omega,R^m),R^m)$, $f \in C(\overline{\Omega},\ R^m)$, $g \in C(\overline{\Omega},\ \overline{\Omega})$, *where* $J_1, \ldots, J_m \subset \boldsymbol{R}$ *are closed and finite intervals* ;

(ii) *there exists* $Q \in M_{mm}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\| \le$$
$$\le Q\left(\|u_1-v_1\|_C + \|u_2-v_2\|_C + \|u_3-v_3\|_{C(\partial\Omega,r^M)}\right)$$

for all $t, s \in \overline{\Omega}$, $u_1, u_2, v_1, v_2 \in J_1\times\ldots\times J_m$, $u_3, v_3 \in C(\partial\Omega, R^m)$;

(iii) $3\cdot mes(\Omega)\cdot Q$ *is a matrix which converges to zero.*

If there exists $r \in M_{m1}(R_+)$ *such that*

$$\left[x \in \tilde{B}(f;r)\right] \Rightarrow \left[x(t) \in J_1\times\cdots\times J_m\right] \tag{10}$$

and the following condition is met:

(iv) $M_K\, mes(\Omega) \le r$,

where $M_K = \begin{pmatrix} M_K^1 \\ \ldots \\ M_K^m \end{pmatrix} \in M_{m1}(R_+)$ *is a matrix with positive constant as elements, such that*

$$\| K(t,s,u,v,w) \|_C \le M_K , \tag{11}$$

for all $t, s \in \overline{\Omega}$, $u, v \in J_1\times\ldots\times J_m$, $w \in C(\partial\Omega,\boldsymbol{R}^m)$,

then the integral equation (3) *has a unique solution* $x^* \in \tilde{B}(f;r) \subset C(\overline{\Omega}, R^m)$, *which can be obtained by the successive approximations method starting at any element from* $\tilde{B}(f;r)$ *sphere. Moreover, if* x_0 *is the start function and* x_k *is the* k^{th} *successive approximation, then the estimation* (9) *is met.*

Proof: From the conditions (i), (ii) and (iii) it results that the operator $A : \tilde{B}(f;r) \to C(\overline{\Omega}, R^m)$, defined by the relation (5) is an contraction with the matrix $3\cdot mes(\Omega)\cdot Q \in M_{mm}(\boldsymbol{R}_+)$.

From the condition (iv) it results that $A\left(\tilde{B}(f;r)\right) \subset \tilde{B}(f;r)$, i.e. $\tilde{B}(f;r) \in I(A)$.

Since $\tilde{B}(f;r)$ is a closed subset in Banach space $C(\overline{\Omega}, R^m)$, it results that *Perov*'s theorem can be applied and the proof is complete. □

IV. Data dependence

In this section we will present one theorem of data dependence of the solution of the integral equation with modified argument (3).

In order to study the data dependence of the solution of integral equation (3) we consider the perturbed integral equation

$$y(t) = \int_\Omega H(t,s,y(s),y(g(s)),y|_{\partial\Omega})ds + h(t) , \tag{12}$$

where $t \in \overline{\Omega}$, $\Omega \subset \boldsymbol{R}^m$ is a bounded domain, $H : \overline{\Omega}\times\overline{\Omega}\times R^m\times R^m\times C(\partial\Omega,R^m) \to R^m$, $h : \overline{\Omega} \to R^m$, $g : \overline{\Omega} \to \overline{\Omega}$.

Applying the *General data dependence theorem*, we have the following data dependence theorem of the solution of integral equation (3):

Theorem 9 *Suppose that:*

(i) *the conditions of the theorem* 7 *are satisfied and we denote with* $x^* \in C(\overline{\Omega}, R^m)$ *the unique solution of integral equation* (3);

(ii) $H \in C(\overline{\Omega}\times\overline{\Omega}\times R^m\times R^m\times C(\partial\Omega,R^m),\ R^m)$, $h \in C(\overline{\Omega},\ R^m)$, $g \in C(\overline{\Omega},\ \overline{\Omega})$;

(iii) *there exists* $T_1, T_2 \in M_{m1}(\boldsymbol{R}_+)$ *such that*

$$\|K(t,s,u,v,w) - H(t,s,u,v,w)\|_C \le T_1$$

for all $t, s \in \overline{\Omega}$, $u, v \in \boldsymbol{R}^m$, $w \in C(\partial\Omega, R^m)$ *and*

$$\|f(t)-h(t)\|_C \le T_2, \quad \textit{for all } t\in\overline{\Omega}.$$

Under these conditions, if $y^*\in C(\overline{\Omega},R^m)$ *is a solution of the integral equation* (12)*, then we have:*

$$\|x^*-y^*\|_{C(\overline{\Omega},R^m)} \le [I-3Q\,mes(\Omega)]^{-1}[T_1\cdot mes(\Omega)+T_2]. \qquad (13)$$

Proof: Let we consider the operator A from the proof of the theorem 6, A: $C(\overline{\Omega},R^m)\to C(\overline{\Omega},R^m)$, attached to the integral equation (3) and defined by the relation (5):

$$A(x)(t)=\int_\Omega K(t,s,x(s),x(g(s)),x\,|_{\partial\Omega})ds+f(t)\ .$$

Also we attach to the integral equation (12) the operator D: $C(\overline{\Omega},R^m)\to C(\overline{\Omega},R^m)$, defined by the relation:

$$D(y)(t)=\int_\Omega H(t,s,y(s),y(g(s)),y\,|_{\partial\Omega})ds+h(t)\,. \qquad (14)$$

From the condition (ii), it results that the operator D is correctly defined.

The set of the solutions of the perturbed equation (12) in $C(\overline{\Omega},R^m)$ space coincides with the fixed points set of the operator D defined by the relation (14).

We have

$$\|x^*-y^*\|_C = \|A(x^*)-D(y^*)\|_C \le$$

$$\le \|A(x^*)-A(y^*)\|_C + \|A(y^*)-D(y^*)\|_C \le$$

$$= \begin{pmatrix} |A_1(x^*)(t)-A_1(y^*)(t)| \\ \cdots \\ |A_m(x^*)(t)-A_m(y^*)(t)| \end{pmatrix} + \begin{pmatrix} |A_1(y^*)(t)-D_1(y^*)(t)| \\ \cdots \\ |A_m(y^*)(t)-D_m(y^*)(t)| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_a^b \left|K_1(t,s,x^*(s),x^*(g(s)),x^*\,|_{\partial\Omega})-K_1(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})\right|ds \\ \cdots\cdots\cdots \\ \int_a^b \left|K_m(t,s,x^*(s),x^*(g(s)),x^*\,|_{\partial\Omega})-K_m(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})\right|ds \end{pmatrix}$$

$$+ \begin{pmatrix} \int_a^b \left|K_1(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})-H_1(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})\right|ds \\ \cdots\cdots\cdots \\ \int_a^b \left|K_m(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})-H_m(t,s,y^*(s),y^*(g(s)),y^*\,|_{\partial\Omega})\right|ds \end{pmatrix}$$

$$+ \begin{pmatrix} |f_1(t)-h_1(t)| \\ \cdots \\ |f_m(t)-h_m(t)| \end{pmatrix}.$$

Since the function K satisfies a generalized Lipschitz conditions with respect to the last three arguments with the matrix Q (theorem 7, condition (ii)) and from the condition (iii) and the generalized norm defined by the relation (4), we obtain

$$\|x^*-y^*\|_C \le 3Qmes(\Omega)\|x^*-y^*\|_C + T_1\cdot mes(\Omega) + T_2\,.$$

Therefore, we have

$$[I-3Qmes(\Omega)]\cdot\|x^*-y^*\|_C \le T_1\cdot mes(\Omega) + T_2 \qquad (15)$$

and now, it results the estimation (13). The proof is complete. □

V. Example

In what follows we present three applications of the theorems established in the previous sections.

A. We consider the integral equation with modified argument

$$x(t)=\int_0^1\left(\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right)ds+\cos t \qquad (16)$$

where $t\in[0,1]$, $K\in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$,

$$K(t,s,u_1,u_2,u_3,u_4)=\frac{\sin u_1+\cos u_2}{7}+\frac{u_3+u_4}{5},$$

$f\in C[0,1]$, $f(t)=\cos t$, $g\in C([0,1],[0,1])$, $g(s)=s/2$, $x\in C[0,1]$

and the conditions of the theorem 5 were verified, in order to prove the existence and uniqueness of the solution in $C[0,1]$ space.

In order to study the existence and uniqueness of the solution of the integral equation (16) in $C[0,1]$ space, we consider the operator $A: C[0,1]\to C[0,1]$ defined by the relation:

$$A(x)(t)=\int_0^1\left(\frac{\sin(x(s))+\cos(x(s/2))}{7}+\frac{x(0)+x(1)}{5}\right)ds+\cos t\,. \qquad (17)$$

The solutions set of the integral equation (16) in $C[0,1]$ space, coincides with the fixed points set of the operator A.

We have

$$|K(t,s,u_1,u_2,u_3,u_4)-K(t,s,v_1,v_2,v_3,v_4)| =$$

$$=\left|\frac{\sin u_1+\cos u_2}{7}+\frac{u_3+u_4}{5}-\frac{\sin v_1+\cos v_2}{7}-\frac{v_3+v_4}{5}\right|\le$$

$$\le\frac{1}{7}|\sin u_1-\sin v_1|+\frac{1}{7}|\sin u_2-\sin v_2|+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\le$$

$$\le\frac{1}{7}2\left|\sin\frac{u_1-v_1}{2}\right|\cdot\left|\cos\frac{u_1+v_1}{2}\right|+\frac{1}{7}2\left|\sin\frac{u_2-v_2}{2}\right|\cdot\left|\cos\frac{u_2+v_2}{2}\right|+$$

$$+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\le$$

$$\le\frac{1}{7}|u_1-v_1|+\frac{1}{7}|u_2-v_2|+\frac{1}{5}|u_3-v_3|+\frac{1}{5}|u_4-v_4|\ ,$$

for all $t, s\in[0,1]$, $u_i, v_i\in\boldsymbol{R}$, $i=\overline{1,4}$.

Now we have

$$|A(x)(t)-A(y)(t)| = \left| \int_0^1 \left(\frac{\sin(x(s))+\cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} - \right.\right.$$

$$\left.\left. - \frac{\sin(y(s))+\cos(y(s/2))}{7} - \frac{y(0)+y(1)}{5} \right) ds \right| \le$$

$$\le \int_0^1 \left| \frac{\sin(x(s))+\cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} - \right.$$

$$\left. - \frac{\sin(y(s))+\cos(y(s/2))}{7} - \frac{y(0)+y(1)}{5} \right| ds \le$$

$$\le \int_0^1 \left(\frac{1}{7}|x(s)-y(s)| + \frac{1}{7}|x(s/2)-y(s/2)| \right.$$

$$\left. + \frac{1}{5}|x(0)-y(0)| + \frac{1}{5}|x(1)-y(1)| \right) ds$$

and using the supremum norm we obtain

$$\|A(x)-A(y)\|_{C[0,1]} \le \left(\frac{1}{7}+\frac{1}{7}+\frac{1}{5}+\frac{1}{5} \right) \cdot \|x-y\|_{C[0,1]} \cdot \int_0^1 ds =$$

$$= \frac{24}{35} \cdot \|x-y\|_{C[0,1]}$$

and therefore it results that the operator A is an α-contraction with the coefficient $\alpha = \frac{24}{35}$.

The conditions of the theorem 5 being satisfied, it results that the integral equation (16) has a unique solution $x^* \in C[0,1]$, which can be obtained by successive approximations method starting at any element $x_0 \in C[0,1]$. Moreover, if x_n is the n[th] successive approximation, then the following estimation is proved:

$$d(x^*, x_n) \le \frac{24^n}{35^{n-1} \cdot 11} d(x_0, x_1) . \tag{18}$$

In order to prove the existence and uniqueness of the solution of integral equation (16) in sphere $\tilde{B}(\cos t; r)$

$$\tilde{B}(\cos t; r) = \{ x \in C[0,1] \mid \|x-\cos t\|_{C[0,1]} \le r,\ r \in R_+ \} ,$$

the conditions of the theorem 6 were verified.

Now, we consider the integral equation (16), and assume that the conditions below are satisfied:

$K \in C([0,1] \times [0,1] \times J^4)$, $J \subset \mathbf{R}$ is compact,

$f \in C[0,1]$,

$g \in C([0,1],[0,1])$

and we verify the conditions of the theorem 6 of existence and uniqueness of the solution of integral equation (16) in sphere $\tilde{B}(\cos t; r) \subset C[0,1]$.

We attach to the integral equation (16), the operator $A: \tilde{B}(\cos t; r) \to C[0,1]$, defined by the relation (17) where r is a positive real number which satisfies the condition below:

$$[\ x \in \tilde{B}(\cos t; r)\ ] \Rightarrow [\ x(t) \in J \subset R\ ]$$

and obviously it is clear that, it exists at least one $r > 0$ with this property.

We have

$$x \in \tilde{B}(\cos t; r) \Rightarrow |x(t)-\cos t| \le r,\quad t \in [0,1] \Rightarrow$$

$$\Rightarrow |x(t)| \le r+1,\quad t \in [0,1]$$

and therefore

$$x \in \tilde{B}(\cos t; r) \quad \Rightarrow \quad \|x\|_{C[0,1]} \le r+1 . \tag{19}$$

We establish under what conditions the sphere $\tilde{B}(\cos t; r)$ is a constant subset for the operator A. We have

$$|K(t,s,u_1,u_2,u_3,u_4)| = \left| \frac{\sin u_1 + \cos u_2}{7} + \frac{u_3+u_4}{5} \right| \le$$

$$\le \frac{1}{7}|\sin u_1| + \frac{1}{7}|\cos u_2| + \frac{1}{5}|u_3| + \frac{1}{5}|u_4| \le$$

$$\le \frac{1}{7}|u_1| + \frac{1}{7}|u_2| + \frac{1}{5}|u_3| + \frac{1}{5}|u_4| ,$$

for all $t, s \in [0,1]$, $u_i, v_i \in J$, $i = \overline{1,4}$, and

$$|A(x)(t)-\cos t| =$$

$$= \left| \int_0^1 \left(\frac{\sin(x(s))+\cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right) ds \right| \le$$

$$\le \int_0^1 \left| \frac{\sin(x(s))+\cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right| ds .$$

Therefore, it results that

$$|A(x)(t)-\cos t| \le \int_0^1 \left(\frac{1}{7}|x(s)| + \frac{1}{7}|x(s/2)| + \frac{1}{5}|x(0)| + \frac{1}{5}|x(1)| \right) ds$$

and using the supremum norm we obtain

$$\|A(x)-\cos t\|_{C[0,1]} \le \frac{24}{35} \cdot \|x\|_{C[0,1]} \cdot \int_0^1 ds = \frac{24}{35} \cdot \|x\|_{C[0,1]} ,$$

which by (19) becomes

$$\|A(x)-\cos t\|_{C[0,1]} \le \frac{24}{35}(r+1) \tag{20}$$

and the condition of the invariability of sphere $\tilde{B}(\cos t; r)$ is

$$\frac{24}{35}(r+1) \le r .$$

Therefore, if $r \geq \frac{24}{11}$, then the sphere $\widetilde{B}(\cos t; r)$ is a constant subset for the operator *A*.

We consider now the operator $A : \widetilde{B}(\cos t; r) \to \widetilde{B}(\cos t; r)$, defined by the relation (17) and where $\widetilde{B}(\cos t; r)$ is a closed subset of the Banach metric space $C[0,1]$.

The solutions set of the integral equation (16) coincides with the fixed points set of this operator *A*.

This operator *A* is an α-contraction with the coefficient $\alpha = \frac{24}{35}$.

Since the conditions of the theorem 6 are satisfied, it results that the integral equation (16) has a unique solution $x^* \in \widetilde{B}(\cos t; r) \subset C[0,1]$, which can be obtained by successive approximations method starting at any element $x_0 \in \widetilde{B}(\cos t; r)$.

Moreover, if x_n is the n^{th} successive approximation, then we have the estimation (18).

B. We consider the following system of integral equations with modified argument

$$\begin{cases} x_1(t) = \int_0^1 \left(\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right) ds + 2t + 1 \\ x_2(t) = \int_0^1 \left(\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right) ds + \sin t \end{cases} \tag{21}$$

where $K \in C([0,1] \times [0,1] \times R^2 \times R^2 \times R^2 \times R^2, R^2)$,

$$K(t,s,u_1,u_2,u_3,u_4) = \left(K_1(t,s,u_1,u_2,u_3,u_4),\, K_2(t,s,u_1,u_2,u_3,u_4)\right),$$

$$K_1(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{15} u_{11} + \frac{2t+1}{15} u_{21} + \frac{1}{5} u_{31} + \frac{1}{5} u_{41},$$

$$K_2(t,s,u_1,u_2,u_3,u_4) = \frac{t+2}{21} u_{12} + \frac{2t+1}{21} u_{22} + \frac{1}{7} u_{32} + \frac{1}{7} u_{42}$$

$f \in C([0,1], R^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 2t + 1$, $f_2(t) = \sin t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$, $x \in C([0,1], R^2)$.

Now, we verify the conditions of the theorem 7 of the existence and uniqueness of the solution of the system of integral equations (21) in the space $C([0,1], R^2)$.

In order to study the existence and uniqueness of the solution of the system of integral equations (21) in $C([0,1], R^2)$ space, we attach to this system, the operator $A: C([0,1],R^2) \to C([0,1],R^2)$, defined by the relation:

$$A(x)(t) = \begin{cases} A_1(x)(t) \\ A_2(x)(t) \end{cases} =$$

$$= \begin{cases} \int_0^1 \left(\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right) ds + 2t + 1 \\ \int_0^1 \left(\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right) ds + \sin t \end{cases}. \tag{22}$$

The solutions set of the system of integral equations (21), in $C([0,1],R^2)$ space, coincides with the fixed points set of the operator *A*, defined by the relation (22).

We have

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4) - K_1(t,s,v_1,v_2,v_3,v_4)| \\ |K_2(t,s,u_1,u_2,u_3,u_4) - K_2(t,s,v_1,v_2,v_3,v_4)| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix} \cdot \begin{pmatrix} |u_{11} - v_{11}| + |u_{21} - v_{21}| + |u_{31} - v_{31}| + |u_{41} - v_{41}| \\ |u_{12} - v_{12}| + |u_{22} - v_{22}| + |u_{32} - v_{32}| + |u_{42} - v_{42}| \end{pmatrix} \tag{23}$$

for all $t, s \in [0,1]$, $u_i, v_i \in \boldsymbol{R}^2$, $i = \overline{1,4}$,

and it results that the function *K* satisfies a Lipschitz condition with respect to the last four arguments, with the matrix

$$Q = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix}, \quad Q \in M_{22}(R_+).$$

Now it is clear that the condition (*ii*) of the theorem 7, is satisfied.

By the estimation of the following difference:

$$|A(x)(t) - A(y)(t)| = \begin{pmatrix} |A_1(x)(t) - A_1(y)(t)| \\ |A_2(x)(t) - A_2(y)(t)| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \left| \int_0^1 \left(K_1(t,s,x(s),x(s/2),x(0),x(1)) - K_1(t,s,y(s),y(s/2),y(0),y(1)) \right) ds \right| \\ \left| \int_0^1 \left(K_2(t,s,x(s),x(s/2),x(0),x(1)) - K_2(t,s,y(s),y(s/2),y(0),y(1)) \right) ds \right| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \int_0^1 |K_1(t,s,x(s),x(s/2),x(0),x(1)) - K_1(t,s,y(s),y(s/2),y(0),y(1))| ds \\ \int_0^1 |K_2(t,s,x(s),x(s/2),x(0),x(1)) - K_2(t,s,y(s),y(s/2),y(0),y(1))| ds \end{pmatrix}$$

and using the relation (23) and the supremum norm, we obtain

$$\| A(x) - A(y) \|_{C([0,1],R^2)} \leq \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix} \cdot \| x - y \|_{C([0,1],R^2)}.$$

Now, it results that the operator *A* satisfies a generalized Lipschitz condition with respect to the last four arguments, with the matrix $\begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix}$, which by theorem 3 converges to zero.

Therefore, the condition (iii) of the theorem 7 and it results that the operator *A* is a contraction.

The conditions of the theorem 7 being satisfied, it results that the system of integral equations with modified argument (21) has a unique solution $x^* \in C([0,1], \boldsymbol{R}^2)$, which can be obtained by the successive approximations method starting at any element $x_0 \in C([0,1],\boldsymbol{R}^2)$. In addition, if x_n is n^{th} successive approximation, then the following estimation is satisfied:

$$\| x^* - x_n \|_C \le \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix}^n \cdot \begin{pmatrix} 1/5 & 0 \\ 0 & 3/7 \end{pmatrix}^{-1} \cdot \| x_0 - x_1 \|_C$$

or

$$\| x^* - x_n \|_C \le 4^n \cdot \begin{pmatrix} \frac{1}{5^{n-1}} & 0 \\ 0 & \frac{1}{3\cdot 7^{n-1}} \end{pmatrix} \cdot \| x_0 - x_1 \|_C . \qquad (24)$$

Next, we will establish the conditions of existence and uniqueness of the solution of the system of integral equations with modified argument (21) in sphere

$$\tilde{B}(f;r) = \{x \in C([0,1], R^2) \mid \|x - f\|_{C([0,1],R^2)} \le r,\ r \in R_+^2\}$$

from the space $C([0,1], \boldsymbol{R^2})$.

We consider the system of integral equations (21), where $K \in C([0,1]\times[0,1]\times J^4, \boldsymbol{R^2})$, $J \subset \boldsymbol{R^2}$ is compact, $f \in C([0,1],\boldsymbol{R^2})$ and $g \in C([0,1],[0,1])$.

In order to verify the conditions of the theorem 8, we attach to the system of integral equations (21), the operator A: $\tilde{B}(f;r) \to C([0,1],\boldsymbol{R^2})$, defined by the relation (22), where $r \in R_+^2$, satisfies the condition:

$$\left[\ x \in \tilde{B}(f;r)\ \right] \Rightarrow \left[\ x(t) \in J \subset R^2\ \right] .,$$

and we prove that there exists at least one r which has this property. We have

$$x \in \tilde{B}(f;r) \Rightarrow |x(t) - f(t)| \le r \Rightarrow \begin{pmatrix} |x_1(t) - (2t+1)| \\ |x_2(t) - \sin t| \end{pmatrix} \le \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

and consequently, it results that

$$x \in \tilde{B}(f;r) \Rightarrow \begin{pmatrix} |x_1(t)| \\ |x_2(t)| \end{pmatrix} \le \begin{pmatrix} r_1 + 3 \\ r_2 + 1 \end{pmatrix}, \quad t \in [0,1]. \qquad (25)$$

In what follows, we determine the conditions which assure that the sphere $\tilde{B}(f;r)$ is a constant subset for the operator A. We have

$$|A(x)(t) - f(t)| =$$

$$= \begin{pmatrix} \left| \int_0^1 \left(\frac{t+2}{15} x_1(s) + \frac{2t+1}{15} x_1(s/2) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1) \right) ds \right| \\ \left| \int_0^1 \left(\frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right) ds \right| \end{pmatrix} \le$$

$$\le \begin{pmatrix} \int_0^1 \left| \frac{t+2}{15} x_2(s) + \frac{2t+1}{15} x_2(s/2) + \frac{1}{5} x_2(0) + \frac{1}{5} x_2(1) \right| ds \\ \int_0^1 \left| \frac{t+2}{21} x_2(s) + \frac{2t+1}{21} x_2(s/2) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1) \right| ds \end{pmatrix} .$$

Also, for the function K we have

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3,u_4)| \\ |K_2(t,s,u_1,u_2,u_3,u_4)| \end{pmatrix} \le$$

$$\le \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix} \cdot \begin{pmatrix} |u_{11}| + |u_{21}| + |u_{31}| + |u_{41}| \\ |u_{12}| + |u_{22}| + |u_{32}| + |u_{42}| \end{pmatrix},$$

for all $t, s \in [0,1]$, $u_i, v_i \in J$, $i = \overline{1,4}$.

So we have

$$|A(x)(t) - f(t)| \le$$

$$\le \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix} \cdot \begin{pmatrix} \int_0^1 \left(|x_1(s)| + |x_1(s/2)| + |x_1(0)| + |x_1(1)| \right) ds \\ \int_0^1 \left(|x_2(s)| + |x_2(s/2)| + |x_2(0)| + |x_2(1)| \right) ds \end{pmatrix}$$

and using the supremum norm, we obtain

$$\| A(x) - f \|_{C([0,1],R^2)} \le \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix} \cdot \| x \|_{C([0,1],R^2)} .$$

By (24) it results that

$$\| A(x) - f \|_{C([0,1],R^2)} \le \begin{pmatrix} 4/5 & 0 \\ 0 & 4/7 \end{pmatrix} \cdot \begin{pmatrix} r_1 + 3 \\ r_2 + 1 \end{pmatrix} = \begin{pmatrix} \frac{4r_1 + 12}{5} \\ \frac{4r_2 + 4}{7} \end{pmatrix}$$

and the invariability condition of sphere $\tilde{B}(f;r) \subset C([0,1],\boldsymbol{R^2})$ is the following inequality

$$\begin{pmatrix} \frac{4r_1 + 12}{5} \\ \frac{4r_2 + 4}{7} \end{pmatrix} \le \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} .$$

Therefore, if $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \ge \begin{pmatrix} 12 \\ 4/3 \end{pmatrix}$, then the sphere $\tilde{B}(f;r)$ is a constant subset for the operator A.

Now, we consider the operator A: $\tilde{B}(f;r) \to \tilde{B}(f;r)$, which is defined also by the relation (22), where $\tilde{B}(f;r)$ is a closed subset of the Banach space ($C[0,1]$, $\boldsymbol{R^2}$).

The solutions set of the system of integral equations (21), in sphere $\tilde{B}(f;r) \subset C([0,1],\boldsymbol{R^2})$ coincides with the fixed points set of the operator A already defined.

By an analogous reasoning with the reasoning from the existence and uniqueness of the solution of the system of integral equations (21) in space $C([0,1],\boldsymbol{R^2})$, it results that the operator A is contraction.

The conditions of the theorem 8 being satisfied, it results that the system of integral equations with modified argument (21) has a unique solution $x^* \in \tilde{B}(f;r) \subset C([0,1],\boldsymbol{R^2})$, which can be obtained by the successive approximations method, starting at any element $x_0 \in C([0,1], \boldsymbol{R}^2)$. In addition, if x_n is the n^{th} successive approximation, then we have:

$$\left\| x^* - x_n \right\|_C \leq 4^n \cdot \begin{pmatrix} \frac{1}{5^{n-1}} & 0 \\ 0 & \frac{1}{3\cdot 7^{n-1}} \end{pmatrix} \cdot \left\| x_0 - x_1 \right\|_C . \qquad (26)$$

C. We consider the following integral equation with modified argument

$$x(t) = \int_0^1 \left[\frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right] ds + 2\cos t + 1 \quad (27)$$

where $K \in C([0,1]\times[0,1]\times\boldsymbol{R}^4)$,

$$K(t,s,u_1,u_2,u_3,u_4) = \frac{\sin(u_1) + \cos(u_2)}{7} + \frac{u_3 + u_4}{5},$$

$f \in C[0,1]$, $f(t) = 2\cos t + 1$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$, and $x \in C[0,1]$

and the conditions of the theorem 8 were verified.

In order to study the data dependence of the solution of the integral equation (27), we consider the following perturbed integral equation

$$y(t) = \int_0^1 \left[\frac{\sin(y(s)) + \cos(y(s/2))}{7} + \frac{y(0)+y(1)}{5} - t - 2 \right] ds + \cos t \quad (28)$$

where $H \in C([0,1] \times [0,1] \times \boldsymbol{R}^4)$,

$$H(t,s,v_1,v_2,v_3,v_4) = \frac{\sin(v_1) + \cos(v_2)}{7} + \frac{v_3 + v_4}{5} - t - 2,$$

$h \in C[0,1]$, $h(t) = \cos t$,

$g \in C([0,1],[0,1])$, $g(s) = s/2$, $y \in C[0,1]$.

The operator $A : C[0,1] \to C[0,1]$, attached to the equation (27), and defined by the relation:

$$A(x)(t) = \int_0^1 \left[\frac{\sin(x(s)) + \cos(x(s/2))}{7} + \frac{x(0)+x(1)}{5} \right] ds + 2\cos t + 1 \quad (29)$$

is an α-contraction with the coefficient $\alpha = \frac{24}{35}$.

Since the conditions of the theorem 5 are satisfied, it results that the integral equation (27) has a unique solution $x^* \in C[0,1]$.

We have:

$$\left| K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4) \right| = \left| t+2 \right| \leq 3$$

for all $t, s \in [0,1]$ and

$$\left| f(t) - h(t) \right| = \left| \cos t + 1 \right| \leq 2, \text{ for all } t \in [0,1].$$

The conditions of the theorem 6 are satisfied and therefore, if $y^* \in C[0,1]$ is a solution of the integral equation (28), then the following estimation is met:

$$\left\| x^* - y^* \right\|_{C[0,1]} \leq \frac{175}{11}.$$

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Brief Biography of the Author:

a) Studies

Educational background:

1974–1978 – Department of Mathematics, Faculty of Mathematics, Babes-Bolyai University of Cluj-Napoca.

1978–1979 – One year degree course in "Numerical Analysis", Department of Mathematics, Faculty of Mathematics, Babes-Bolyai University of Cluj-Napoca.

1996–2000 – Department of Management, Faculty of Science, University of Petrosani.

2000 – Degree course in "Human Resources Management", Faculty of Science, University of Petrosani.

Professional experience:

1979–2001 – IT specialist and manager of the IT systems research and programming department of Electronic Center of Computer Science.

From 1992 until 2001, affiliated member of the teaching staff of Mathematics and Computer Science Department of University of Petrosani

From 2001, member of the teaching staff of Mathematics and Computer Science Department of University of Petrosani. Fields of work*: Differential and integral equations, Statistics in Sociology, Basis of Mathematics*.

Fields of work:

– **Mathematics**: Differential equations, Integral equations with modified argument, Numerical analysis, IT (analysis and programming), Linear algebra and geometry, Statistical control of quality, Statistics, Mathematical analysis.

– **Computer science** applied in: Economic statistics, Economy, Engineering and Topography.

b) Academic Positions

2001–2004 - assistant

2004–present - lecturer

2008, PhD at Babes-Bolyai University of Cluj-Napoca under the guidance of PhD. Professor Ioan A. Rus. Theme of the doctorate thesis: "Integral equations with modified argument".

c) Scientific Activities (research, publications, projects, etc….)

– 25 scientific papers presented and published in proceedings or volums of national and international scientific conferences.

– 1 scientific paper presented to international scientific conferences (to appear).

– 8 scientific papers presented to national and international scientific conferences.

– I took part in 8 national conferences and in 15 international conferences

– 8 didactic books and books of problems.

– 8 specialized papers (studies) worked out on the basis of the relation with the research, designing and production units.

– 10 packages of software.

d) WSEAS Activities (papers, sessions, organization of sessions, organization of conferences, books, special issues in the journals etc… within WSEAS)*

Paper ID: 518-219, presented at 8th WSEAS International Conference on MATHEMATICAL METHODS AND COMPUTATIONAL TECHNIQUES IN ELECTRICAL ENGINEERING (MMACTEE '06), Bucharest, Romania, October 16-18, 2006 (published in WSEAS Transactions on Mathematics

Paper ID: 602-279, presented at 10th WSEAS International Conference on MATHEMATICAL and COMPUTATIONAL METHODS in SCIENCE and ENGINEERING (MACMESE '08), Bucharest, Romania, November 7–9, 2008 (published in Proceedings of the 10th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering)

e) Others:

– member of **Roumanian Mathematical Society**

– **Reviewer** for **World Multi-Conference on Systemics, Cybernetics and Informatics** WMSCI, Florida, USA, 2005, 2006, 2007, 2008;

– **Reviewer** for **World Scientific and Engineering Academy and Society,** WSEAS conferences, Greece, 2007, 2008.