Boundary Stabilization of a String with Two Rigid Loads: Calculation of Optimal Feedback Gain Based on a Finite Difference Approximation

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***Abstract*—** This paper is concerned with the boundary stabilization problem of a string with two rigid loads which is described by two kinds of hyperbolic equations. In our previous work, a control law that made an energy function of the system non-increasing was derived, and the asymptotic stability of the closed-loop system with the controller was proved by using the LaSalle's invariance principle. Moreover, a simplified lumped parameter model was considered in connection with the string with two rigid loads, and the design method to determine an optimal feedback gain for the model was proposed. In this paper, it is shown that the controller with an optimal gain based on a finite differnce approximation works more effectively than the one based on our previous method for the original system through numerical simulations.

***Keywords*—** Boundary stabilization problem, C_0-semigroup, descriptor system, finite difference approximation, hyperbolic equation, LaSalle's invariance principle, string.

I. Introduction

In the field of distributed parameter systems, the stabilization and optimal control of strings without natural damping are challenging topics. Since old times, several types of problems in this direction have been investigated by many researchers (see, for example, [1]–[3], [4, Chapter 6], [5], [6, Chapter 6], [7]–[13], and the references therein). Especially, from the engineering point of view, Rao has treated the stabilization problem of suppressing the vibration of a distributed parameter overhead crane model with one rigid load [8]. In that paper, after deriving a control law, the energy multiplier method is applied to the closed-loop system and the exponential stability of the energy is proved. d'Andréa-Novel and Coron have proposed a back-stepping approach for the similar problem [2]. Also, Grabowski and Żołopa have solved the output motion planning problem [5].

In this paper, we study the stabilization problem of a string with two rigid loads which is described by two kinds of hyperbolic equations. Especially, the system can be regarded as a distributed parameter overhead crane model with two rigid loads. Here, it is supposed that the flexible cable of the overhead crane has a constant length. For the string with two rigid loads, an energy function is defined and a control law such that the energy becomes non-increasing is derived. In [9], we have given the proof of the asymptotic stability for the closed-loop system with the controller, by using the LaSalle's invariance principle. However, how to determine an optimal gain of the controller remains an open problem. There, in our previous work [10], we considered a simplified lumped parameter model in connection with the string with two rigid loads, and proposed the design method to determine an optimal feedback gain for the model. In the paper, it was shown that the controller with the optimal gain works effectively to some extent for the original system through numerical simulations. The purpose of this paper is to improve the method, namely, to propose a method to determine a more optimal feedback gain based on a finite difference approximation.

This paper is organized as follows: In Section II, we explain the model treated in this paper. In Section III, we review our previous results concerning control law, closed-loop stability, optimal feedback gain based on a simplified model. Section IV is our main part of this paper, in which the calculation method of an optimal feedback gain based on a finite difference approximation is proposed. In Section V, the result of numerical simulations is given. Finally, this paper is summarized in Section VI.

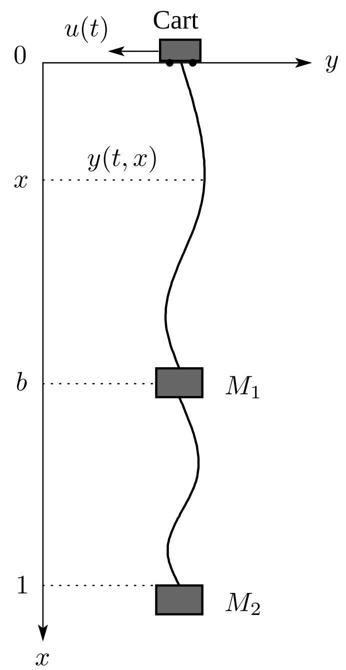

Fig. 1. Distributed parameter overhead crane model with two rigid loads.

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II. System Description

Let b be a constant such that $0 < b < 1$. We shall consider the following string with two rigid loads:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(t,x) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial y}{\partial x}(t,x)\right), \\ \qquad\qquad\qquad\qquad t>0,\ x\in(0,b), \\ \dfrac{\partial^2 y}{\partial t^2}(t,x) = \dfrac{\partial}{\partial x}\left(\tilde{a}(x)\dfrac{\partial y}{\partial x}(t,x)\right), \\ \qquad\qquad\qquad\qquad t>0,\ x\in(b,1), \\ a(0)\dfrac{\partial y}{\partial x}(t,0) = u(t), \quad t>0, \\ y(t,b^-) = y(t,b^+), \quad t\geq 0, \\ M_1\dfrac{\partial^2 y}{\partial t^2}(t,b) = \tilde{a}(b^+)\dfrac{\partial y}{\partial x}(t,b^+) \\ \qquad\qquad -a(b^-)\dfrac{\partial y}{\partial x}(t,b^-), \quad t>0, \\ M_2\dfrac{\partial^2 y}{\partial t^2}(t,1) = -\tilde{a}(1)\dfrac{\partial y}{\partial x}(t,1), \quad t>0, \\ y(0,x) = p(x), \ \ \dfrac{\partial y}{\partial t}(0,x) = q(x), \ \ x\in[0,1]. \end{cases} \tag{1}$$

In the above, $u(t)$ denotes the control force, and $a(x) := g(M_1 + M_2 + 1 - x)$ $(0 \leq x < b)$ and $\tilde{a}(x) := g(M_2 + 1 - x)$ $(b < x \leq 1)$ the tension force of the cable at the point x, where g is the gravitational acceleration, and M_1 and M_2 the masses of rigid loads. System (1) expresses a distributed parameter overhead crane model with two rigid loads (see Fig. 1). Here, it is assumed that each load is a mass point and that the mass of the cart, which is sufficiently small compared with the one of each load, is neglected. Moreover, it is supposed that the displacement $y(t,x)$ and its derivative $y_x(t,x)$ are small through the cable and that the total mass of the cable is 1, i.e., the line density is equal to 1.

Remark 1: We note that, under zero control input, the vibration does not decay at all, since no damping terms are contained in system (1) with $u(t) \equiv 0$.

III. Derivation of Control Law and Closed-Loop Stability

A. Derivation of Control Law

Let us define the following energy function $E(t)$ for system (1):

$$\begin{aligned} E(t) := \frac{1}{2}\Bigg[&\int_0^b \{a(x)y_x^2(t,x) + y_t^2(t,x)\}dx \\ &+ \int_b^1 \{\tilde{a}(x)y_x^2(t,x) + y_t^2(t,x)\}dx \\ &+\alpha y^2(t,0) + M_1 y_t^2(t,b) + M_2 y_t^2(t,1)\Bigg], \end{aligned}$$

where $\alpha > 0$. Differentiating $E(t)$ with respect to t, and using (1) and integration by parts yields

$$\frac{d}{dt}E(t) = -y_t(t,0)(u(t) - \alpha y(t,0)).$$

Here, choosing the control input $u(t)$ as

$$u(t) = \alpha y(t,0) + \gamma y_t(t,0), \quad \gamma > 0, \tag{2}$$

we have

$$\frac{d}{dt}E(t) = -\gamma y_t^2(t,0) \leq 0.$$

Consequently, the energy $E(t)$ for system (1) becomes non-increasing under the control law (2).

Remark 2: In [8], an energy function of this type is used and the same control law is derived for a distributed parameter overhead crane model with one rigid load. In that paper, the energy multiplier method is applied to the closed-loop system and the exponential stability of the energy is proved.

B. Closed-Loop Stability

The closed-loop system consisting of (1) and (2) is described by

$$\begin{cases} y_{tt}(t,x) = (a(x)y_x(t,x))_x, \quad t>0,\ x\in(0,b), \\ y_{tt}(t,x) = (\tilde{a}(x)y_x(t,x))_x, \quad t>0,\ x\in(b,1), \\ a(0)y_x(t,0) = \alpha y(t,0) + \gamma y_t(t,0), \quad t>0, \\ y(t,b^-) = y(t,b^+), \quad t\geq 0, \\ M_1 y_{tt}(t,b) = \tilde{a}(b^+)y_x(t,b^+) - a(b^-)y_x(t,b^-), \\ \qquad\qquad\qquad\qquad\qquad\qquad t>0, \\ M_2 y_{tt}(t,1) = -\tilde{a}(1)y_x(t,1), \quad t>0, \\ y(0,x) = p(x), \quad y_t(0,x) = q(x), \quad x\in[0,1]. \end{cases} \tag{3}$$

In order to formulate this closed-loop system in an abstract space, let us introduce the Hilbert space

$$X = \left\{ \begin{bmatrix} y(\cdot) \\ z(\cdot) \\ \tilde{y}(\cdot) \\ \tilde{z}(\cdot) \\ \xi \\ \eta \end{bmatrix} \in H^1(0,b)\times L^2(0,b)\times H^1(b,1) \times L^2(b,1)\times \mathbf{R}\times\mathbf{R}\ ;\ y(b) = \tilde{y}(b) \right\}$$

with the inner product

$$\begin{aligned} &\langle f_1, f_2\rangle_X \\ &= \int_0^b \{a(x)y_{1x}(x)y_{2x}(x) + z_1(x)z_2(x)\}dx \\ &\quad + \int_b^1 \{\tilde{a}(x)\tilde{y}_{1x}(x)\tilde{y}_{2x}(x) + \tilde{z}_1(x)\tilde{z}_2(x)\}dx \\ &\quad +\alpha y_1(0)y_2(0) + M_1\xi_1\xi_2 + M_2\eta_1\eta_2, \\ &\text{for} \quad f_1 = [y_1, z_1, \tilde{y}_1, \tilde{z}_1, \xi_1, \eta_1]^T \in X, \\ &\qquad\quad f_2 = [y_2, z_2, \tilde{y}_2, \tilde{z}_2, \xi_2, \eta_2]^T \in X. \end{aligned}$$

In the above, $H^1(0,b)$ and $H^1(b,1)$ are the usual Sobolev spaces. Here, we define the operator $A : D(A) \subset X \to X$ as follows:

$$A\begin{bmatrix} y(\cdot) \\ z(\cdot) \\ \tilde{y}(\cdot) \\ \tilde{z}(\cdot) \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -z(\cdot) \\ -(a(\cdot)y_x(\cdot))_x \\ -\tilde{z}(\cdot) \\ -(\tilde{a}(\cdot)\tilde{y}_x(\cdot))_x \\ -\dfrac{\tilde{a}(b^+)\tilde{y}_x(b^+) - a(b^-)y_x(b^-)}{M_1} \\ \dfrac{\tilde{a}(1)\tilde{y}_x(1)}{M_2} \end{bmatrix},$$

$$\text{for } \begin{bmatrix} y(\cdot) \\ z(\cdot) \\ \tilde{y}(\cdot) \\ \tilde{z}(\cdot) \\ \xi \\ \eta \end{bmatrix} \in D(A),$$

$$D(A) = \left\{ \begin{bmatrix} y(\cdot) \\ z(\cdot) \\ \tilde{y}(\cdot) \\ \tilde{z}(\cdot) \\ \xi \\ \eta \end{bmatrix} \in H^2(0,b) \times H^1(0,b) \right.$$
$$\times H^2(b,1) \times H^1(b,1) \times \mathbf{R} \times \mathbf{R}\,;$$
$$\xi = z(b) = \tilde{z}(b),\ \eta = \tilde{z}(1),$$
$$\left. a(0)y_x(0) = \alpha y(0) + \gamma z(0),\ y(b) = \tilde{y}(b) \right\}.$$

And, denoting $y(t,x)$ restricted to $0 \le x \le b$ by $y(t,x)$, and $y(t,x)$ restricted to $b < x \le 1$ by $\tilde{y}(t,x)$, and moreover introducing the new variables $z(t,\cdot) = y_t(t,\cdot)$, $\tilde{z}(t,\cdot) = \tilde{y}_t(t,\cdot)$, $\xi(t) = y_t(t,b)(= y_t(t,b^-) = \tilde{y}_t(t,b^+))$, $\eta(t) = \tilde{y}_t(t,1)$, the closed-loop system (3) can be written as

$$\frac{d}{dt}f(t) = -Af(t), \tag{4}$$

where

$$f(t) := [y(t,\cdot), z(t,\cdot), \tilde{y}(t,\cdot), \tilde{z}(t,\cdot), \xi(t), \eta(t)]^T.$$

Then, the solution of (4) is expressed as

$$f(t) = e^{-tA}f(0),$$

since the operator $-A$ generates a C_0-semigroup of contractions e^{-tA} on X [9]. Moreover, the operator $(I + \lambda A)^{-1} : X \to X$ is compact for every $\lambda > 0$ [9]. Based on these facts, the following fact has been proved by using the LaSalle's invariance principle.

Theorem 1: ([9]) The closed-loop system (4) is asymptotically stable, namely, for every $f(0) \in X$, $\|e^{-tA}f(0)\|_X$ goes to zero as t goes to infinity.

C. Optimal Feedback Gain Based on a Simplified Model (Former Method [10])

In connection with system (1), we shall consider the following lumped parameter overhead crane model with two rigid loads (see Fig. 2):

$$\begin{cases} \epsilon\ddot{y}_0 = T_1\sin\theta_1 - u, \\ M_1\ddot{y}_1 = T_3\sin\theta_2 - T_2\sin\theta_1, \\ M_2\ddot{y}_2 = -T_4\sin\theta_2, \\ y_0 + b\sin\theta_1 = y_1, \\ y_1 + (1-b)\sin\theta_2 = y_2, \end{cases} \tag{5}$$

where ϵ is a sufficiently small positive constant which expresses the mass of cart. $T_1 := g(M_1 + M_2 + 1)$, $T_2 := g(M_1 + M_2 + 1 - b)$, $T_3 := g(M_2 + 1 - b)$, $T_4 := gM_2$ denote the tension forces.

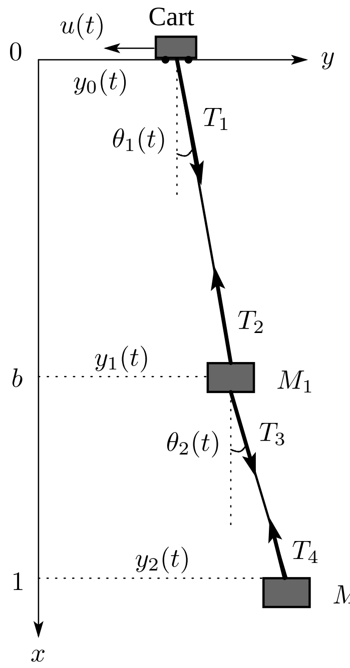

Fig. 2. Lumped parameter overhead crane model with two rigid loads I.

When $|\theta_1|$ and $|\theta_2|$ are sufficiently small, system (5) is approximated as

$$\begin{cases} \epsilon\ddot{y}_0 = -\hat{a}_1y_0 + \hat{a}_1y_1 - u, \\ \ddot{y}_1 = \hat{a}_2y_0 - \hat{a}_3y_1 + \hat{a}_4y_2, \\ \ddot{y}_2 = \hat{a}_5y_1 - \hat{a}_5y_2, \end{cases}$$

where $\hat{a}_1 := T_1/b$, $\hat{a}_2 := T_2/M_1b$, $\hat{a}_3 := T_2/M_1b + T_3/M_1(1-b)$, $\hat{a}_4 := T_3/M_1(1-b)$, $\hat{a}_5 := T_4/M_2(1-b)$. Here, by letting ϵ go to zero, we have the following descriptor system:

$$\mathrm{E}\dot{\mathrm{z}} = \mathrm{Az} + \mathrm{B}u, \tag{6}$$

where

$$\mathrm{z} := [y_0, \dot{y}_0, y_1, \dot{y}_1, y_2, \dot{y}_2]^T,$$

$$\mathrm{E} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{A} := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\hat{a}_1 & 0 & \hat{a}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hat{a}_2 & 0 & -\hat{a}_3 & 0 & \hat{a}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \hat{a}_5 & 0 & -\hat{a}_5 & 0 \end{bmatrix},$$

$$\mathrm{B} := [0, -1, 0, 0, 0, 0]^T.$$

When we apply the control law

$$u = \alpha y_0 + \gamma \dot{y}_0 = \mathrm{Fz}$$

to system (6), the closed-loop system becomes

$$\mathrm{E\dot{z}} = (\mathrm{A} + \mathrm{BF})\mathrm{z}, \tag{7}$$

where

$$\mathrm{F} := [\alpha, \gamma, 0, 0, 0, 0].$$

Now, let us consider the functional

$$J[u] := \int_0^\infty (\mathrm{z}^T(t)\mathrm{Qz}(t) + \mathrm{R}u^2(t))dt \tag{8}$$

for system (6), where Q is a positive definite matrix and R is a positive number. The problem of determing the control law for system (6) such that $J[u]$ is minimized is called the linear quadratic regulator problem. Using the admissible solution (X, Y) to the generalized algebraic Riccati equation

$$\mathrm{A}^T\mathrm{X} + \mathrm{YA} + \mathrm{Q} - \mathrm{YBR}^{-1}\mathrm{B}^T\mathrm{X} = 0,$$
$$\mathrm{E}^T\mathrm{X} = \mathrm{YE},$$

the solution to the problem is given by

$$u = \mathrm{Kz}, \quad \mathrm{K} := -\mathrm{R}^{-1}\mathrm{B}^T\mathrm{X}$$

(see [14]). In this, (X, Y) is calculated according to the way stated in [14]. Then, the closed-loop system consisting system (6) and the optimal control is given by

$$\mathrm{E\dot{z}} = (\mathrm{A} + \mathrm{BK})\mathrm{z}. \tag{9}$$

Then, we can determine the parameters α and γ such that the characteristic polynomial of (7)

$$\begin{aligned} \varphi_{\mathrm{F}}(s) &= \det(s\mathrm{E} - (\mathrm{A} + \mathrm{BF})) \\ &= f_1 s^5 + f_2 s^4 + f_3 s^3 + f_4 s^2 + f_5 s + f_6 \end{aligned}$$

approaches the one of (9)

$$\begin{aligned} \varphi_{\mathrm{K}}(s) &= \det(s\mathrm{E} - (\mathrm{A} + \mathrm{BK})) \\ &= k_1 s^5 + k_2 s^4 + k_3 s^3 + k_4 s^2 + k_5 s + k_6 \end{aligned}$$

as much as possible, by solving the following optimization problem:

Find the pair (α, γ) *that attains*

$$\min\{\, \|f - k\| \,;\, \alpha > 0,\, \gamma > 0 \,\}, \tag{10}$$

where $f := [f_1, f_2, f_3, f_4, f_5, f_6]^T$, $k := [k_1, k_2, k_3, k_4, k_5, k_6]^T$, *and the norm* $\|\cdot\|$ *is the Euclidean norm. Hereafter, we denote the solution to the optimization problem (10) by* $(\overline{\alpha}, \overline{\gamma})$.

IV. Optimal Feedback Gain Based on a Finite Difference Appro- ximation (Proposed Method)

Let I, J be positive integers. Assume that b is chosen such that

$$b = I\Delta x,$$

where $\Delta x := 1/(I + J)$. Hereafter, we set $a_i := a(i\Delta x)$ for $i = 0, 1, \cdots, I$, and $\tilde{a}_{I+i} := \tilde{a}((I+i)\Delta x)$ for $i = 0, 1, \cdots, J$.

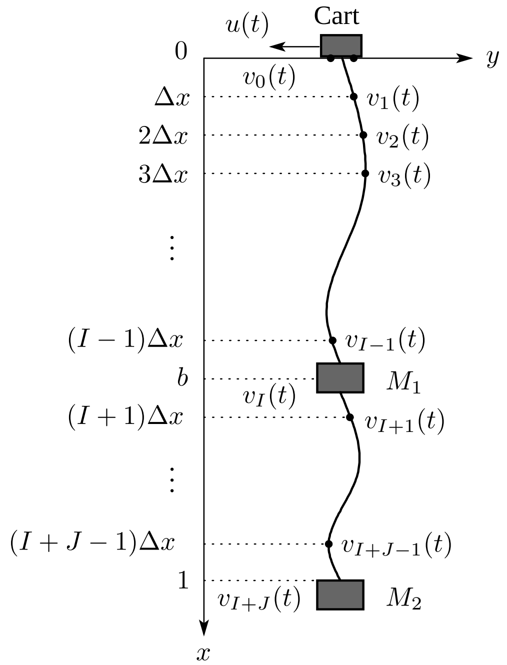

Fig. 3. Lumped parameter overhead crane model with two rigid loads II.

Introducing $v = y$, $w = \dot{y}$, the first equation of system (1) is written as

$$\begin{cases} \dot{v} = w, \\ \dot{w} = a(x)v'' - gv', \end{cases} \tag{11}$$

where $\dot{} = \partial/\partial t$, $' = \partial/\partial x$. Similarly, the second equation of system (1) is written as

$$\begin{cases} \dot{v} = w, \\ \dot{w} = \tilde{a}(x)v'' - gv'. \end{cases} \tag{12}$$

Here, defining $v_i(t) := v(t, i\Delta x)$, $w_i(t) := w(t,\, i\Delta x)$ (see Fig. 3) and approximating $v'(t, i\Delta x)$, $v''(t, i\Delta x)$ as

$$v'(t, i\Delta x) \cong \frac{v_{i+1}(t) - v_{i-1}(t)}{2\Delta x},$$

$$v''(t, i\Delta x) \cong \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{\Delta x^2},$$

it follows from (11) that

$$\begin{cases} \dot{v}_i = w_i, \\ \dot{w}_i = \alpha_i v_{i-1} + \beta_i v_i + \gamma_i v_{i+1}, \end{cases} \tag{13}$$

for $i = 1, 2, \cdots, I-1$, where

$$\alpha_i := \frac{a_i}{\Delta x^2} + \frac{g}{2\Delta x}, \quad \beta_i := -\frac{2a_i}{\Delta x^2},$$
$$\gamma_i := \frac{a_i}{\Delta x^2} - \frac{g}{2\Delta x}.$$

Similarly, from (12) we have

$$\begin{cases} \dot{v}_{I+i} = w_{I+i}, \\ \dot{w}_{I+i} = \tilde{\alpha}_{I+i} v_{I+i-1} + \tilde{\beta}_{I+i} v_{I+i} + \tilde{\gamma}_{I+i} v_{I+i+1}, \end{cases} \tag{14}$$

for $i = 1, 2, \cdots, J-1$, where

$$\tilde{\alpha}_{I+i} := \frac{\tilde{a}_{I+i}}{\Delta x^2} + \frac{g}{2\Delta x}, \quad \tilde{\beta}_{I+i} := -\frac{2\tilde{a}_{I+i}}{\Delta x^2},$$
$$\tilde{\gamma}_{I+i} := \frac{\tilde{a}_{I+i}}{\Delta x^2} - \frac{g}{2\Delta x}.$$

For the time being, we suppose that the cart has a sufficiently small mass $\epsilon > 0$. Then, the third equation of system (1) is replaced by

$$\epsilon y_{tt}(t, 0) = a(0) y_x(t, 0) - u(t).$$

Here, by approximating $y_x(t, 0)$ as

$$y_x(t, 0) \cong \frac{v_1(t) - v_0(t)}{\Delta x},$$

we have

$$\begin{cases} \dot{v}_0 = w_0, \\ \epsilon \dot{w}_0 = -c v_0 + c v_1 - u, \end{cases} \tag{15}$$

where $c := a_0/\Delta x$. Moreover, by approximating $y_x(t, b^+)$, $y_x(t, b^-)$ in the fifth equation as

$$y_x(t, b^+) \cong \frac{v_{I+1}(t) - v_I(t)}{\Delta x},$$
$$y_x(t, b^-) \cong \frac{v_I(t) - v_{I-1}(t)}{\Delta x},$$

and, $y_x(t, 1)$ in the sixth equation as

$$y_x(t, 1) \cong \frac{v_{I+J}(t) - v_{I+J-1}(t)}{\Delta x},$$

we get

$$\begin{cases} \dot{v}_I = w_I, \\ \dot{w}_I = d v_{I-1} - (d + \tilde{d}) v_I + \tilde{d} v_{I+1}, \end{cases} \tag{16}$$

$$\begin{cases} \dot{v}_{I+J} = w_{I+J}, \\ \dot{w}_{I+J} = \tilde{e} v_{I+J-1} - \tilde{e} v_{I+J}, \end{cases} \tag{17}$$

where

$$d := \frac{a_I}{M_1 \Delta x}, \quad \tilde{d} := \frac{\tilde{a}_I}{M_1 \Delta x}, \quad \tilde{e} := \frac{\tilde{a}_{I+J}}{M_2 \Delta x}.$$

Combining (13), (14), (15), (16), and (17), we get

$$\mathrm{E}_\epsilon \dot{\mathrm{z}} = \mathrm{A}\mathrm{z} + \mathrm{B}u, \tag{18}$$

where

$$\mathrm{z} := [v_0, v_1, \cdots, v_{I+J}, w_0, w_1, \cdots, w_{I+J}]^T,$$

$$\mathrm{E}_\epsilon := \begin{bmatrix} I_{I+J+1} & O_{I+J+1} \\ O_{I+J+1} & \begin{bmatrix} \epsilon & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{bmatrix},$$

$$\mathrm{A} := \begin{bmatrix} O_{I+J+1} & I_{I+J+1} \\ \mathrm{A}_{21} & O_{I+J+1} \end{bmatrix},$$

$$\mathrm{A}_{21} := \begin{bmatrix} -c & c & 0 & \cdots & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \tilde{\alpha}_{I+J-1} & \tilde{\beta}_{I+J-1} & \tilde{\gamma}_{I+J-1} \\ 0 & \cdots & 0 & \tilde{e} & -\tilde{e} \end{bmatrix},$$

$$\mathrm{B} := \begin{bmatrix} O_{(I+J+1)\times 1} \\ \mathrm{B}_2 \end{bmatrix}, \quad \mathrm{B}_2 := [-1, 0, \cdots, 0]^T.$$

In this, I_{I+J+1} denotes the $(I+J+1) \times (I+J+1)$ unit matrix, and O_{I+J+1} the $(I+J+1) \times (I+J+1)$ zero matrix. Here, by letting $\epsilon \searrow 0$ in (18), we obtain

$$\mathrm{E}\dot{\mathrm{z}} = \mathrm{A}\mathrm{z} + \mathrm{B}u, \tag{19}$$

where

$$\mathrm{E} := \begin{bmatrix} I_{I+J+1} & O_{I+J+1} \\ O_{I+J+1} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{bmatrix}.$$

In this way, we have a finite difference approximated model for system (1), which is a descriptor system since the matrix E is singular.

When we apply the control law

$$u = \alpha v_0 + \gamma w_0 = \mathrm{F}\mathrm{z}$$

to system (19), the closed-loop system becomes

$$\mathrm{E}\dot{\mathrm{z}} = (\mathrm{A} + \mathrm{B}\mathrm{F})\mathrm{z}, \tag{20}$$

where

$$\mathrm{F} := [\alpha, 0, \cdots, 0, \gamma, 0, \cdots, 0].$$

Therefore, we can calculate the gain (α, γ) such that the closed-loop system (20) approaches a specified optimal closed-loop system as much as possible, by using the method explained in Subsection III-C.

V. Numerical Simulations

Let us set $g = 9.8$, $M_1 = 0.7$, $M_2 = 1.0$, and $b = 0.7$ in system (1). And, we set the initial conditions as $p(x) = 0.001(e^{5x} - 1) + 0.1$ and $q(x) = 0.0$ for $x \in [0, 1]$. In the following subsections, we give the simulation results by the two design methods.

A. Former Method

According to the design method explained in Subsection III-C, we calculate the optimal parameters α and γ of the control law (2). In functional (8), let us set the positive definite matrix Q and the positive number R as

$$\mathrm{Q} = 3.9 \times I_6, \quad \mathrm{R} = 1.0,$$

where I_6 denotes the 6×6 unit matrix. Then, the admissible solution (X, Y) to the generalized algebraic Riccati equation is computed as

$$\mathrm{X} = \left[\begin{array}{rrrr} 82.3290 & -0.0000 & -80.9112 & 3.8153 \\ 4.4200 & 2.0000 & -3.6929 & 1.9566 \\ -80.9112 & 0.0000 & 274.4085 & -0.0837 \\ 3.8153 & 0.0000 & -0.0837 & 4.0363 \\ 5.3371 & 0.0000 & -186.8889 & -0.2641 \\ 3.3132 & -0.0000 & 3.5054 & 1.6151 \end{array}\right.$$

$$\left.\begin{array}{rr} 5.3371 & 3.3132 \\ 2.7370 & 1.6991 \\ -186.8889 & 3.5054 \\ -0.2641 & 1.6151 \\ 187.2904 & -0.5427 \\ -0.5427 & 8.5941 \end{array}\right],$$

$$\mathrm{Y} = \left[\begin{array}{rrrr} 82.3290 & 3.3645 & -80.9112 & 3.8153 \\ -0.0000 & 1.9500 & 0.0000 & 0.0000 \\ -80.9112 & -2.6556 & 274.4085 & -0.0837 \\ 3.8153 & 1.9077 & -0.0837 & 4.0363 \\ 5.3371 & 2.6686 & -186.8889 & -0.2641 \\ 3.3132 & 1.6566 & 3.5054 & 1.6151 \end{array}\right.$$

$$\left.\begin{array}{rr} 5.3371 & 3.3132 \\ 0.0000 & -0.0000 \\ -186.8889 & 3.5054 \\ -0.2641 & 1.6151 \\ 187.2904 & -0.5427 \\ -0.5427 & 8.5941 \end{array}\right].$$

Therefore, the optimal feedback gain is calculated as

$$\mathrm{K} = \left[\begin{array}{cccc} 4.4200 & 2.0000 & -3.6929 & 1.9566 \end{array}\right.$$
$$\left.\begin{array}{cc} 2.7370 & 1.6991 \end{array}\right],$$

and the characteristic polynomial of the optimal closed-loop system is calculated as

$$\varphi_{\mathrm{K}}(s) = \det(s\mathrm{E} - (\mathrm{A} + \mathrm{BK})) = [s^5, s^4, s^3, s^2, s, 1]k,$$

where

$$k = 10^3 \times \left[\begin{array}{cccc} 0.0020 & 0.0422 & 0.3449 & 3.9696 \end{array}\right.$$
$$\left.\begin{array}{cc} 7.3901 & 4.5264 \end{array}\right]^T.$$

Hence, the optimal parameters are determined as $\overline{\alpha} = 3.46$, $\overline{\gamma} = 5.62$. Then, $\|f - k\|$ takes the minimum value 407.6082 at the point $(\overline{\alpha}, \overline{\gamma})$. The broken lines of Figs. 4–6 show the evolution of input $u(t)$ and displacements $y(t, 0.7)$, $y(t, 1)$ in the closed-loop system (3), respectively.

B. Proposed Method

In order to apply our proposed method to the problem, we first set

$$I = 7, \quad J = 3,$$

namely, $\Delta x = 0.1$. Since we have set $b = 0.7$, the condition

$$b = I\Delta x$$

is satisfied. In this case, the matrices A and E are of 22×22 size, and the matrix B is of 22×1 size. According to the design method proposed in Section IV, we calculate the optimal parameters α and γ of the control law (2). In functional (8), let us set the positive definite matrix Q and the positive number R as

$$\mathrm{Q} = 3.9 \times I_{22}, \quad \mathrm{R} = 1.0,$$

where I_{22} denotes the 22×22 unit matrix. Here, we note that the same coefficient 3.9 is used for the matrix Q to compare with the result in Subsection V-A. Then, the admissible solution (X, Y) to the generalized algebraic Riccati equation is computed as

$$\mathrm{X} = 10^4 \times \left[\begin{array}{rrrr} 0.0644 & -0.0600 & -0.0083 & 0.0057 \\ -0.0600 & 0.2567 & -0.2340 & 0.0486 \\ -0.0083 & -0.2340 & 0.4959 & -0.3118 \\ 0.0057 & 0.0486 & -0.3118 & 0.5025 \\ -0.0037 & -0.0144 & 0.0766 & -0.2997 \\ 0.0024 & -0.0008 & -0.0215 & 0.0728 \\ -0.0004 & 0.0042 & -0.0001 & -0.0236 \\ 0.0032 & -0.0069 & 0.0098 & 0.0172 \\ 0.0041 & 0.0390 & -0.0386 & -0.0469 \\ -0.0151 & -0.0522 & 0.0576 & 0.0603 \\ 0.0090 & 0.0221 & -0.0261 & -0.0248 \\ 0.0065 & -0.0043 & -0.0043 & 0.0029 \\ 0.0007 & 0.0004 & -0.0016 & 0.0007 \\ -0.0002 & 0.0013 & -0.0002 & -0.0015 \\ 0.0001 & -0.0005 & 0.0014 & -0.0002 \\ -0.0000 & 0.0003 & -0.0007 & 0.0012 \\ 0.0000 & -0.0001 & 0.0004 & -0.0005 \\ 0.0000 & 0.0001 & -0.0002 & 0.0003 \\ 0.0003 & 0.0004 & -0.0003 & -0.0003 \\ 0.0002 & -0.0001 & -0.0004 & 0.0008 \\ -0.0002 & 0.0009 & 0.0007 & -0.0009 \\ 0.0004 & -0.0000 & -0.0006 & 0.0004 \end{array}\right.$$

$$\begin{array}{rrrrr} -0.0037 & 0.0024 & -0.0004 & 0.0032 & 0.0041 \\ -0.0144 & -0.0008 & 0.0042 & -0.0069 & 0.0390 \\ 0.0766 & -0.0215 & -0.0001 & 0.0098 & -0.0386 \\ -0.2997 & 0.0728 & -0.0236 & 0.0172 & -0.0469 \\ 0.4686 & -0.2760 & 0.0634 & -0.0287 & 0.0231 \\ -0.2760 & 0.4293 & -0.2318 & 0.0020 & 0.0651 \\ 0.0634 & -0.2318 & 0.3419 & -0.1501 & 0.0061 \\ -0.0287 & 0.0020 & -0.1501 & 0.5048 & -0.6396 \\ 0.0231 & 0.0651 & 0.0061 & -0.6396 & 1.3704 \\ -0.0055 & -0.0604 & -0.0215 & 0.3393 & -1.0375 \\ -0.0035 & 0.0189 & 0.0122 & -0.0499 & 0.2556 \\ -0.0019 & 0.0012 & -0.0002 & 0.0016 & 0.0021 \\ -0.0004 & 0.0002 & -0.0000 & 0.0001 & 0.0005 \\ 0.0007 & -0.0004 & 0.0002 & 0.0001 & 0.0002 \\ -0.0012 & 0.0005 & -0.0003 & 0.0004 & -0.0007 \\ -0.0002 & -0.0009 & 0.0003 & -0.0000 & -0.0005 \\ 0.0008 & -0.0002 & -0.0005 & -0.0005 & 0.0009 \\ -0.0003 & 0.0005 & -0.0002 & -0.0008 & 0.0013 \\ 0.0000 & 0.0005 & 0.0009 & -0.0001 & -0.0012 \\ 0.0005 & -0.0009 & -0.0013 & 0.0014 & -0.0001 \\ -0.0011 & -0.0001 & 0.0007 & 0.0002 & -0.0001 \\ 0.0005 & 0.0003 & -0.0000 & -0.0002 & 0.0002 \end{array}$$

$$
\begin{array}{rrrrr}
-0.0151 & 0.0090 & -0.0000 & 0.0007 & -0.0002 \\
-0.0522 & 0.0221 & 0.0000 & 0.0004 & 0.0013 \\
0.0576 & -0.0261 & -0.0000 & -0.0016 & -0.0002 \\
0.0603 & -0.0248 & 0.0000 & 0.0007 & -0.0015 \\
-0.0055 & -0.0035 & -0.0000 & -0.0004 & 0.0007 \\
-0.0604 & 0.0189 & 0.0000 & 0.0002 & -0.0004 \\
-0.0215 & 0.0122 & -0.0000 & -0.0000 & 0.0002 \\
0.3393 & -0.0499 & 0.0000 & 0.0001 & 0.0001 \\
-1.0375 & 0.2556 & 0.0000 & 0.0005 & 0.0002 \\
1.1739 & -0.4395 & 0.0000 & -0.0012 & -0.0007 \\
-0.4395 & 0.2273 & -0.0000 & 0.0006 & 0.0004 \\
-0.0077 & 0.0046 & 0.0002 & 0.0004 & -0.0001 \\
-0.0012 & 0.0006 & 0.0000 & 0.0001 & -0.0000 \\
-0.0007 & 0.0004 & -0.0000 & -0.0000 & 0.0001 \\
0.0009 & -0.0003 & 0.0000 & 0.0000 & -0.0000 \\
0.0011 & -0.0006 & -0.0000 & 0.0000 & 0.0000 \\
0.0001 & -0.0003 & -0.0000 & 0.0000 & 0.0000 \\
-0.0007 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-0.0003 & 0.0005 & -0.0000 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\
-0.0002 & 0.0001 & -0.0000 & -0.0000 & 0.0000 \\
-0.0002 & -0.0000 & 0.0000 & 0.0000 & 0.0000
\end{array}
$$

$$
\begin{array}{rrrrr}
0.0001 & -0.0000 & 0.0000 & 0.0000 & 0.0003 \\
-0.0005 & 0.0003 & -0.0001 & 0.0001 & 0.0004 \\
0.0014 & -0.0007 & 0.0004 & -0.0002 & -0.0003 \\
-0.0002 & 0.0012 & -0.0005 & 0.0003 & -0.0003 \\
-0.0012 & -0.0002 & 0.0008 & -0.0003 & 0.0000 \\
0.0005 & -0.0009 & -0.0002 & 0.0005 & 0.0005 \\
-0.0003 & 0.0003 & -0.0005 & -0.0002 & 0.0009 \\
0.0004 & -0.0000 & -0.0005 & -0.0008 & -0.0001 \\
-0.0007 & -0.0005 & 0.0009 & 0.0013 & -0.0012 \\
0.0009 & 0.0011 & 0.0001 & -0.0007 & -0.0003 \\
-0.0003 & -0.0006 & -0.0003 & 0.0000 & 0.0005 \\
0.0001 & -0.0000 & 0.0000 & 0.0000 & 0.0002 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0001 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0001 & -0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0001 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & -0.0000 & 0.0001 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0006 \\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0002 \\
0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0001 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0002
\end{array}
$$

$$
\left.
\begin{array}{rrr}
0.0002 & -0.0002 & 0.0004 \\
-0.0001 & 0.0009 & -0.0000 \\
-0.0004 & 0.0007 & -0.0006 \\
0.0008 & -0.0009 & 0.0004 \\
0.0005 & -0.0011 & 0.0005 \\
-0.0009 & -0.0001 & 0.0003 \\
-0.0013 & 0.0007 & -0.0000 \\
0.0014 & 0.0002 & -0.0002 \\
-0.0001 & -0.0001 & 0.0002 \\
-0.0000 & -0.0002 & -0.0002 \\
0.0000 & 0.0001 & -0.0000 \\
0.0001 & -0.0001 & 0.0002 \\
0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0000 \\
-0.0000 & -0.0000 & 0.0000 \\
-0.0002 & 0.0001 & 0.0002 \\
0.0005 & -0.0002 & 0.0001 \\
-0.0002 & 0.0004 & -0.0001 \\
0.0001 & -0.0001 & 0.0009
\end{array}
\right],
$$

$$
\mathrm{Y} = 10^4 \times \left[
\begin{array}{rrrr}
0.0644 & -0.0600 & -0.0083 & 0.0057 \\
-0.0600 & 0.2567 & -0.2340 & 0.0486 \\
-0.0083 & -0.2340 & 0.4959 & -0.3118 \\
0.0057 & 0.0486 & -0.3118 & 0.5025 \\
-0.0037 & -0.0144 & 0.0766 & -0.2997 \\
0.0024 & -0.0008 & -0.0215 & 0.0728 \\
-0.0004 & 0.0042 & -0.0001 & -0.0236 \\
0.0032 & -0.0069 & 0.0098 & 0.0172 \\
0.0041 & 0.0390 & -0.0386 & -0.0469 \\
-0.0151 & -0.0522 & 0.0576 & 0.0603 \\
0.0090 & 0.0221 & -0.0261 & -0.0248 \\
-0.0000 & 0.0000 & -0.0000 & 0.0000 \\
0.0007 & 0.0004 & -0.0016 & 0.0007 \\
-0.0002 & 0.0013 & -0.0002 & -0.0015 \\
0.0001 & -0.0005 & 0.0014 & -0.0002 \\
-0.0000 & 0.0003 & -0.0007 & 0.0012 \\
0.0000 & -0.0001 & 0.0004 & -0.0005 \\
0.0000 & 0.0001 & -0.0002 & 0.0003 \\
0.0003 & 0.0004 & -0.0003 & -0.0003 \\
0.0002 & -0.0001 & -0.0004 & 0.0008 \\
-0.0002 & 0.0009 & 0.0007 & -0.0009 \\
0.0004 & -0.0000 & -0.0006 & 0.0004
\end{array}
\right.
$$

$$
\begin{array}{rrrrr}
-0.0037 & 0.0024 & -0.0004 & 0.0032 & 0.0041 \\
-0.0144 & -0.0008 & 0.0042 & -0.0069 & 0.0390 \\
0.0766 & -0.0215 & -0.0001 & 0.0098 & -0.0386 \\
-0.2997 & 0.0728 & -0.0236 & 0.0172 & -0.0469 \\
0.4686 & -0.2760 & 0.0634 & -0.0287 & 0.0231 \\
-0.2760 & 0.4293 & -0.2318 & 0.0020 & 0.0651 \\
0.0634 & -0.2318 & 0.3419 & -0.1501 & 0.0061 \\
-0.0287 & 0.0020 & -0.1501 & 0.5048 & -0.6396 \\
0.0231 & 0.0651 & 0.0061 & -0.6396 & 1.3704 \\
-0.0055 & -0.0604 & -0.0215 & 0.3393 & -1.0375 \\
-0.0035 & 0.0189 & 0.0122 & -0.0499 & 0.2556 \\
-0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \\
-0.0004 & 0.0002 & -0.0000 & 0.0001 & 0.0005 \\
0.0007 & -0.0004 & 0.0002 & 0.0001 & 0.0002 \\
-0.0012 & 0.0005 & -0.0003 & 0.0004 & -0.0007 \\
-0.0002 & -0.0009 & 0.0003 & -0.0000 & -0.0005 \\
0.0008 & -0.0002 & -0.0005 & -0.0005 & 0.0009 \\
-0.0003 & 0.0005 & -0.0002 & -0.0008 & 0.0013 \\
0.0000 & 0.0005 & 0.0009 & -0.0001 & -0.0012 \\
0.0005 & -0.0009 & -0.0013 & 0.0014 & -0.0001 \\
-0.0011 & -0.0001 & 0.0007 & 0.0002 & -0.0001 \\
0.0005 & 0.0003 & -0.0000 & -0.0002 & 0.0002
\end{array}
$$

$$
\begin{array}{rrrrr}
-0.0151 & 0.0090 & 0.0057 & 0.0007 & -0.0002 \\
-0.0522 & 0.0221 & -0.0035 & 0.0004 & 0.0013 \\
0.0576 & -0.0261 & -0.0042 & -0.0016 & -0.0002 \\
0.0603 & -0.0248 & 0.0029 & 0.0007 & -0.0015 \\
-0.0055 & -0.0035 & -0.0019 & -0.0004 & 0.0007 \\
-0.0604 & 0.0189 & 0.0012 & 0.0002 & -0.0004 \\
-0.0215 & 0.0122 & -0.0002 & -0.0000 & 0.0002 \\
0.3393 & -0.0499 & 0.0016 & 0.0001 & 0.0001 \\
-1.0375 & 0.2556 & 0.0021 & 0.0005 & 0.0002 \\
1.1739 & -0.4395 & -0.0075 & -0.0012 & -0.0007 \\
-0.4395 & 0.2273 & 0.0045 & 0.0006 & 0.0004 \\
0.0000 & -0.0000 & 0.0002 & 0.0000 & -0.0000 \\
-0.0012 & 0.0006 & 0.0003 & 0.0001 & -0.0000 \\
-0.0007 & 0.0004 & -0.0001 & -0.0000 & 0.0001 \\
0.0009 & -0.0003 & 0.0001 & 0.0000 & -0.0000 \\
0.0011 & -0.0006 & -0.0000 & 0.0000 & 0.0000 \\
0.0001 & -0.0003 & 0.0000 & 0.0000 & 0.0000 \\
-0.0007 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-0.0003 & 0.0005 & 0.0002 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0001 & 0.0000 & -0.0000 \\
-0.0002 & 0.0001 & -0.0001 & -0.0000 & 0.0000 \\
-0.0002 & -0.0000 & 0.0002 & 0.0000 & 0.0000
\end{array}
$$

$$
\begin{array}{rrrrr}
0.0001 & -0.0000 & 0.0000 & 0.0000 & 0.0003 \\
-0.0005 & 0.0003 & -0.0001 & 0.0001 & 0.0004 \\
0.0014 & -0.0007 & 0.0004 & -0.0002 & -0.0003 \\
-0.0002 & 0.0012 & -0.0005 & 0.0003 & -0.0003 \\
-0.0012 & -0.0002 & 0.0008 & -0.0003 & 0.0000 \\
0.0005 & -0.0009 & -0.0002 & 0.0005 & 0.0005 \\
-0.0003 & 0.0003 & -0.0005 & -0.0002 & 0.0009 \\
0.0004 & -0.0000 & -0.0005 & -0.0008 & -0.0001 \\
-0.0007 & -0.0005 & 0.0009 & 0.0013 & -0.0012 \\
0.0009 & 0.0011 & 0.0001 & -0.0007 & -0.0003 \\
-0.0003 & -0.0006 & -0.0003 & 0.0000 & 0.0005 \\
0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0001 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0001 & -0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0001 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & -0.0000 & 0.0001 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0006 \\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0002 \\
0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0001 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0002
\end{array}
$$

$$
\left.\begin{array}{rrr}
0.0002 & -0.0002 & 0.0004 \\
-0.0001 & 0.0009 & -0.0000 \\
-0.0004 & 0.0007 & -0.0006 \\
0.0008 & -0.0009 & 0.0004 \\
0.0005 & -0.0011 & 0.0005 \\
-0.0009 & -0.0001 & 0.0003 \\
-0.0013 & 0.0007 & -0.0000 \\
0.0014 & 0.0002 & -0.0002 \\
-0.0001 & -0.0001 & 0.0002 \\
-0.0000 & -0.0002 & -0.0002 \\
0.0000 & 0.0001 & -0.0000 \\
0.0000 & -0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0000 \\
-0.0000 & -0.0000 & 0.0000 \\
-0.0002 & 0.0001 & 0.0002 \\
0.0005 & -0.0002 & 0.0001 \\
-0.0002 & 0.0004 & -0.0001 \\
0.0001 & -0.0001 & 0.0009
\end{array}\right] .
$$

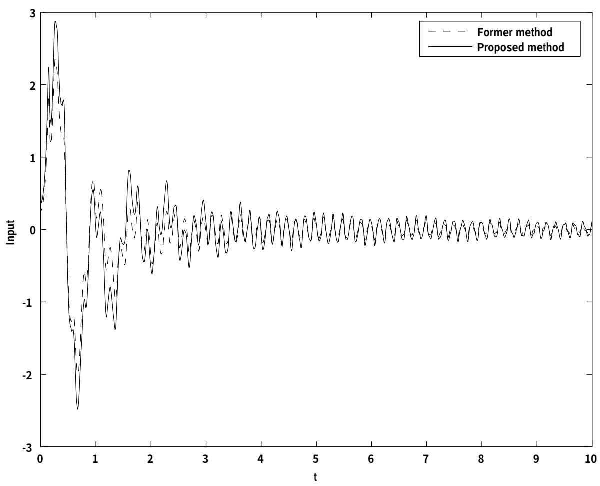

Fig. 4. Control input $u(t)$. $\alpha = 3.46$, $\gamma = 5.62$ (broken line). $\alpha = 6.65$, $\gamma = 9.28$ (solid line).

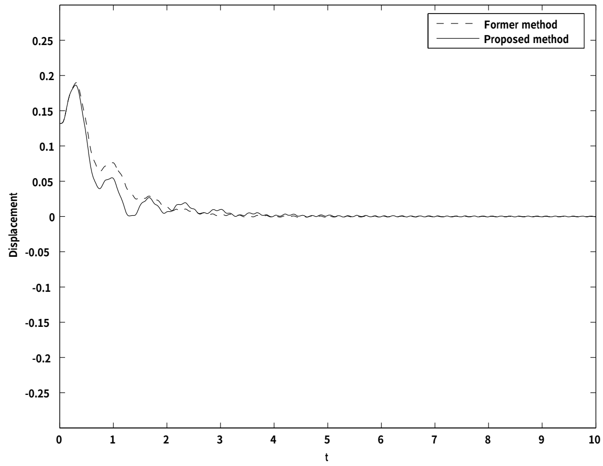

Fig. 5. Displacement $y(t, 0.7)$ under the control law. $\alpha = 3.46$, $\gamma = 5.62$ (broken line). $\alpha = 6.65$, $\gamma = 9.28$ (solid line).

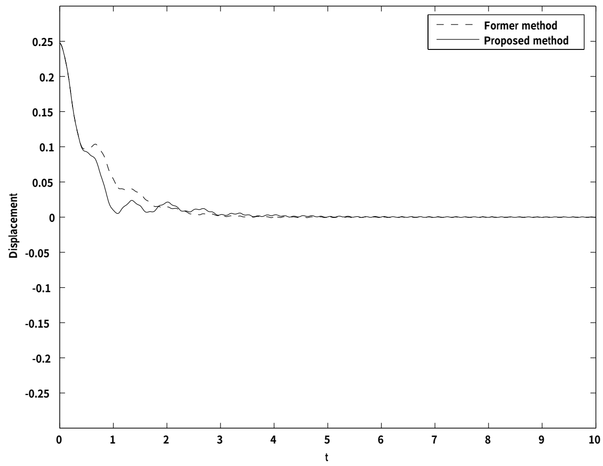

Fig. 6. Displacement $y(t, 1)$ under the control law. $\alpha = 3.46$, $\gamma = 5.62$ (broken line). $\alpha = 6.65$, $\gamma = 9.28$ (solid line).

Therefore, the optimal feedback gain is calculated as

$$
\begin{aligned}
\mathrm{K} = \big[\; & 65.4234 \quad -42.8421 \quad -42.7786 \quad 29.3219 \\
& -19.0760 \quad 12.2423 \quad -2.0152 \quad 16.4422 \quad 21.2656 \\
& -77.3222 \quad 45.9720 \quad 2.0000 \quad 3.5717 \quad -0.9250 \\
& 0.5276 \quad -0.0517 \quad 0.1224 \quad 0.2131 \quad 1.6375 \\
& 1.1730 \quad -0.7989 \quad 1.8481 \;\big],
\end{aligned}
$$

and the characteristic polynomial of the optimal closed-loop system is calculated as

$$
\varphi_{\mathrm{K}}(s) = \det(s\mathrm{E} - (\mathrm{A} + \mathrm{BK})) = [s^{21}, s^{20}, \cdots, s^2, s, 1]k,
$$

where

$$
\begin{aligned}
k = 10^{31} \times \big[\; & 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000 \\
& 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000 \\
& 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0000 \\
& 0.0000 \quad 0.0003 \quad 0.0038 \quad 0.0536 \quad 0.3613 \\
& 2.2526 \quad 5.8852 \quad 4.1896 \;\big]^T .
\end{aligned}
$$

See Appendix for the MATLAB program to solve the admissible solution (X, Y) to the generalized algebraic Riccati equation, the optimal feedback gain K, and the characteristic polynomial of the optimal closed-loop system. Hence, the optimal parameters are determined as $\overline{\alpha} = 6.65$, $\overline{\gamma} = 9.28$. Then, $\|f - k\|$ takes the minimum value 2.6628×10^{30} at the point $(\overline{\alpha}, \overline{\gamma})$. The solid lines of Figs. 4–6 show the evolution of input $u(t)$ and displacements $y(t, 0.7)$, $y(t, 1)$ in the closed-loop system (3), respectively. Thus, for the string with two rigid loads, we see that this control law has a better performance than that in Subsection V-A in the sense that the vibration is suppressed more fast.

VI. Conclusion

In this paper, we studied the stabilization problem of a string with two rigid loads which was described by two kinds of hyperbolic equations. Since the controller that assured the closed-loop stability contained free parameters $\alpha, \gamma > 0$, we considered a finite difference approximated model and proposed the design method to determine an optimal feedback gain $(\overline{\alpha}, \overline{\gamma})$. Through numerical simulations, it was shown that the performance of the control law which was designed based on the descriptor expression of the finite difference approximated model was better than that of the previous control law. How to determine the optimal feedback gain by the direct method, namely, by the one not based on the approximated model is still remained as an open problem.

Appendix

Based on [14], we can write the MATLAB program to solve the admissible solution (X, Y) to the generalized algebraic Riccati equation, the optimal feedback gain K, and the characteristic polynomial of the optimal closed-loop system.

```
% MATLAB program

g=9.8;
M1=0.7;
M2=1.0;

I=7;
J=3;

L=I+J;
dx=1/L;

a_0=g*(M1+M2+1-dx*0);

for i=1:I
    a(i)=g*(M1+M2+1-dx*i);
end

for i=0:J
    a_tilde(I+i)=g*(M2+1-dx*(I+i));
end

for i=1:I-1
    alpha(i)=a(i)/(dx^2)+g/(2*dx);
    beta(i)=-2*a(i)/(dx^2);
    gamma(i)=a(i)/(dx^2)-g/(2*dx);
end

for i=1:J-1
    alpha_tilde(I+i)
        =a_tilde(I+i)/(dx^2)+g/(2*dx);
    beta_tilde(I+i)=-2*a_tilde(I+i)/(dx^2);
    gamma_tilde(I+i)
        =a_tilde(I+i)/(dx^2)-g/(2*dx);
end

c=a_0/dx;
d=a(I)/(M1*dx);
d_tilde=a_tilde(I)/(M1*dx);
e_tilde=a_tilde(I+J)/(M2*dx);

A11=zeros(L+1);
A12=eye(L+1);
A22=zeros(L+1);

A21(1,1)=-c;
A21(1,2)=c;
A21(I+1,I)=d;
A21(I+1,I+1)=-(d+d_tilde);
A21(I+1,I+2)=d_tilde;
A21(I+J+1,I+J)=e_tilde;
A21(I+J+1,I+J+1)=-e_tilde;

for i=2:I
    A21(i,i-1)=alpha(i-1);
end

for i=2:I
    A21(i,i)=beta(i-1);
end

for i=2:I
    A21(i,i+1)=gamma(i-1);
end

for i=2:J
    A21(I+i,I+i-1)=alpha_tilde(I+i-1);
end

for i=2:J
    A21(I+i,I+i)=beta_tilde(I+i-1);
end

for i=2:J
    A21(I+i,I+i+1)=gamma_tilde(I+i-1);
end

A=[A11 A12; A21 A22];

B1=zeros(L+1,1);
B2=zeros(L+1,1);
B2(1,1)=-1;
B=[B1; B2];

E11=eye(L+1);
E12=zeros(L+1);
E21=zeros(L+1);
E22=eye(L+1);
E22(1,1)=0;
E=[E11 E12; E21 E22];

q=3.9;
r=1;
Q=q*eye(2*L+2);
R=r*eye(1);

Etilde=[E zeros(2*L+2) zeros(2*L+2,1);
    zeros(2*L+2) E' zeros(2*L+2,1);
    zeros(1,2*L+2) zeros(1,2*L+2) zeros(1)];

Atilde=[A zeros(2*L+2) B;
    -Q -A' zeros(2*L+2,1);
    zeros(1,2*L+2) B' R];

[T,D]=eig(Atilde,Etilde);
TT=[T(:,1) T(:,3:4) T(:,7:8) T(:,11:12)
    T(:,15:18) T(:,23:24) T(:,27:28) T(:,31:32)
    T(:,35:36) T(:,39) T(:,41)];
U1=TT(1:2*L+2,:);
```

```
U2=TT(2*L+3:4*L+4,:);
U3=TT(4*L+5,:);

V1=zeros(2*L+2,1);
V1(L+2,1)=1;
V2=V1;

h=2;
Hx=h;
Hy=q*r/h;

X=real([U2 V2*Hx]*inv([U1 V1]))
Y=real(([U2 V2*Hy]*inv([U1 V1]))')
K=-inv(R)*B'*X

[U,S,V]=svd(E);
EE=S(1:2*L+1,1:2*L+1);
AA=U'*(A+B*K)*V;
AA11=AA(1:2*L+1,1:2*L+1);
AA12=AA(1:2*L+1,2*L+2);
AA21=AA(2*L+2,1:2*L+1);
AA22=AA(2*L+2,2*L+2);

k=det(-AA22)*det(EE)*poly(inv(EE)
    *(AA11-AA12*inv(AA22)*AA21))
```

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