The influence of noise kurtosis on the dynamics of a harmonic oscillator with fluctuating frequency

Katrin Laas, Romi Mankin, Astrid Rekker

***Abstract*—The influence of noise kurtosis on underdamped motion of a harmonic oscillator with fluctuating frequency subjected to an external periodic force and an additive thermal noise is considered. The colored fluctuations of the oscillator frequency are modeled as a trichotomous noise. It is established that the spectral amplification and variance of the output signal exhibits a nonmonotonic dependence on the noise kurtosis, thus demonstrating the phenomenon of noise kurtosis controlled stochastic resonance. Some unexpected effects such as hypersensitive response of spectral amplification to small variations of noise amplitude, encountered in the case of a large kurtosis of colored noise are also discussed.**

***Index Terms*—Hypersensitive response, noise kurtosis, thermal noise, trichotomous noise, stochastic resonance, stochastic oscillator.**

I. Introduction

One of the key issues in ecology is how environmental fluctuations and species interactions may determine the oscillations in population sizes displayed by many organisms in nature as well as in laboratory cultures [1]- [6]. Ecologists have mainly been interested in the dynamical consequences of population interaction, often ignoring environmental variability altogether. However, the essential role of environmental fluctuations has recently been recognized in theoretical ecology. Noise-induced effects on population dynamics have been subject to intense theoretical investigations [7]- [11]. Moreover, ecological investigations suggest that population dynamics is sensitive to noise correlation time (noise color) [11]- [16]. In spite of the obvious significance of this circumstance, the role of nonequilibrium fluctuations (colored noise) of environmental parameters has not been much investigated in the context of ecosystems [15]- [17].

Recently, noise-induced anomalous transport phenomena of Brownian particles in nonlinear periodic structures have been the topic of a number of physical investigations. Among them, we can mention the ratchet effect [18], hypersensitive response [19], noise-enhanced stability [20], and absolute negative mobility [21], to name but a few. Active analytical and numerical investigations of various models in this field have been stimulated by their possible applications in chemical physics, molecular biology, nanotechnology, and for separation techniques of nanoobjects [18], [22]–[24]. For example, the feasibility of particle transport by man-made devices has been experimentally demonstrated for several ratchet types [25]. One of the objects of special attention in this context is the noise-driven harmonic oscillator. The harmonic oscillator is the simplest toy model for different phenomena in nature and as such it is a typical theoretician's paradigm for various fundamental conceptions [26].

The problem of noise-driven dynamics of a Brownian harmonic oscillator was earlier formulated and solved by Chandrasekhar [27], using the Langevin and Fokker-Planck equations. Since then the Chandrasekhar model and its many variants have been reappearing in literature. For example, the study of a harmonic oscillator with random frequency is a subject that has been extensively investigated in different fields including physics [28], biology [29], chemistry [30], etc. In most of the previous analysis the influence of white noise is considered. However, more realistic models of physical systems, such as, e.g., the dynamics of a dye laser and the transport of proteins in cells in the presence of thermal noise and colored noise of biological origin, require considering a system simultaneously driven by white noise and colored noise. It has been shown that the influence of colored noise on the oscillator frequency may lead to different resonant phenomena.

- First, it may cause energetic instability, which manifests itself in an unlimited increase of the second-order moments of the output with time, while the mean value of the oscillator displacement remains finite [28], [31]. This phenomenon is a stochastic counterpart of classical parametric resonance [28], [32].
- Second, if the oscillator is subjected to an external periodic force and the fluctuations of the oscillator frequency are colored, the behavior of the amplitude of the first moment shows a nonmonotonic dependence on noise parameters, i.e., stochastic resonance [33], [34].

To avoid misunderstandings, let us mention that we use the term stochastic resonance (SR) in the wide sense, meaning the nonmonotonic behavior of the output signal or some function of it (moments, autocorrelation functions, signal-to-noise ratio) in response to noise parameters [33].

It is generally acknowledged that kurtosis is an important characteristic of environmental variability, or noise. Theoretical investigations suggest that population dynamics is sensitive to noise kurtosis [15], [35]. Surprisingly, in spite of the obvious significance of this circumstance, the existence of SR in the case of a stochastic harmonic oscillator has never been linked to the potential influences of changes in noise kurtosis.

Katrin Laas, PhD student, Insitute of Mathematics and Natural Sciences, Tallinn University, Narva Road 25, 10120 Tallinn, Estonia (e-mail: katrin.laas@tlu.ee).

Romi Mankin, Professor, Insitute of Mathematics and Natural Sciences, Tallinn University, Narva Road 25, 10120 Tallinn, Estonia (e-mail: romi.mankin@tlu.ee).

Astrid Rekker, Associate Professor, Insitute of Mathematics and Natural Sciences, Tallinn University, Narva Road 25, 10120 Tallinn, Estonia (e-mail: astrid.rekker@tlu.ee).

Thus motivated, we consider a model similar to the one presented in [33], except for some details of the noises, i.e., a harmonic oscillator with fluctuating frequency subjected to an external sinusoidal force and an additive thermal noise. The fluctuations of the frequency are modeled as a three-level Markovian noise (trichotomous noise) [35]. Note that in the model presented in [33] the thermal noise is absent and the colored fluctuations of the frequency are assumed to be a dichotomous noise. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noise. Furthermore, it is remarkable that for trichotomous noises the kurtosis κ, in contrast to the Gaussian colored noise, $\kappa = 0$, and symmetric dichotomous noise, $\kappa = -2$, can be anything from -2 to ∞. This extra degree of freedom can prove useful in modeling actual fluctuations.

The main contribution of this paper is as follows. We provide exact formulas for the analytic treatment of the dependence of SR characteristics (variance of the output signal, and spectral amplification) on various system parameters: viz. temperature, correlation time, kurtosis, noise amplitude, and frequency of the input signal. On the basis of exact expressions for the SR characteristics we find a number of cooperation effects arising as a consequence of interplay between multiplicative trichotomous noise, thermal noise and a deterministic force, e.g.:

- a resonant-like behavior versus the noise kurtosis of the output variance and spectral amplification (SPA);
- for large values of the noise kurtosis the SR characteristics are very sensitive to small variation of noise amplitude – a phenomenon called hypersensitive response;
- the noise-induced doubly unidirectional transitions between the stable regimes and unstable energetic states of the oscillator.

The structure of the paper is as follows. In Section 2 we present the model investigated. A description of the output SR quantifiers is given and exact formulas are found for analysing the long-time behavior of SPA and variance. In Section 3 the conditions of energetic instability are considered. The SR phenomenon versus noise kurtosis is demonstrated in Section 4. In Section 5 we analyze the dependence of the SR characteristics on noise amplitude, while the conditions for the appearance of hypersensitivity are also discussed. Section 6 contains some brief concluding remarks.

II. Model

As an archetypical model for an oscillatory system strongly coupled with a noisy environment, we consider the stochastically perturbed harmonic oscillator with a random frequency

$$\ddot{X} + \gamma\dot{X} + [\omega^2 + Z(t)]X = A_0 \sin\Omega t + \xi(t), \qquad (1)$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, γ is a damping parameter, and the driving force $\xi(t)$ is a Gaussian white noise with a zero mean and with a delta-correlated correlation function given by

$$\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'). \qquad (2)$$

Fluctuations of the frequency ω^2 are expressed by a Markovian trichotomous noise $Z(t)$, which consists of jumps between three values: $z_1 = a$, $z_2 = 0$, $z_3 = -a$, $a > 0$ [16]. The jumps follow, in time, the pattern of a Poisson process, the values occurring with the stationary probabilities

$$p_s(a) = p_s(-a) = q, \qquad p_s(0) = 1 - 2q, \qquad (3)$$

where $0 < q < 1/2$. In a stationary state the fluctuation process $Z(t)$ satisfies

$$\langle Z(t)\rangle = 0, \qquad \langle Z(t+\tau)Z(t)\rangle = 2qa^2 e^{-\nu\tau}, \qquad (4)$$

where the switching rate ν is the reciprocal of the noise correlation time

$$\tau_c = 1/\nu, \qquad (5)$$

i.e., $Z(t)$ is a symmetric zero-mean exponentially correlated noise. The trichotomous process is a particular case of a kangaroo process [36] with the kurtosis

$$\kappa = \frac{\langle Z^4(t)\rangle}{\langle Z^2(t)\rangle^2} - 3 = \frac{1}{2q} - 3. \qquad (6)$$

In this work we will restrict ourselves to the case where, for all states of the trichotomous noise, the frequency of the oscillator is positive, i.e.,

$$a < \omega^2. \qquad (7)$$

To find the first and second moments of X we use the well-known Shapiro-Loginov procedure [37], which for an exponentially correlated noise $Z(t)$ yields

$$\frac{d}{dt}\langle Zm\rangle = \langle Z\frac{dm}{dt}\rangle - \nu\langle Zm\rangle, \qquad (8)$$

where m is some function of the noise, $m = m(Z)$.

From Eqs. (1) and (8), we thus obtain an exact linear system of six first-order differential equations for six variables, $M_{1,1} \equiv \langle X\rangle$, $M_{1,2} \equiv \langle\dot{X}\rangle$, $M_{1,3} \equiv \langle ZX\rangle$, $M_{1,4} \equiv \langle Z\dot{X}\rangle$, $M_{1,5} \equiv \langle Z^2X\rangle$, $M_{1,6} \equiv \langle Z^2\dot{X}\rangle$:

$$\begin{aligned}
\dot{M}_{1,1} &= M_{1,2}, \\
\dot{M}_{1,2} &= -\gamma M_{1,2} - \omega^2 M_{1,1} - M_{1,3} \\
&\quad + A_0\sin(\Omega t), \\
\dot{M}_{1,3} &= M_{1,4} - \nu M_{1,3}, \\
\dot{M}_{1,4} &= -(\gamma+\nu)M_{1,4} - \omega^2 M_{1,3} - M_{1,5}, \\
\dot{M}_{1,5} &= M_{1,6} - \nu M_{1,5} + 2qa^2\nu M_{1,1}, \\
\dot{M}_{1,6} &= -(\gamma+\nu)M_{1,6} - \omega^2 M_{1,5} - a^2 M_{1,3} \\
&\quad + 2qa^2\nu M_{1,2} + 2qa^2 A_0\sin(\Omega t), \qquad (9)
\end{aligned}$$

where $\dot{M} \equiv dM/dt$. The solution of equations (9) can be represented in the form

$$M_{1,i} = A_{2i-1}\sin(\Omega t) + A_{2i}\cos(\Omega t) + \sum_{j=1}^{6} C_j L_{i,j} e^{\rho_j t}, \qquad (10)$$

where the coefficients L_{ij}, $i,j = 1,\ldots,6$, are given by

$$\begin{aligned}
L_{1,j} &= 1, \quad L_{2,j} = \rho_j, \\
L_{3,j} &= -[\omega^2 + \rho_j(\rho_j+\gamma)], \\
L_{4,j} &= (\rho_j+\nu)L_{3,j}, \\
L_{5,j} &= -L_{3,j}[(\rho_j+\nu)(\rho_j+\nu+\gamma)+\omega^2], \\
L_{6,j} &= (\rho_j+\nu)L_{5,j} - 2qa^2\nu, \qquad (11)
\end{aligned}$$

C_j are constants of integration determined by the initial conditions, and $\{\rho_j, j = 1, \ldots, 6\}$ is the set of roots of the algebraic equation

$$\begin{aligned}&[\omega^2 + \rho(\rho+\gamma)][\omega^2 + (\rho+\nu)(\rho+\nu+\gamma)]^2\\ &-a^2[\omega^2 + \rho(\rho+\gamma) + 2q\nu(2\rho+\gamma+\nu)] = 0. \qquad (12)\end{aligned}$$

Note that the moments $M_{1,j}$ are independent of the thermal noise $\xi(t)$. One can check up the stability of the solution (10), which means that the solution of Eq. (12) cannot have roots with a positive real part. According to the Routh-Hurwitz theorem this requirement is met by the sixth-order polynomial in ρ in Eq. (12) for all values of the parameters, if the inequality (7) holds. Thus in the long time limit, $t \to \infty$, the moments $M_{1,j}$ are given by

$$M_{1,i|t\to\infty} = M_{1,i}^{(as)} = A_{2i-1}\sin(\Omega t) + A_{2i}\cos(\Omega t), \quad (13)$$

where $i = 1, \ldots, 6$. Particularly, the first moment $M_{1,1}^{(as)} = \langle X\rangle_{as}$ reads

$$\langle X\rangle_{as} = A\sin(\Omega t + \varphi), \qquad (14)$$

where

$$A^2 = A_1^2 + A_2^2 = \frac{A_0^2[f_1^2 + (f_2 + 2qa^2)^2]}{f_3^2 + f_4^2}, \qquad (15)$$

$$\tan\varphi = \frac{f_1 f_3 - f_4(f_2 + 2qa^2)}{f_1 f_4 + f_3(f_2 + 2qa^2)}, \qquad (16)$$

and the quantities f_i, ($i = 1, \ldots, 4$) are determined by

$$\begin{aligned}f_1 &= 2\Omega(\gamma + 2\nu)[\nu(\gamma+\nu) + \omega^2 - \Omega^2],\\ f_2 &= [\nu(\gamma+\nu) + \omega^2 - \Omega^2]^2\\ &\quad -\Omega^2(\gamma+2\nu)^2 - a^2,\\ f_3 &= [f_2(\omega^2 - \Omega^2) - \Omega\gamma f_1] - 2qa^2\nu(\nu+\gamma),\\ f_4 &= [\Omega\gamma f_2 + (\omega^2 - \Omega^2)f_1] - 4qa^2\nu\Omega. \qquad (17)\end{aligned}$$

Our next task is to evaluate the long-time behavior of the moments: $M_{2,1} \equiv \langle X^2\rangle$, $M_{2,2} \equiv \langle X\dot{X}\rangle$, $M_{2,3} \equiv \langle ZX^2\rangle$, $M_{2,4} \equiv \langle ZX\dot{X}\rangle$, $M_{2,5} \equiv \langle Z^2X^2\rangle$, $M_{2,6} \equiv \langle Z^2X\dot{X}\rangle$, $M_{2,7} \equiv \langle \dot{X}^2\rangle$, $M_{2,8} \equiv \langle Z\dot{X}^2\rangle$, $M_{2,9} \equiv \langle Z^2\dot{X}^2\rangle$.

From Eqs. (1) and (8) nine linear differential equations can be obtained for the moments $M_{2,i}$, $i = 1, \ldots, 9$.

$$\begin{aligned}\dot{M}_{2,1} &= 2M_{2,2},\\ \dot{M}_{2,2} &= M_{2,7} - \gamma M_{2,2} - \omega^2 M_{2,1} - M_{2,3}\\ &\quad +A_0 M_{1,1}\sin(\Omega t),\\ \dot{M}_{2,7} &= -2\gamma M_{2,7} - 2\omega^2 M_{2,2} - 2M_{2,4}\\ &\quad +2A_0 M_{1,2}\sin(\Omega t) + 2D,\\ \dot{M}_{2,3} &= -\nu M_{2,3} + 2M_{2,4},\\ \dot{M}_{2,4} &= M_{2,8} - (\gamma+\nu)M_{2,4} - \omega^2 M_{2,3} - M_{2,5}\\ &\quad +A_0 M_{1,3}\sin(\Omega t),\\ \dot{M}_{2,8} &= -(2\gamma+\nu)M_{2,8} - 2\omega^2 M_{2,4} - 2M_{2,6}\\ &\quad +2A_0 M_{1,4}\sin(\Omega t),\\ \dot{M}_{2,5} &= -\nu M_{2,5} + 2M_{2,6} + 2qa^2\nu M_{2,1},\\ \dot{M}_{2,6} &= M_{2,9} - (\gamma+\nu)M_{2,6} - \omega^2 M_{2,5} - a^2 M_{2,3}\\ &\quad +2qa^2\nu M_{2,2} + A_0 M_{1,5}\sin(\Omega t),\\ \dot{M}_{2,9} &= -(2\gamma+\nu)M_{2,9} - 2\omega^2 M_{2,6}\\ &\quad -2a^2 M_{2,4} + 2qa^2\nu M_{2,7}\\ &\quad +2A_0 M_{1,6}\sin(\Omega t) + 4qa^2 D \qquad (18)\end{aligned}$$

with $\dot{M} \equiv dM/dt$.

Starting from Eqs. (18) and (13), we obtain that in the limit $t \to \infty$ the moments $M_{2,i}^{(as)} = M_{2,i|t\to\infty}$ are given by

$$\begin{aligned}M_{2,i}^{(as)} &= N_i + J_{2i-1}\sin(2\Omega t) + J_{2i}\cos(2\Omega t),\\ &\quad i = 1, \ldots, 9, \qquad (19)\end{aligned}$$

where the constants N_i and J_k are determined with sets of algebraic linear equations. Note that the result (19) is correct only under the implicit assumption of energetic stability, i.e., the roots of the characteristic polynomial equation of the nine first-order differential equations determining the moments $M_{2,i}$, $i = 1, \ldots, 9$, cannot have positive real parts. This characteristic polynomial equation reads

$$\begin{aligned}&(\rho+\gamma+\nu)\Big\{(\rho+\gamma)(\rho+\gamma+\nu)[4\omega^2\\ &+\rho(\rho+2\gamma)]\{[4\omega^2 + (\rho+\nu)(\rho+\nu+2\gamma)]^2\\ &-16a^2\} - 8qa^2\nu\{\nu[4\omega^2 + \rho(\rho+2\gamma)]\\ &+(2\rho+2\gamma+\nu)^3\}\Big\} = 0. \qquad (20)\end{aligned}$$

By the condition (7), Eq. (20) and the Routh-Hurwitz theorem yield the necessary and sufficient condition for energetic stability, namely

$$a^2 < a_{cr}^2 = \frac{\omega^2\gamma(\gamma+\nu)[4\omega^2 + \nu(2\gamma+\nu)]^2}{16\omega^2\gamma(\gamma+\nu) + 2q\nu[4\omega^2\nu + (2\gamma+\nu)^3]}. \quad (21)$$

We note that in the case of dichotomous noise this condition for stability is in accordance with the results of [31]. Henceforth in this Section we shall assume that the condition (21) is fulfilled. Turning now to Eq. (19), we consider the quantity N_1 in more detail. It follows from Eqs. (18) and (19) that the time-homogeneous part of the second moment $M_{2,1}^{(as)} = \langle X^2\rangle_{as}$ is given by

$$\begin{aligned}N_1 &= \frac{1}{S_2}\big\{S_1 + 2D(\nu+\gamma)\big[[4\omega^2 + \nu(\nu+2\gamma)]^2\\ &\quad -16a^2(1-2q)\big]\big\}, \qquad (22)\end{aligned}$$

where

$$\begin{aligned}S_1 &= A_0\big\{(2\gamma+\nu)\{8A_{11} + 4(2\gamma+\nu)A_9\\ &\quad -[4\omega^2 + \nu(2\gamma+\nu)][2A_7 + (2\gamma+\nu)A_5]\}\\ &\quad +(\gamma+\nu)(\gamma A_1 + A_3)\{[4\omega^2 + \nu(2\gamma+\nu)]^2\\ &\quad -16a^2\} + 8qa^2\nu(2A_3 - \nu A_1)\big\},\end{aligned}$$

$$S_2 = 2\omega^2\gamma(\gamma+\nu)[4\omega^2 + \nu(2\gamma+\nu)]^2\left[1 - \left(\frac{a}{a_{cr}}\right)^2\right], \quad (23)$$

and

$$
\begin{aligned}
A_1 &= \frac{A_0[f_1 f_4 + f_3(f_2 + 2qa^2)]}{f_3^2 + f_4^2}, \\
A_2 &= \frac{A_0[f_1 f_3 - f_4(f_2 + 2qa^2)]}{f_3^2 + f_4^2}, \\
A_3 &= -\Omega A_2, \quad A_4 = \Omega A_1, \\
A_5 &= A_0 + (\Omega^2 - \omega^2)A_1 + \Omega\gamma A_2, \\
A_6 &= (\Omega^2 - \omega^2)A_2 - \Omega\gamma A_1, \\
A_7 &= \nu A_0 + [\nu(\Omega^2 - \omega^2) + \gamma\Omega^2]A_1 \\
&\quad +\Omega[\nu\gamma - (\Omega^2 - \omega^2)]A_2, \\
A_8 &= \Omega A_0 + \Omega[(\Omega^2 - \omega^2) - \nu\gamma]A1 \\
&\quad +[\gamma\Omega^2 + \nu(\Omega^2 - \omega^2)]A_2, \\
A_9 &= \Omega A_8 - \omega^2 A_5 - (\nu + \gamma)A_7, \\
A_{10} &= -\Omega A_7 - \omega^2 A_6 - (\nu + \gamma)A_8, \\
A_{11} &= \nu A_9 - \Omega A_{10} - 2qa^2\nu A_1, \\
A_{12} &= \Omega A_9 + \nu A_{10} - 2qa^2\nu A_2. \qquad (24)
\end{aligned}
$$

Particularly, the time-homogeneous part of the variance of the oscillator displacement X can be expressed as

$$\sigma^2(X) = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}(\langle X^2\rangle_{as} - \langle X\rangle_{as}^2)dt = N_1 - \frac{A^2}{2}. \quad (25)$$

Evidently, if the noise amplitude a tends to the critical value a_{cr}, the second moment $\langle X^2\rangle_{as}$ diverges.

Finally, we emphasize that for all figures throughout this work we use a dimensionless formulation of the dynamics with $\omega = 1$ and $A_0 = 1$.

III. Variance instability

Our next task is to find the boundaries of the region of energetic instability in the parameter space (ν, γ). From Eq. (21) one can discern two cases (see Fig. 1). First, if the damping is sufficiently strong, $\gamma > \gamma^*$, then the function $a_{cr}^2(\nu)$ increases monotonically from the value ω^4 to infinity as the switching rate ν increases. Thus, by the condition (7) the system is stable, i.e., no energetic instability can occur.

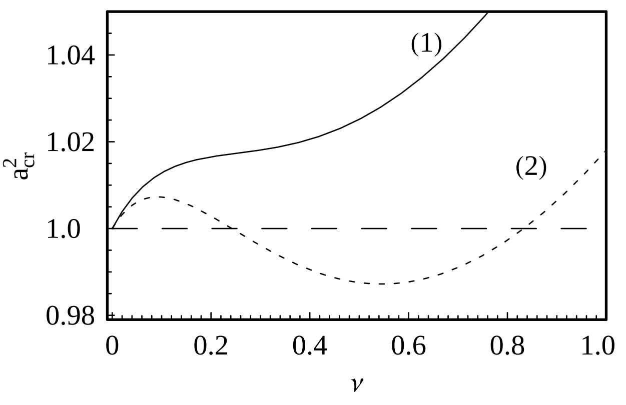

Fig. 1. Dependence of the critical noise amplitude a_{cr}^2 on the noise switching rate ν, obtained from Eq. (21). The parameter values: $\omega = 1$, $q = 0.35$. The solid curve (1) and the dashed curve (2) correspond to the values of the damping parameter $\gamma = 0.335$ and $\gamma = 0.295$, respectively. Note the nonmonotonous dependence of a_{cr}^2 on ν for $\gamma = 0.295$.

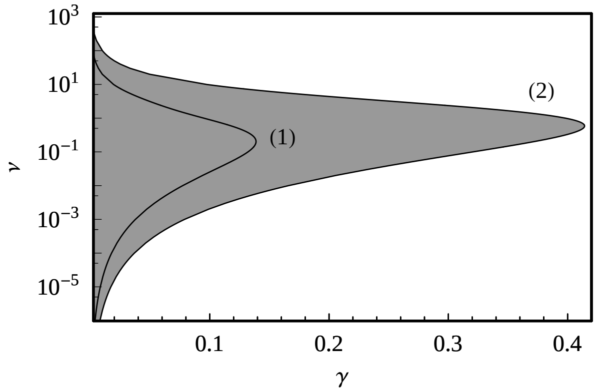

Fig. 2. A plot of the phase diagram in the $\nu - \gamma$ plane at $\omega = 1$. The shaded domain in the figure corresponds to the region where noise-induced energetic instability is possible. The lines depict the borders of the energetic instability regions for two values of the noise parameter q [see Eq. (29)]. The curves (1) and (2) correspond to the values of the parameter $q = 0.1$ and $q = 0.5$, respectively.

The critical damping parameter γ^* is given by the system of algebraic equations:

$$\frac{d}{d\nu}a_{cr}^2(\nu) = 0, \qquad \frac{d^2}{d\nu^2}a_{cr}^2(\nu) = 0, \qquad (26)$$

where $a_{cr}^2(\nu)$ is given by Eq. (21). For example, in the case of dichotomous noise, $q = 1/2$, the critical parameter

$$(\gamma^*)^2 = (3\sqrt{3} - 5)\omega^2. \qquad (27)$$

Second, in the case of $\gamma < \gamma^*$, Eq. (21) demonstrates that the functional dependence of a_{cr}^2 on the noise correlation time $\tau_c = 1/\nu$ exhibits a resonance form as τ_c is varied [cf. curve (2) in Fig. 1]. For increasing values of ν, the critical noise amplitude a_{cr} starts from the value ω^2, increasing to a local maximum $a_{cr\ max}$, next it decreases, attaining a local minimum $a_{cr\ min}$, and then a_{cr} tends to infinity as $\nu \to \infty$. Relying on Eqs. (7) and (21) one can find the necessary and sufficient conditions for the emergence of energetic instability (and reentrant transition) due to noise correlation time variations. Namely, energetic instability appears for the parameter values:

$$\gamma < \gamma^*, \quad a_{cr}^2 < \omega^4, \quad a_{cr\ min}^2 < a^2 < \omega^4, \qquad (28)$$

where $a_{cr\ min}^2$ corresponds to the local minimum of the function $a_{cr}^2(\nu)$. This case is characterized by the following scenario: For small values of the switching rate, $\nu < \nu_1$, where $a < a_{cr}(\nu)$, the system is stable. At $\nu = \nu_1$, i.e., $a = a_{cr}(\nu_1)$, the system becomes unstable. In the interval $\nu_1 < \nu < \nu_2$ of the switching rate there appears an instability, where the second moments of the oscillator displacements diverge. At $\nu = \nu_2$, where $a = a_{cr}(\nu_2)$, the energetic instability disappears and the system approaches the stable regime, thus making a reentrant transition. Now, we will briefly consider the necessary condition for instability, $a_{cr}^2 < \omega^4$.

Figure 2 shows a phase diagram in the $\nu - \gamma$ plane at two values of q. As the damping parameter γ increases the region of instability narrows down and disappears at the critical value of the damping parameter $\gamma_{cr}(q)$. Hence, there is an upper

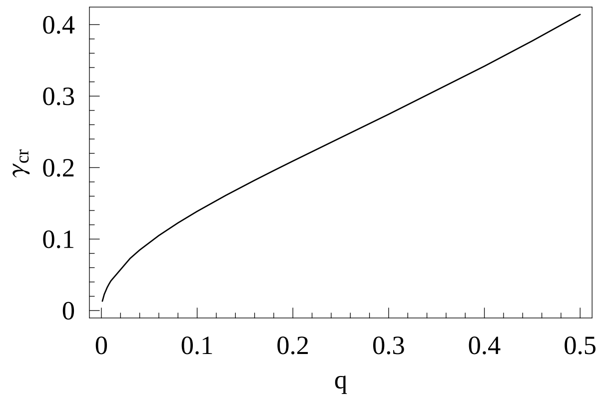

Fig. 3. Dependence of the critical damping parameter γ_{cr} on the noise parameter q at $\omega = 1$.

limit $\gamma_{cr}(q)$ for the damping parameter at greater values of which the instability cannot occur. The boundary of the region of the instability phase and the critical parameter $\gamma_{cr}(q)$ are given by the fourth-order polynomial equation

$$\begin{aligned}&[\nu(\nu+2\gamma)+4\omega^2][\nu\omega^2-\gamma(\nu+\gamma)(\nu+2\gamma)]\\&\quad=(1-2q)\omega^2[4\omega^2\nu+(\nu+2\gamma)^3].\end{aligned} \tag{29}$$

In the case of dichotomous noise, $q = 1/2$, Eq. (29) reduces to a second-order equation and the boundary of the instability $\nu_{\pm}(\gamma)$ reads

$$\nu_{\pm}(\gamma) = \frac{1}{2\gamma}\left(\omega^2 - 3\gamma^2 \pm \sqrt{\omega^4+\gamma^4-6\omega^2\gamma^2}\right). \tag{30}$$

Thus, in this case $\gamma_{cr}(\frac{1}{2}) = \omega(\sqrt{2}-1)$. The tendency apparent in Fig. 2, viz. a decrease of γ_{cr} as the kurtosis $\kappa = \frac{1}{2q} - 3$ of the noise Z increases, is a general feature of $\gamma_{cr}(q)$ (see Fig. 3). Thus, energetic instability is possible only if

$$\gamma < \omega(\sqrt{2}-1). \tag{31}$$

Another important critical parameter is $a_{cr\ min}$, because of the conditions (28) for the occurrence of energetic instability. It follows from Eq. (21) that a_{cr}^2 decreases monotonically as the noise parameter q increases or as the damping coefficient decreases. The functional dependence of a_{cr} on the noise switching rate ν is more complicated, exhibiting several extrema. To get more information, we shall study it in the asymptotic limit of low damping. In general, the parameter $a_{cr\ min}$ can be found by numerical calculations from Eq. (21). In the low-damping limit, we allow γ to become small $\gamma \ll \gamma_{cr}$, and use γ as a perturbation parameter. In this case the critical parameter $a_{cr\ min}$ and the corresponding switching rate ν_m can be given as

$$a_{cr\ min}^2 \approx \frac{2\gamma\omega^3}{q}, \quad \nu_m \approx 2\omega - \frac{(2-q)\gamma}{q}. \tag{32}$$

The interesting feature of the result (32) is that Eq. (32) establishes a quantitative connection between stochastic oscillator instability and the parametric instability of a deterministic oscillator. Note that one of the trademarks of parametric resonance of the deterministic harmonic oscillator with a periodic perturbed frequency ω is that the most pronounced instability is induced by a superharmonic perturbation with the frequency $\omega_p \approx 2\omega$, [32]. Thus, the minimal value of the noise amplitude at which the instability of the oscillator develops corresponds to the noise switching rate, which coincides with the leading frequency of the parametric resonance for a deterministic oscillator.

IV. Response to noise kurtosis

The qualitative behavior of the SR characteristics A^2 and σ^2 versus the noise kurtosis κ is sensitive to values of other system parameters. In the case exposed in Fig. 4 the variance exhibits a single-peak form of SR at small and moderate values of the noise switching rate ν. As ν increases the SR phenomenon disappears and in this case the variance is rather an increasing function of $q = 1/[2(\kappa+3)]$. It is remarkable that in the transition regime ($\nu \approx 0.25$ in Fig. 4) the variance is nearly constant over a finite range of κ values. In contrast to the variance, SPA (A^2) is a monotonically decreasing function of q, i.e., in this parameter regime the SR phenomenon for SPA is absent (see Fig. 5).

The phenomenon of noise-kurtosis-induced SR is not restricted to the simple-peak form of SR. Figure 6(a) depicts a more complicated behavior of the variance as a function of noise kurtosis for different values of the noise amplitude. In the parameter regime considered by the curve (1) in Fig. 6(a), for increasing values of q, the variance starts from zero, increasing to a local maximum, next it decreases, attaining a local minimum, and then σ^2 tends to infinity as q tends to the value $q_{cr} \approx 0.325$. Such a combined SR phenomenon, i.e., first an enhacement, next a suppression and finally a rapid increase of the output variance, is significantly associated with the critical characteristics of stochastic parametric resonance. Namely, the critical value q_{cr} of the noise parameter q at which the variance tends to infinity corresponds to the appearance of noise-induced energetic instability [cf. Eq. (21)]. Hence, the key factor for the appearance of SR with two local extrema in σ^2 versus κ is the occurrence of energetic instability at some values of the noise kurtosis κ. As a rule, in the parameter regimes considered in Fig. 6 the SR phenomenon for SPA is absent [see Fig. 6(b)].

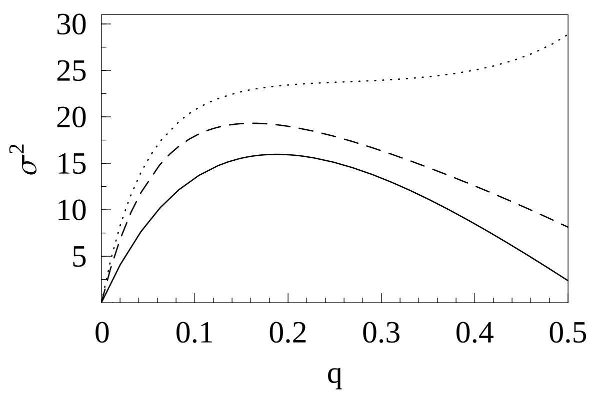

Fig. 4. Variance of the output signal (σ^2) versus the noise parameter q for different values of the noise switching rate ν [Eq. (25)]. The system parameter values: $A_0 = \omega = \Omega = 1$, $D = 0, a = 0.8, \gamma = 0.1$. Solid line, $\nu = 0.05$; dashed line, $\nu = 0.15$; dotted line, $\nu = 0.25$

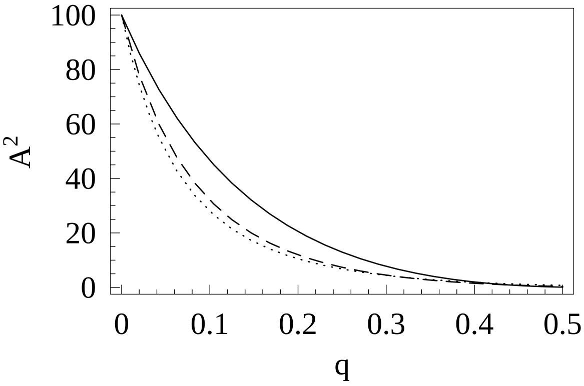

Fig. 5. Dependence of the SPA (A^2) computed from Eqs. (15) and (17) for $A_0 = \omega = \Omega = 1$, $a = 0.8$, $\gamma = 0.1$, on the noise parameter q at several values of the noise switching rate ν. Solid line, $\nu = 0.05$; dashed line, $\nu = 0.15$; dotted line, $\nu = 0.25$.

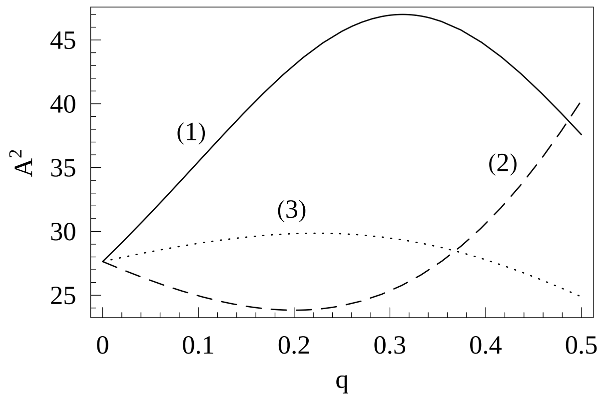

Fig. 7. Dependence of the SPA (A^2) computed from Eqs. (15) and (17) for $A_0 = \omega = 1$, $\Omega = 0.9$, $\gamma = 0.01$, on the noise parameter q. Solid line, $a = 0.9$, $\nu = 1.6$; dashed line, $a = 0.25$, $\nu = 0.058$; dotted line, $a = 0.3$, $\nu = 0.2$.

As mentioned above, there are certain ranges of system parameters for which the behavior of SR characteristics can be qualitatively different. A plot (Fig. 7) of SPA versus the noise parameter q for different parameters shows a typical resonance with nonmonotonic behavior of the function $A^2(q)$. From Fig. 7 one can discern two cases. First, if the noise switching rate ν is relatively small, then the SR phenomenon for SPA exhibits in the form of suppression of A^2 at some values of q [cf. curve (2) in Fig. 7]. Actually, in the case of very small values of the damping parameter γ and the switching rate ν the effect of suppression can be very strong, i.e., at the local minimum of the function $A^2(q)$ SPA tends to zero. Second, in the case of moderate values of the switching rate ν a local enhancement of SPA versus q occurs [curve (1) in Fig. 7]. It is remarkable that the peak of $A^2(q)$ quite strongly depends on ν as both its magnitude and its position change. For example, if ν increases, the position of the peak shifts towards greater values of the noise parameter q.

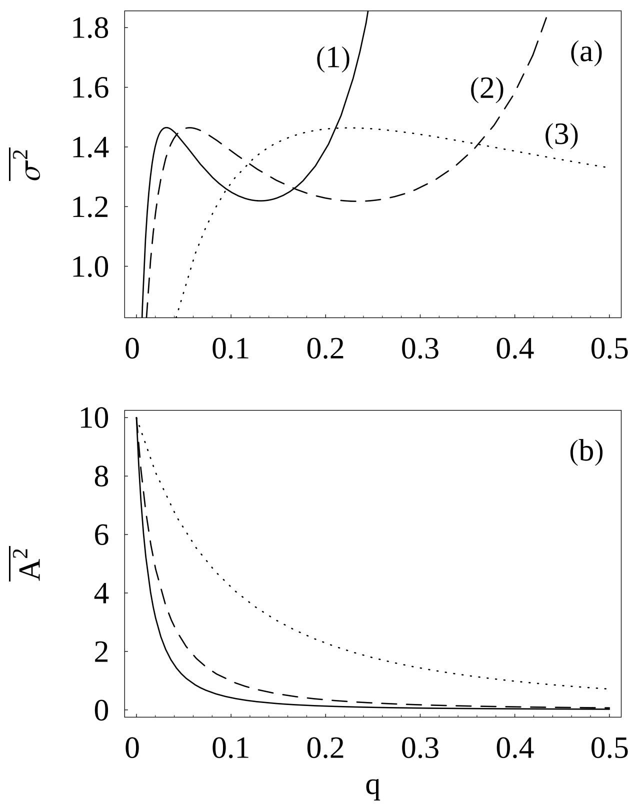

Fig. 6. SR characteristics [(a) variance (σ^2) and (b) SPA (A^2)] as functions of the noise parameter q for different values of the noise amplitude a [Eqs. (22), (25), (15), and (17)]. The system parameter values: $A_0 = \omega = 1$, $D = 0$, $\Omega = 1$, $\gamma = 10^{-4}$, $\nu = 0.4$. Solid line, $a = 0.04$; dashed line, $a = 0.03$; dotted line, $a = 0.015$. Note that energetic instability occurs for the curve (1) in the panel (a). The critical value of q at which the energetic instability appears is $q_{cr} \approx 0.325$. $\bar{A^2} \equiv 10^{-7} A^2$ and $\bar{\sigma^2} \equiv 10^{-7}\sigma^2$.

Let us note that the SR phenomenon versus kurtosis also appears in the case of adiabatic noise. At the long-correlation-time limit, $\nu \to 0$, the SPA and variance saturate at the values:

$$
\begin{aligned}
A^2 = A_0^2 \Big\{ & [(\omega^2-\Omega^2)^2+\Omega^2\gamma^2-(1-2q)a^2]^2 \\
& +4a^2(1-2q)\Omega^2\gamma^2 \Big\} \\
\times \Big\{ & [(\omega^2-\Omega^2)^2+\Omega^2\gamma^2][(\omega^2-\Omega^2-a)^2 \\
& +\Omega^2\gamma^2][(\omega^2-\Omega^2+a)^2+\Omega^2\gamma^2] \Big\}^{-1},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\sigma^2 \;=\; & A_0^2 q a^2 [(\omega^2-\Omega^2)^2+\Omega^2\gamma^2+a^2 \\
& \times(1-2q)] \Big\{ [(\omega^2-\Omega^2)^2+\Omega^2\gamma^2] \\
& \times[(\omega^2-\Omega^2-a)^2+\Omega^2\gamma^2][(\omega^2-\Omega^2 \\
& +a)^2+\Omega^2\gamma^2] \Big\}^{-1},
\end{aligned} \tag{34}
$$

respectively. From (33) it follows that by the conditions

$$
\begin{gathered}
a^2 > (\omega^2-\Omega^2)^2-\Omega^2\gamma^2 > 0, \\
\gamma^2 > \Omega^2 - 2\omega^2
\end{gathered} \tag{35}
$$

the SPA reaches the minimum

$$\begin{aligned} A^2_{min} &= 4A_0^2\Omega^2\gamma^2(\omega^2-\Omega^2)^2\Big\{[(\omega^2-\Omega^2)^2 \\ &\quad +\Omega^2\gamma^2][(\omega^2-\Omega^2-a)^2+\Omega^2\gamma^2] \\ &\quad \times[(\omega^2-\Omega^2+a)^2+\Omega^2\gamma^2]\Big\}^{-1} \end{aligned} \quad (36)$$

at

$$q = q_m \equiv \frac{1}{2a^2}[a^2+\Omega^2\gamma^2-(\Omega^2-\omega^2)^2]. \quad (37)$$

Note that the inequalities (35) are the necessary and sufficient conditions for the SR phenomenon of SPA in the adiabatic limit. Evidently, if the damping parameter γ is low, the suppression of SPA at $q = q_m$ is very pronounced, i.e., A^2_{min} tends to zero as γ vanishes [see Eq. (36)]. The necessary and sufficient conditions for the existence of a resonant-like amplification of the output variance σ^2 read as

$$\begin{aligned} &\omega^4 > a^2 > (\omega^2-\Omega^2)^2+\Omega^2\gamma^2, \\ &0<\gamma^2<2\omega^2-\Omega^2. \end{aligned} \quad (38)$$

Relying on Eq. (34) we obtain that the maximum of the output variance exhibits at

$$q = q_{max} \equiv \frac{1}{4a^2}[a^2+\Omega^2\gamma^2+(\Omega^2-\omega^2)^2]. \quad (39)$$

It is seen from Eq. (39) that in the adiabatic regime the SR phenomenon for the variance σ^2 is possible only if the values of the noise kurtosis κ are in the interval (-2,-1).

V. Hypersensitive response

Next we consider the dependence of SR characteristics on the noise amplitude a. In Fig. 8 we depict, on two panels, the behavior of A^2 and σ^2, for various values of the temperature D. In the case considered, the critical noise amplitude ($a_{cr} \approx 0.9619$) is very close to the maximal value of the noise amplitude, $a = 1$. Both SR characteristics exhibit a nonmonotonic dependence on the noise amplitude, i.e., a typical SR phenomenon continues to increase a. Clearly, additive thermal noise does not affect the spectral amplification A^2, but the variance σ^2 increases rapidly as the temperature D increases. For the parameter regime $a \ll a_{cr}$, the main contribution of the temperature appears as an additive term D/γ in σ^2 [cf. Eqs. (22)-(25)].

As shown in Fig. 8 (a) the curve A^2 vs a first exhibits a maximum and then a minimum appears, that is to say, the SR exhibited first is followed by a suppression. Most important, we observe that the resonance of SPA occurs for the noise amplitude $a \approx |\Omega^2-\omega^2|$, thus the resonance of SPA corresponds to the resonance frequency of the deterministic system for the fixed colored noise state $z_3 = -a$. A resonance-like peak of the variance σ^2 at $a \approx |\Omega^2-\omega^2|$ is also observed [Fig. 8(b)]. This behavior of the variance, i.e., a strong amplification of the variance at the resonance peak of SPA, is quite robust and occurs within a broad range of system parameters. With increasing the noise amplitude, one observes another region at the critical noise amplitude ($a \approx a_{cr}$), where the enhancement of the variance vs a is extremely rapid.

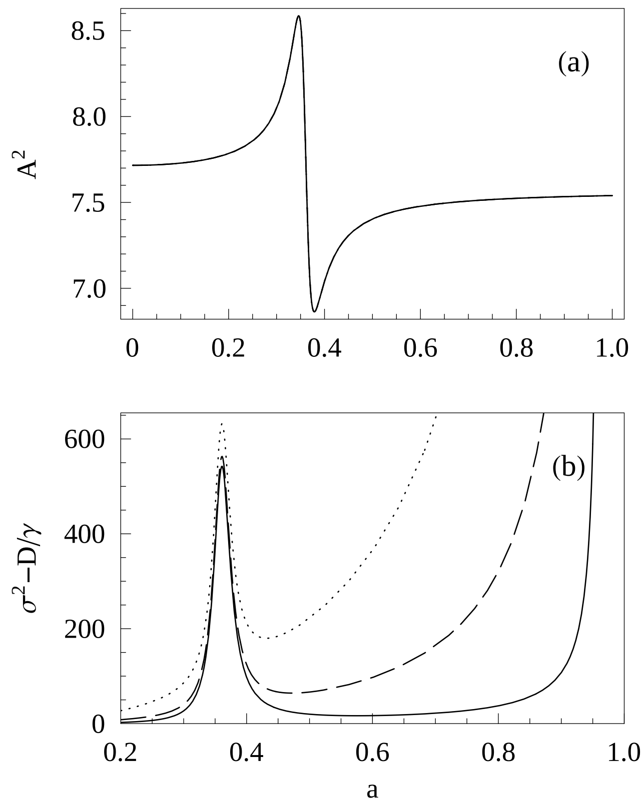

Fig. 8. SR characteristics [(a) SPA (A^2) and (b) variance (σ^2)] versus the noise amplitude a at several temperatures: $D_1 = 0.05$ (solid line), $D_2 = 0.5$ (dashed line), $D_3 = 2$ (dotted line) [Eqs. (15) and (25)]. Parameter values: $\gamma = 3*10^{-4}$, $\nu = 0.01$, $q = 0.005$, $\Omega = 0.8$, $\omega = A_0 = 1$. Note that the critical noise amplitude $a_{cr} \approx 0.9619$.

Particularly, σ^2 increases unrestrictedly as energetic instability appears.

An interesting peculiarity of Fig. 8 (a) is the rapid decrease of SPA from the maximum to the minimum as a increases. It is noteworthy that in the case of dichotomous noise such an effect is absent. The effect is very pronounced at low values of the switching rate ν and a low damping γ (see Fig. 9). To throw some light on the physics of the above-mentioned new effect, we shall now briefly consider the behavior of the SR characteristics A^2 and σ^2 in the parameter regime

$$\nu^2 \ll \gamma^2 \ll q|\omega^2-\Omega^2| \ll \omega^2, \quad q \ll 1. \quad (40)$$

In this case, it follows from Eqs. (17) and (15) that SPA reaches the maximum

$$A^2_{max} \approx A_0^2 q^2/(\Omega^2\gamma^2) \quad (41)$$

at

$$a = a_{max} \approx |\Omega^2-\omega^2|, \quad (42)$$

and the minimum

$$A^2_{min} \approx A_0^2\Omega^2\gamma^2/[q^2(\Omega^2-\omega^2)^4] \quad (43)$$

at

$$a = a_{min} \approx |\Omega^2-\omega^2|/\sqrt{1-2q}. \quad (44)$$

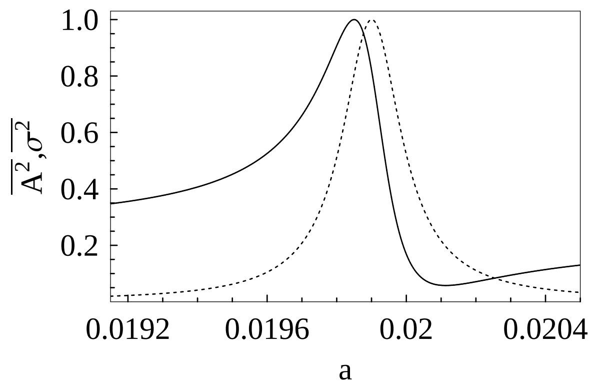

Fig. 9. A plot of the dependence of SPA (A^2) and variance (σ^2) on the noise amplitude a in a region of hypersensitive response [Eqs. (15) and (25)]. System parameter values: $\gamma = 5{*}10^{-6}$, $\nu = 5{*}10^{-5}$, $q = 0.008$, $\Omega = 0.99$, $D = 0.1$, and $A_0 = \omega = 1$. The values of A^2 and σ^2 at the local maximum are $A_m^2 = 10421$, $\sigma_m^2 = 7.78{*}10^6$. The solid line and dotted line correspond to $\bar{A}^2 \equiv A^2/A_m^2$ and $\bar{\sigma}^2 \equiv \sigma^2/\sigma_m^2$, respectively.

For sufficiently strong inequalities (40), A^2_{min} tends to zero and A^2_{max} grows up to very large values. Thus in the case considered SPA is extremely sensitive to small variation of a: $\Delta a = a_{min} - a_{max} \approx q|\Omega^2 - \omega^2|$. Note that this small interval of the noise amplitude,

$$a \in (|\Omega^2 - \omega^2|, (1+q)|\Omega^2 - \omega^2|), \tag{45}$$

also contains a very narrow and high SR peak of the variance with the maximal value

$$\sigma^2_{max} \approx A_0^2 q/(2\Omega^2\gamma^2). \tag{46}$$

The above formulas for A^2_{max} and σ^2_{max} indicate that the main mechanism for the formation of SR in the SPA and σ^2 is the conventional amplitude-resonance generated by an external periodic forcing with the frequency $\Omega = \sqrt{\omega^2 \pm a}$. More precisely, consider an ensemble of realizations of the stochastic oscillator for each of which a particular sequence of switching times, between the states of the nonequilibrium noise $Z(t)$, is chosen from the distribution of switching times. For a given time moment t the relative amount of realizations with the noise state $z_3 = -a$ is q. As the switching rate ν and the damping coefficient γ are small (the noise correlation time is long) there is, between two switchings of the noise $Z(t)$, enough time for a very strong amplification of the amplitude of $X(t)$, which happens in the noise state $z_3 = -a$ due to the conventional resonance at $\Omega = \sqrt{\omega^2 - a}$. Particularly, in the state $z_3 = -a$ all these realizations are strongly synchronized because of the phase lag $\varphi = -\pi/2$ between the periodic driving force and the periodic response of the system by resonance. Therefore, although the fraction of such realizations is low ($q \ll 1$), the contribution of these realizations still dominates by the formation of SR in the SPA because of high amplitudes and synchronization. As the noise amplitude increases, the drastic decrease of SPA and also the appearance of the resonant amplification of the variance (see Fig. 9) indicate a rapid desynchronization of the realizations of the stochastic oscillator. Note that the above scenario accords with calculations of the phase lag φ for the mean displacement of the oscillator [Eq. (16)]. For example, in the case of the system parameters used in Fig. 9 the result is as follows: as a is gradually increased from zero and swept through the resonant amplitude $a \approx 0.02$, φ first decreases very slowly and later quickly from zero, passes through $-\pi/2$ when $a \approx 0.02$, and quickly approaches zero when $a > 0.02$.

VI. Conclusions

In the present work, we have analysed the phenomenon of stochastic parametric resonance within the context of a noisy, harmonic oscillator with a fluctuating frequency driven by sinusoidal forcing and by an additive thermal noise. The frequency fluctuations are modeled as a colored three-level Markovian noise. The Shapiro-Loginov formula [37] allows us to find a closed system of equations for the first-order and second-order cumulants and the exact expressions for the long-time behavior of several SR characteristics, such as SPA, and variance.

As the main result we have established the phenomenon of noise-kurtosis-controlled stochastic resonance. The phenomenon is more pronounced for moderate values of the noise correlation time. Notably, in the fast-noise limit the effect is absent. Depending on the values of the noise parameters the SR versus noise kurtosis appears as an enhancement or as a suppression of the output SR characteristics. For example, the enhancement of SPA occurs, as a rule, at smaller values of the noise correlation time as a suppression of SPA. To our knowledge, neither the resonant-like enhancement nor the suppresion of output SR characteristics versus noise kurtosis have been noticed or discussed before.

It is interesting that the results of the present paper can be iterpreted in terms of cross-correlation intensity between two dichotomous noises. Namely, the trichotomous noise $Z(t)$ in Eq. (1) can be presented as the sum of two cross-correlated zero-mean symmetric dichotomous noises $Z_1(t)$ and $Z_2(t)$, i.e., $Z(t) = Z_1(t) + Z_2(t)$. The dichotomous noises $Z_1(t)$ and $Z_2(t)$ are characterized as follows: $z_1, z_2 \in \{(1/2)a, -(1/2)a\}$ with $\nu_1 = \nu_2 = \nu$ and the correlation function

$$\langle Z_i(t) Z_j(t')\rangle = \rho_{ij}\, \frac{a^2}{4}\, e^{-\nu|t-t'|}, \;\; i,j = 1,2, \tag{47}$$

where $\rho_{i,i} = 1$, and $\rho_{i,j} = \rho \in (-1,1)$ with $i \neq j$ is the cross-correlation intensity of the noises Z_1 and Z_2. In this case the probability $q = (1+\rho)/4$, where it follows that the correlation coefficient ρ and the kurtosis κ of the trichotomous noise $Z(t)$ must be related as $\kappa = -(1+3\rho)/(1+\rho)$. It is obvious that the noise kurtosis $\kappa = -1$ corresponds to $\rho = 0$, i.e., to the case of two statistically independent dichotomous noises. Let us note that such a cross-correlation between dichotomous noises may result from the following two reasons: the two noises are either partly of the same origin or are influenced by the same factors. Notably, some cross-correlation-induced effects have earlier been considered in the context of ratchet models [38], [39], where it has also been suggested that cross-correlation between colored noises may provide some understanding as to why structurally very similar motor proteins with two heads, such as kinesin and dynein motor families, move in opposite directions on the micro-tubules despite sharing the same environment and experiencing the same periodicity, like in the case of the conventional kinesin and ncd [40].

Another result is that for a harmonic oscillator colored fluctuations of the frequency can cause correlation-time-induced transitions from energetic stability to instability as well as in the opposite direction. Furthermore, the transition is found to be reentrant, e.g., if the damping coefficient is lower than a certain threshold value, then the energetic instability appears above a critical value of the noise correlation time, but disappears again through a reentrant transition to the energetically stable state at a higher value of the noise correlation time.

As the third main result we have established the effect of a very sensitive response of SR characteristics to small variations of noise amplitude (see Fig. 9), where, e.g., SPA displays a quick jump from a very high value to a low one as the noise amplitude increases but a little. It is remarkable that in the case of dichotomous noise $(q = 1/2)$ such an effect is absent. This feature of the stochastic oscillator suggests that investigation of output SR characteristics versus noise amplitude can reveal important information about input signal in oscillator-devices, even in the case of a small input signal-to-noise ratio, $A_0^2/D \ll 1$ [see Eqs. (1) and (2)]. This conjecture presents an objective that is worthwhile to be addressed in greater detail in some future.

We believe that the results obtained are of interest also in population biology, where the proposed model can be applied for investing the influence of a fluctuating environment on the oscillatory dynamics of predator–prey communities [17], [41], [42].

Acknowledgements: The research was partly supported by the Estonian Science Foundation Grant No. 7319, the International Atomic Energy Agency Grant No. 14797 and by the Estonian Ministry of Education and Research under project SF0132723s06.

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