A multi-item production lot size inventory model with cycle dependent parameters

Zaid T. Balkhi, Abdelaziz Foul

Abstract-In this paper, a multi-item production inventory model is considered within a given time horizon that consists of different time periods. For each product, production, demand, and deterioration rates in each period are known. Shortage for each product is allowed but it is completely backlogged . The objective is to find the optimal production and restarting times for each product in each period so that the overall total inventory cost for all products is minimized. In this paper, a formulation of the problem is developed and optimization techniques are performed to show uniqueness and global optimality of the solution.

Keywords - Multi-item production, Inventory, Varying demand, Deterioration, Optimality.

I. INTRODUCTION

Inventory is known as materials, commodities**,** products,..etc, which are usually carried out in stocks in order to be consumed or benefited from when needed. In fact, most of economic, trading, manufacturing, administrative,…etc, systems regardless of its size, needs to deal with its own Inventory Control System . Keeping inventory in stores has its own various costs which may, sometimes, be more than the value of the commodity being carried out in stores. As examples, nuclear and biological weapons, blood in blood banks, and some kinds of sensitive medications. However, any inventory system must answer the following two main questions. (i) How much to order or to produce for each inventory cycle?. (ii) When to order or to produce a new quantity?. Answering these two questions for certain inventory system leads to the so called " Optimal Inventory Policies" which " Minimize the Total Inventory Costs" of this system. It is expected that all systems, in which controlling and managing inventory is an important factor that has great effects on its performance, can greatly benefit from this research results so as to minimize their relevant inventory cost operations. In fact, many classical inventory models concern with single item. Among these are Resh, Friedman, and Barbosa [19] who considered a classical lot size inventory model with linearly increasing demand. Hong, Standroporty, and Hayya [15] considered an inventory model in which the production rate is uniform and finite where he introduced three production policies for linearly increasing demand. A new inventory model in which products deteriorate at a constant rate and in which demand , production rates are allowed to vary with time has been introduced by Balkhi and Benkherouf [2]. In this model, an optimal production policy that minimizes the total relevant cost is established.Subsequently, Balkhi [1], [3], [4], [5], [6], [8], [9], and Balkhi, Goyal, and Giri [7] have introduced several inventory models in each of which, the demand, production, and deterioration rates are arbitrary functions of times, and in some of which, shortages are allowed but are completely backlogged. In each of the last mentioned seven papers, closed forms of the total inventory cost was established, a solution procedure was introduced and the conditions that guarantee the optimality of the solution for the considered inventory system were introduced. Though so many papers have dealt with single item optimal inventory policy and though the literature concerned with multi-item are sparse, the analysis of multi-item optimal inventory policies, is, almost, parallel to that of single item.The multi-item inventory classical inventory models under resource constraints are available in the well known books of Hadley and Whitin [14] and in Nador [18] . Ben-Daya and Raouf [11] have developed an

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approach for more realistic and general single period for multi-item with budgetary and floor ,or shelf space constraints , where the demand of items follows a uniform probability distribution subject to the restrictions on available space and budget. Bhattacharya [12] has studied two-item inventory model for deteriorating items . Lenard and Roy [17] have used different approaches for the determination of optimal inventory policies based on the notion of efficient policy and extended this notion to multi-item inventory control by defining the concept of family and aggregate items. Kar, Bhunia, and Maiti [16] obtained some interesting results about multi deteriorating items with constraint space and investment. Rosenblatt [20] has discussed multi-item inventory system with budgetary constraint comparison between the Lagrangan and the fixed cycle approach, whereas, Rosenblatt and Rothblum [21] have studied a single resource capacity where this capacity was treated as a decision variable . Recently, Balkhi and Foul [10] have applied a multi-item production inventory model to the Saudi Basic Industries Corporation (SABIC), which is one of the world's leading manufacturers of fertilizers, plastics, chemicals ,and metals ,in Saudi Arabia.For more details about multi-item inventory system, the readers are advised to consult the survey of Yasemin and Erenguc [23] and the references therein.

Our main concern in this study is to find the optimal production and restarting times for each product in each period so that the overall total inventory cost for all products is minimized. In this paper, a formulation of the problem is developed and optimization techniques are performed to show uniqueness and global optimality of the solution. Optimal number of units to be produced from each of the products are determined by a simple linear program. Having found these optimal numbers of units, we establish optimal inventory policies for the different products, which means that we determine the optimal stopping and restarting production times for each produced item so that the total relevant inventory cost of all items is minimum. The paper is organized as follows. First, we introduce our assumptions and notations, then we build the mathematical model of the underlying problem. The solution procedure of the developed model is established in section 4 , and the optimality of the obtained solution is proved in section 5. Finally we introduce a conclusion in which we summarize the main results of the paper as well as our proposals for further research .

II. ASSUMPTIONS AND NOTATIONS

Our assumptions and notations for our model are as follows:

1. m different items are produced and held in stock over a known and finite planning horizon of H units long which is divide into n different cycles.
2. The items are subject to deterioration when they are effectively in stock and there is no repair or replacement of deteriorated items.
3. The demand, production and deterioration rates of item i in cycle j are item and cycle dependent , and are denoted by D_{ij}, P_{ij} and θ_{ij} respectively.
4. Shortages are allowed for all items, but are completely backlogged.
5. The following notations are used in the sequel , where $i = 1,2,\dots m$ and $j = 1,2,\dots n$

$I_{ij}(t)$ Inventory level of item i in cycle j at time t.

t_{ij} Time at which the inventory level of item i in cycle j reaches its maximum.

T_{ij} Time at which the inventory level of item i in cycle j starts to fall below zero and shortage starts to accumulate.

S_{ij} Beginning of cycle j for item i, with S_{io}=0 and S_{in}=H.

s_{ij} Time at which the shortage for item i reaches its maximum in cycle j.

h_{ij} Inventory holding cost in cycle j per unit of item i per unit of time .

b_{ij} Shortage cost in cycle j per unit of item i per unit of time.

k_{ij} Set up cost for item i in cycle j.

c_{ij} Unit production cost of item i in cycle j.

III. MODEL FORMULATION

For each item i ($i = 1,2,\dots m$) and each cycle time j ($j = 1,2,\dots n$), the system operates as follows. The production starts at time S_{ij-1} to build up the inventory level at a rate $P_{ij} - D_{ij} - \theta_{ij} I_{ij}(t)$ until time t_{ij} where the production stops. Then the stock level depletes at a rate $D_{ij} - \theta_{ij} I_{ij}(t)$ until it reaches zero at time T_{ij} where shortages start to accumulate with rate $-D_{ij}$ up to time s_{ij}, after which the production is restarted with rate $P_{ij} - D_{ij}$ until time S_{ij} to fulfill both the shortage and the demand. A typical behavior of the system is shown in Fig. 1.

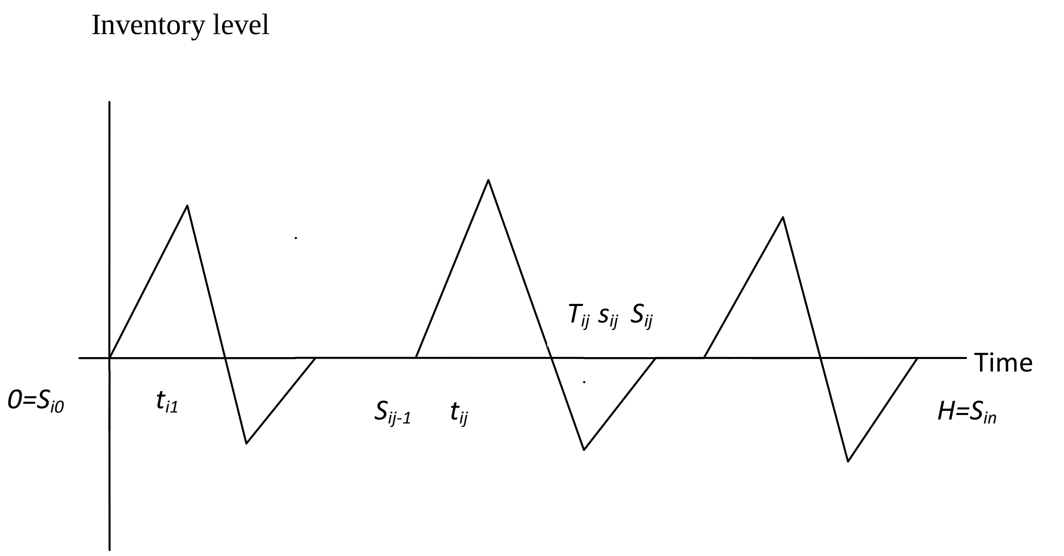

Fig1. The variation of the inventory system in the given period

Before writing the mathematical formulation of the problem, we note that the jth cycle can be divided into four intervals. That is, $S_{ij-1} \le t < t_{ij}$, $t_{ij} \le t < T_{ij}$, $T_{ij} \le t < s_{ij}$, $s_{ij} \le t < S_{ij}$. For item i and cycle j, the inventory level $I_{ij}(t)$ is governed by the following differential equations :

$$d\, I_{ij}(t)/dt = P_{ij} - D_{ij} - \theta_{ij}\, I_{ij}(t) \qquad S_{ij-1} \le t < t_{ij}, \qquad (1)$$

with initial condition $I_{ij}(S_{ij-1}) = 0$,

$$d\, I_{ij}(t)/dt = - D_{ij} - \theta_{ij}\, I_{ij}(t) \qquad t_{ij} \le t < T_{ij} \qquad (2)$$

with ending condition $I_{ij}(T_{ij}) = 0$,

$$d\, I_{ij}(t)/dt = - D_{ij} \qquad T_{ij} \le t < s_{ij} \qquad (3)$$

with initial condition $I_{ij}(T_{ij}) = 0$, and

$$d\, I_{ij}(t)/dt = P_{ij} - D_{ij} - \theta_{ij}\, I_{ij}(t) \qquad s_{ij} \le t < S_{ij} \qquad (4)$$

with ending condition $I_{ij}(S_{ij}) = 0$,

The solution to (1), (2), (3), and (4) is given by:

$$I_{ij}(t) = e^{-\theta_{ij} t} \int_{S_{ij-1}}^{t} (P_{ij} - D_{ij})\, e^{\theta_{ij} u} du \; ;$$

$$S_{ij-1} \le t < t_{ij}$$

$$I_{ij}(t) = e^{-\theta_{ij} t} \int_{t}^{T_{ij}} D_{ij}\, e^{\theta_{ij} u} du \;\; ; \; t_{ij} \le t < T_{ij}$$

$$I_{ij}(t) = -\int_{T_{ij}}^{t} D_{ij}\, du \quad ; \qquad T_{ij} \le t < s_{ij}$$

$$I_{ij}(t) = -\int_{t}^{S_{ij}} (P_{ij} - D_{ij})\, du \; ; \qquad s_{ij} \le t < S_{ij}$$

respectively .Integrating the right hand side of the last four equations, we obtain :

$$I_{ij}(t) = \frac{(P_{ij} - D_{ij})}{\theta_{ij}} \left(1 - e^{\theta_{ij}(S_{ij-1} - t)}\right) \; ;$$

$$S_{ij-1} \le t < t_{ij} \qquad (5)$$

$$I_{ij}(t) = \frac{-D_{ij}}{\theta_{ij}} \left(1 - e^{\theta_{ij}(T_{ij} - t)}\right) \quad ;$$

$$t_{ij} \le t < T_{ij} \qquad (6)$$

$$I_{ij}(t) = -D_{ij}(t - T_{ij}) \; ; T_{ij} \le t < s_{ij} \qquad (7)$$

$$I_{ij}(t) = -(P_{ij} - D_{ij})(S_{ij} - t);$$

$$s_{ij} \le t < S_{ij} \qquad (8)$$

Now, the amount of item i being held in stock in period $[S_{ij-1} , t_{ij}]$ is given by

$$I_{ij}(S_{ij-1} , t_{ij}) = \int_{S_{ij-1}}^{t_{ij}} I_{ij}(t) dt = \frac{(P_{ij} - D_{ij})}{\theta_{ij}} \times$$

$$\left(t_{ij} + \frac{1}{\theta_{ij}} e^{\theta_{ij}(S_{ij-1} - t_{ij})} - S_{ij-1} - \frac{1}{\theta_{ij}}\right)$$

Hence, the holding cost is $h_{ij} I_{ij}(S_{ij-1} , t_{ij})$.

Similarly the amount of item i being held in stock in period $[t_{ij} , T_{ij}]$ is given by

$I_{ij}(t_{ij}, T_{ij}) = \int_{t_{ij}}^{T_{ij}} I_{ij}(t)dt = \frac{-D_{ij}}{\theta_{ij}} \times$

$(-t_{ij} - \frac{1}{\theta_{ij}} e^{\theta_{ij}(T_{ij} - t_{ij})} + T_{ij} + \frac{1}{\theta_{ij}})$

And hence the holding cost is $h_{ij} I_{ij}(t_{ij}, T_{ij})$.

The amount of shortage of item *i* in period [T_{ij}, s_{ij}] is given by

$I_{ij}(T_{ij}, s_{ij}) = \int_{T_{ij}}^{s_{ij}} I_{ij}(t)dt$

$= D_{ij}\left(\frac{1}{2}(s_{ij}^2 + T_{ij}^2) - s_{ij}T_{ij}\right)$

Hence the shortage cost in period $[T_{ij}, s_{ij}]$ is

$b_{ij} I_{ij}(T_{ij}, s_{ij})$. Similarly , the shortage cost in period $[s_{ij}, S_{ij}]$ is $b_{ij} I_{ij}(s_{ij}, S_{ij})$, where

$I_{ij}(s_{ij}, S_{ij}) = \int_{s_{ij}}^{S_{ij}} I_{ij}(t)dt$

$= (P_{ij} - D_{ij})(\frac{1}{2}(s_{ij}^2 + S_{ij}^2) - s_{ij}S_{ij})$

Finally, the number of units produced from item *i* in cycle *j* is equal to

$\int_{S_{ij-1}}^{t_{ij}} P_{ij}(t)dt + \int_{s_{ij}}^{S_{ij}} P_{ij}(t)dt$.

Hence, the production cost of item *i* in cycle *j*

is given by $c_{ij} P_{ij}((t_{ij} - S_{ij-1}) + (S_{ij} - s_{ij}))$

Thus the total net inventory cost of item *i* in cycle *j* , say W_{ij} is given by

$W_{ij} = k_{ij} + h_{ij}[\frac{(P_{ij} - D_{ij})}{\theta_{ij}}(t_{ij} + \frac{1}{\theta_{ij}} e^{\theta_{ij}(S_{ij-1} - t_{ij})} -$

$S_{ij-1} - \frac{1}{\theta_{ij}}) - \frac{D_{ij}}{\theta_{ij}}(-t_{ij} - \frac{1}{\theta_{ij}} e^{\theta_{ij}(T_{ij} - t_{ij})} +$

$T_{ij} + \frac{1}{\theta_{ij}})] + b_{ij}[D_{ij}\left(\frac{1}{2}(s_{ij}^2 + T_{ij}^2) - s_{ij}T_{ij}\right)$

$+ (P_{ij} - D_{ij})(\frac{1}{2}(s_{ij}^2 + S_{ij}^2) - s_{ij}S_{ij})] +$

$c_{ij} P_{ij}((t_{ij} - S_{ij-1}) + (S_{ij} - s_{ij})) \qquad (9)$

The total net inventory cost for all items in the given time horizon of *n* cycles is given by

$$W = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} \qquad (10)$$

Thus, our problem is to minimize W as a function of S_{ij-1} , t_{ij}, T_{ij}, s_{ij}, and S_{ij} subject to the following constraints :

$$S_{ij-1} < t_{ij} < T_{ij} < s_{ij} < S_{ij} \qquad (11)$$

$\frac{(P_{ij} - D_{ij})}{\theta_{ij}}(1 - e^{\theta_{ij}(S_{ij-1} - t_{ij})}) =$

$\frac{-D_{ij}}{\theta_{ij}}(1 - e^{\theta_{ij}(T_{ij} - t_{ij})})$

(12)

$D_{ij}(s_{ij} - T_{ij}) = (P_{ij} - D_{ij})(S_{ij} - s_{ij}) \ (13)$

$i = 1,2,\ldots m$. and $j = 1,2,\ldots n$

Constraint (11) is a natural constraint which must be satisfied, otherwise the whole problem will be meaningful. Constraint (12) says that the inventory levels given by (5), and (6) at $t = t_{ij}$ are equal. Similarly, constraint (13) says that the inventory levels given by (6), and (7) at $t = s_{ij}$ are equal. Thus, our problem, call it *(Q)*, is :

$$\left.\begin{array}{l} \text{Minimize } W = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} \\ \textit{Subject to}: \ (11), (12) \textit{ and } (13) \end{array}\right\} (Q)$$

For convenience, let us rewrite (12) and (13) as equality constraints :

$f_{ij} = \frac{(P_{ij} - D_{ij})}{\theta_{ij}}(1 - e^{\theta_{ij}(S_{ij-1} - t_{ij})})$

$+ \frac{D_{ij}}{\theta_{ij}}(1 - e^{\theta_{ij}(T_{ij} - t_{ij})}) = 0 \qquad (14)$

$g_{ij} = -D_{ij}(s_{ij} - T_{ij}) + (P_{ij} - D_{ij}) \times$

$(S_{ij} - s_{ij}) = 0 \qquad (15)$

IV. SOLUTION PROCEDURE

Consider problem *(Q)* where constraint (11) is ignored and suppose that the number of cycles *n* is fixed. Let's call the new problem as *(P)*. Clearly problem *(P)* is a nonlinear program with equality constraints. Therefore the solution procedure used is the Lagrangean technique. The Lagrangean function for problem *(P)* is given by

$$L(\bar{t},\bar{T},\bar{s},\bar{S},\bar{\lambda},\bar{\mu}) = W + \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij} f_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij}\, g_{ij} \qquad (16)$$

Where $\bar{t} = (t_{ij})$, $\bar{T} = (T_{ij})$, $\bar{s} = (s_{ij})$, $\bar{S} = (S_{ij})$, $\bar{\lambda} = (\lambda_{ij})$, $\bar{\mu} = (\mu_{ij})$.

For $i = 1,2,\ldots m$, and $j = 1,2,\ldots n$.

The first order necessary conditions for having a minimum are :

$$\frac{\partial L}{\partial t_{ij}} = 0, \quad \frac{\partial L}{\partial T_{ij}} = 0, \quad \frac{\partial L}{\partial s_{ij}} = 0,$$

$$\frac{\partial L}{\partial \lambda_{ij}} = 0, \quad and \quad \frac{\partial L}{\partial \mu_{ij}} = 0 \qquad (17)$$

Conditions (17) can be written explicitly as :

$$\frac{\partial L}{\partial t_{ij}} = \frac{h_{ij}(P_{ij} - D_{ij})}{\theta_{ij}}(1 - e^{\theta_{ij}(S_{ij-1} - t_{ij})}) + \frac{h_{ij}D_{ij}}{\theta_{ij}}(1 - e^{\theta_{ij}(T_{ij} - t_{ij})}) + c_{ij}P_{ij} + \lambda_{ij}\left((P_{ij} - D_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} + D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})}\right) = 0 \qquad (18)$$

$$\frac{\partial L}{\partial T_{ij}} = \frac{h_{ij}D_{ij}}{\theta_{ij}}(e^{\theta_{ij}(T_{ij} - t_{ij})} - 1) + b_{ij}D_{ij}(T_{ij} - s_{ij}) - \lambda_{ij}D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})}) + \mu_{ij} D_{ij} = 0 \qquad (19a)$$

$$\frac{\partial L}{\partial s_{ij}} = b_{ij}D_{ij}(s_{ij} - T_{ij}) + b_{ij}(P_{ij} - D_{ij})(s_{ij} - S_{ij}) -$$

$$c_{ij}P_{ij} + \mu_{ij}P_{ij} = 0 \qquad (20)$$

$$\frac{\partial L}{\partial \lambda_{ij}} = \frac{(P_{ij} - D_{ij})}{\theta_{ij}}(1 - e^{\theta_{ij}(S_{ij-1} - t_{ij})}) + \frac{D_{ij}}{\theta_{ij}}(1 - e^{\theta_{ij}(T_{ij} - t_{ij})}) = 0 \qquad (21)$$

$$\frac{\partial L}{\partial \mu_{ij}} = -D_{ij}(s_{ij} - T_{ij}) + (P_{ij} - D_{ij})(S_{ij} - s_{ij}) = 0 \qquad (22)$$

for $i = 1,2,\ldots m$, and $j = 1,2,\ldots n\text{-}1$.

$$\frac{\partial L}{\partial S_{ij}} = b_{ij}(P_{ij} - D_{ij})(S_{ij} - s_{ij}) + c_{ij}P_{ij} + \mu_{ij}(P_{ij} - D_{ij}) = 0 \qquad (23a)$$

for $i = 1,2,\ldots m$ and $j = 1,2,\ldots,n\text{-}1$(see model assumptions) . Using (21) and (22), we can express equations (18) and (20) as :

$$\frac{\partial L}{\partial t_{ij}} = c_{ij}P_{ij} + \lambda_{ij} P_{ij} = 0 \quad \text{and}$$

$$\frac{\partial L}{\partial s_{ij}} = c_{ij}P_{ij} + \mu_{ij}P_{ij} = 0$$

From which, we get

$$\lambda_{ij} = \mu_{ij} = -c_{ij} \qquad (24)$$

(24) implies that (19a) and (23a) can be expressed as :

$$\frac{\partial L}{\partial T_{ij}} = b_{ij}D_{ij}(T_{ij} - s_{ij}) + (c_{ij} + \frac{h_{ij}}{\theta_{ij}})D_{ij} \times e^{\theta_{ij}(T_{ij} - t_{ij})} + c_{ij}\, D_{ij} - \frac{h_{ij}D_{ij}}{\theta_{ij}} = 0 \quad (19b)$$

$$\frac{\partial L}{\partial S_{ij}} = b_{ij}(P_{ij} - D_{ij})(S_{ij} - s_{ij})D_{ij} \quad (23b)$$

Let us denote by *(S)*, the nonlinear system consisting of equations (19b), ,(21),(22), (23b), and (24). Next we show that any solution that satisfies system *(S)* satisfies constraint (11).

Theorem 1 . Any solution that satisfies system *(S)* satisfies constraint (11).

Proof : From (14), we have:

$$0 < (P_{ij} - D_{ij})(1 - e^{-\theta_{ij}(S_{ij-1} - t_{ij})}) =$$

$$D_{ij}(e^{\theta_{ij}(T_{ij} - t_{ij})} - 1)$$

which implies that

$$S_{ij-1} < t_{ij} \Leftrightarrow T_{ij} > t_{ij} \quad (a).$$

Now, for j =1, we have
$S_{ij-1} = 0 < t_{i1}$ which implies , from (a),

that $t_{i1} < T_{i1}$, therefore *(a)* is true for j=1. By induction on j, we can show that *(a)* holds for any $j \geq 1$.From (15), we have:

$$D_{ij}(s_{ij} - T_{ij}) = (P_{ij} - D_{ij})(S_{ij} - s_{ij})$$

which implies that

$$s_{ij} > T_{ij} \Leftrightarrow S_{ij} > s_{ij} \quad (b).$$

Substituting (24) into (19a) and recalling *(a)*, we get :

$$b_{ij}(s_{ij} - T_{ij}) = \frac{h_{ij}}{\theta_{ij}}(e^{\theta_{ij}(T_{ij} - t_{ij})} - 1) +$$

$$c_{ij}(e^{\theta_{ij}(T_{ij} - t_{ij})} + 1) > 0 \Rightarrow$$

$$s_{ij} > T_{ij} \qquad (c)$$

But , the relation $t_{i1} > 0$ needs not to be considered since the corresponding multiplier equals zero as an implication of Kuhn-Tucker optimality conditions. Combining *(a)*, *(b)*, and *(c)*, we get: $S_{ij-1} < t_{ij} < T_{ij} < s_{ij} < S_{ij}$. This completes the proof of the theorem.

As a consequence of theorem 1, any solution to system *(S)* is a feasible solution to problem *(P)*.

V. OPTIMALITY OF THE SOLUTION

In this section, we derive conditions that guarantee the existence, uniqueness, and global optimality of solution to problem (Q). For that purpose, we first establish sufficient conditions under which the Hessian matrix of the Lagrangean function $L(\overline{t}, \overline{T}, \overline{s}, \overline{S}, \overline{\lambda}, \overline{\mu})$ is positive definite at any feasible solution of (P). To compute the Hessian matrix of L we consider the following block matrices :

$$L_{\overline{t}} = \left[\frac{\partial^2 L}{\partial \overline{t}^2}\right], \qquad L_{\overline{T}} = \left[\frac{\partial^2 L}{\partial \overline{T}^2}\right],$$

$$L_{\overline{S}} = \left[\frac{\partial^2 L}{\partial \overline{S}^2}\right], L_{\overline{s}} = \left[\frac{\partial^2 L}{\partial \overline{s}^2}\right], \quad L_{XY} = \left[\frac{\partial^2 L}{\partial X_{ij} \partial Y_{ij}}\right]$$

After some calculations, we can easily show that the Hessian matrix has the following form :

$$\begin{bmatrix}
L_{t_{i1}{}^2} & L_{t_{i1}T_{i1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
L_{t_{i1}T_{i1}} & L_{T_1^2} & L_{T_{i1}s_{i1}} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & L_{T_{i1}s_{i1}} & L_{s_{i1}^2} & L_{s_{i1}S_{i1}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & L_{s_{i1}S_{i1}} & L_{S_{i1}^2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & L_{S_{ij-1}t_{ij}} & L_{t_{ij}^2} & L_{T_{ij}t_{ij}} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & L_{t_{ij}T_{ij}} & L_{T_{ij}^2} & L_{s_{ij}T_{ij}} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & L_{T_{ij}s_{ij}} & L_{s_{ij}^2} & L_{S_{ij}s_{ij}} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & L_{S_{ij}s_{ij}} & L_{S_{ij}^2} & L_{S_{ij}t_{j+1}} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & L_{T_{in}s_{in}} & L_{s_{in}^2}
\end{bmatrix}$$

By Balkhi and Benkherouf [2, Theorem 2], and Stewart [22], the above matrix is positive definite if

$L_{t_{i1}^2} \geq |L_{t_{i1}T_{i1}}|$ (25a) , and

$L_{t_{ij}^2} \geq |L_{S_{ij-1}t_{ij}}| + |L_{T_{ij}t_{ij}}|$, j=2,3….n (25b)

$L_{T_{ij}^2} \geq |L_{t_{ij}T_{ij}}| + |L_{s_{ij}T_{ij}}|$, j=1,2, ….n (26)

$L_{s_{ij}^2} \geq |L_{T_{ij}s_{ij}}| + |L_{s_{ij}S_{ij}}|$, j=1,2, ….n-1 (27a)

and $L_{s_{in}^2} \geq |L_{T_{in}s_{in}}|$ (27b)

$L_{S_{ij}^2} \geq |L_{S_{ij}s_{ij}}| + |L_{t_{ij+1}S_{ij}}|$,

j=1,2, ….n-1 (28)

Recalling (14) and (15), we have :

$$\frac{\partial^2 L}{\partial t_{ij}^2} = h_{ij}P_{ij} + c_{ij}\theta_{ij}P_{ij} = P_{ij}(h_{ij} + c_{ij}\theta_{ij}) > 0$$

$$\frac{\partial^2 L}{\partial T_{ij}\partial t_{ij}} = -D_{ij}(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(T_{ij} - t_{ij})}$$

For $i = 1,2,\ldots m$, and $j = 2,3,\ldots n$, from (16) we have:

$$\frac{\partial L}{\partial S_{ij-1}} = -c_{ij}P_{ij} - \lambda_{ij}(P_{ij} - D_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} - h_{ij}(P_{ij} - D_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} = 0$$

$$\frac{\partial^2 L}{\partial S_{ij-1}\partial t_{ij}} = -(P_{ij} - D_{ij})(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} \quad (29)$$

Hence (25a) $\Leftrightarrow$

$$P_{ij}(h_{ij} + c_{ij}\theta_{ij}) \geq D_{ij}(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(T_{ij} - t_{ij})}$$

$$\Leftrightarrow P_{ij} \geq D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})} \quad (30)$$

and (25b) $\Leftrightarrow$

$$P_{ij}(h_{ij} + c_{ij}\theta_{ij}) \geq (P_{ij} - D_{ij})(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} + D_{ij}(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(T_{ij} - t_{ij})} \Leftrightarrow$$

$$\boldsymbol{P_{ij} \geq (P_{ij} - D_{ij})e^{\theta_{ij}(S_{ij-1} - t_{ij})} + D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})}} \quad \textbf{(31)}$$

(Here we note that (31)⇒ (30), so no need for (30)). From (19a) :

$$\frac{\partial^2 L}{\partial T_{ij}^2} = D_{ij}(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(T_{ij} - t_{ij})} + b_{ij}D_{ij} > 0$$

$$\frac{\partial^2 L}{\partial s_{ij}\partial T_{ij}} = -b_{ij}D_{ij}$$

Thus (26) $\Leftrightarrow$

$$D_{ij}(h_{ij} + c_{ij}\theta_{ij})e^{\theta_{ij}(T_{ij} - t_{ij})} + b_{ij}D_{ij} \geq P_{ij}(h_{ij} + c_{ij}\theta_{ij}) + b_{ij}D_{ij} \Leftrightarrow$$

$$D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})} \geq P_{ij} \quad (32)$$

Combining (30) with (32), we obtain :

$$P_{ij} = D_{ij}e^{\theta_{ij}(T_{ij} - t_{ij})} > D_{ij} \quad (33)$$

(33) is a significant relation between P_{ij} and D_{ij}. However, we must take P_{ij} which satisfies (31) so that all second order conditions of optimality are fulfilled. From (20), we have

$$\frac{\partial^2 L}{\partial T_{ij}\partial s_{ij}} = -b_{ij}D_{ij}, \quad \frac{\partial^2 L}{\partial s_{ij}^2} = b_{ij}P_{ij} > 0,$$

$$\frac{\partial^2 L}{\partial S_{ij}\partial s_{ij}} = -b_{ij}(P_{ij} - D_{ij})$$

Thus (27a) $\Leftrightarrow$ $b_{ij}P_{ij} \geq b_{ij}D_{ij} + b_{ij}(P_{ij} - D_{ij})$

$\Leftrightarrow$ $P_{ij} \geq P_{ij}$ which is always satisfied.

Similarly, we can easily verify that (27b) is always satisfied. Finally, from (23) we have

$$\frac{\partial^2 L}{\partial S_{ij}^2} = b_{ij}(P_{ij} - D_{ij}) \ > 0$$

$$\frac{\partial^2 L}{\partial S_{ij} \partial s_{ij}} = -b_{ij}(P_{ij} - D_{ij})$$

Recalling (24), and replacing j by $j+1$ in (29), we obtain

$$\frac{\partial^2 L}{\partial S_{ij}\,\partial t_{ij+1}} = - \ c_{ij+1}(P_{ij+1} - D_{ij+1})\theta_{ij+1}e^{\theta_{ij+1}(S_{ij-1} - t_{ij+1})} + h_{ij+1}(P_{ij+1} - D_{ij+1})\theta_{ij+1}e^{\theta_{ij+1}(S_{ij} - t_{ij+1})} = 0$$

So (28) $\Leftrightarrow$ $b_{ij}(P_{ij} - D_{ij}) \ \geq b_{ij}(P_{ij} - D_{ij}) \ +$

$$\left| h_{ij+1}(P_{ij+1} - D_{ij+1})\theta_{ij+1}e^{\theta_{ij+1}(S_{ij} - t_{ij+1})} - c_{ij+1}(P_{ij+1} - D_{ij+1})\theta_{ij+1}e^{\theta_{ij+1}(S_{ij} - t_{ij+1})} \right|$$

$\Leftrightarrow$ $c_{ij+1} \geq h_{ij+1}$ which always hold since the holding cost is usually less than item production cost. Thus we have the following result:

Theorem 2 . Any solution for which (31) holds is a minimizing solution for problem (*P*).

Lemma 1 . All turning points are functions of t_{i1}.

Proof : Given that $t_{i1} > s_{i0} = 0$. Then from (14), T_{i1} is a function of t_{i1}, which in turn implies that (recall (19a)) s_{i1} is a function of t_{i1} and that (recall(15)) S_{i1} is a function of t_{i1}. Now for j=2 and by the relation $\frac{\partial L}{\partial S_{ij-1}} = 0 \ and$ (24), t_{i2} is a function of S_{i1} and therefore is a function of t_{i1}. From (14), T_{i2} is a function of t_{i1} and from (19a), s_{i2} is a function of T_{i2} and t_{i2}, hence a function of t_{i1}. Substituting in (15), we get that S_{i2} is a function of t_{i1}. Repeating the same process, we obtain that all variables are functions of t_{i1}. This completes the proof of the lemma. Now let

$$t_{i1}' = \frac{dt_{i1}}{dt_{i1}} = 1, \ t_{ij}' = \frac{dt_{ij}}{dt_{i1}},$$

$$S_{ij-1}' = \frac{dS_{ij-1}}{dt_{i1}}, T_{ij}' = \frac{dT_{ij}}{dt_{i1}},$$

$$s_{ij}' = \frac{ds_{ij}}{dt_{i1}}, \quad S_{ij}' = \frac{dS_{ij}}{dt_{i1}},$$

Then we have the following important result :

Lemma 2 : $0 \leq S_{ij-1}' < t_{ij}' < T_{ij}' < s_{ij}' < S_{ij}'$, for j=1,2,….n-1 and

$s_{ij+1}' < S_{ij+1}'$,for j=1,2,….n-2. (34)

Proof : Since $S_{ij-1} = 0$ for j=1 $\Rightarrow$ $S_{ij-1}' = 0 < t_{i1}' = 1$ for j=1. From (14) and with j=1, we have

$0 < (P_{i1} - D_{i1})t_{i1}'e^{-\theta_{i1}t_{i1}}) =$

$D_{i1}(T_{i1}' - t_{i1}')e^{\theta_{ij}(T_{i1} - t_{i1})}$.

Hence $T_{i1}' - t_{i1}' > 0 \ \Rightarrow T_{i1}' > t_{i1}'$.

Also from (19a) and (24) with j=1, we have

$$h_{i1}D_{i1}(T_{i1}' - t_{i1}')e^{\theta_{i1}(T_{i1} - t_{i1})} + c_{i1}D_{i1}(T_{i1}' - t_{i1}')e^{\theta_{i1}(T_{i1} - t_{i1})} = -b_{i1}D_{i1}(T_{i1}' - s_{i1}')$$

$\Rightarrow$ $(T_{i1}' - s_{i1}') < 0$ $\Rightarrow$ $T_{i1}' - s_{i1}'$.

Finally, from(20) with j=1, we have:

$0 < b_{i1}D_{i1}(s_{i1}' - T_{i1}') = -b_{i1}(P_{i1} - D_{i1})(s_{i1}' - S_{i1}')$

$\Rightarrow$ $s_{i1}' - S_{i1}' < 0$ $\Rightarrow$ $s_{i1}' < S_{i1}'$. Thus (34) holds for some $k \geq 1$ and by induction we can then show that (34) holds for *k+1*. This completes the proof of the lemma.

Corollary 1 : All turning points are increasing functions of t_{i1} and of each other.

Proof : We have shown in lemma 1 that the turning points S_{ij-1} , t_{ij}, T_{ij}, s_{ij}, and S_{ij} are functions of each other and that all of them are functions of t_{i1}. Hence, by the chain rule of differentiation and the result of lemma 2, we can conclude that all these turning points are increasing functions of t_{i1} and that T_{ij} is an increasing functions of t_{ij}, s_{ij} is an increasing function of T_{ij}, and S_{ij} is an increasing function of s_{ij} . This is so, since for example, we have :

$$0 < T_{ij}^{'} = \frac{dT_{ij}}{dt_{i1}} = \frac{\partial T_{ij}}{\partial t_{ij}} \frac{\partial t_{ij}}{\partial t_{i1}} \quad \Rightarrow \quad \frac{\partial T_{ij}}{\partial t_{ij}} > 0$$

$$0 < \frac{dT_{ij}}{dt_{i1}} = \frac{\partial T_{ij}}{\partial S_{ij}} \frac{\partial S_{ij}}{\partial t_{i1}} \quad \Rightarrow \quad \frac{\partial T_{ij}}{\partial S_{ij}} > 0$$

$$0 < \frac{dT_{ij}}{dt_{i1}} = \frac{\partial T_{ij}}{\partial s_{ij}} \frac{\partial s_{ij}}{\partial t_{i1}} \quad \Rightarrow \quad \frac{\partial T_{ij}}{\partial s_{ij}} > 0$$

Hence, all turning points are increasing functions of t_{i1} and of each other. This completes the proof of the corollary.

Theorem 3 : Under condition (31), problem *(P)* has a unique and global optimal solution.

Proof: Since all turning points are functions of t_{i1}, we conclude that if t_{i1} has been chosen adequately, then all other turning points are also chosen adequately and we then must have $S_{in} = H$. Now, let us consider arbitrary starting point t_{i1}. If t_{i1} is near the correct value, then S_{in} will be near *H*. However, for any choice of t_{i1} and for any item *i*, we always must have

$$\sum_{j=1}^{n}\{(t_{ij} - S_{ij-1}) + (T_{ij} - t_{ij}) + (s_{ij} - S_{ij}) + (S_{ij} - s_{ij})\} = H \qquad (35)$$

(Recall (11)). Recalling that all turning points are functions of t_{i1} , and that if t_{i1}=0 implies (from(14)) that $T_{i1} = 0$ which in turn implies (from(19a)) that $s_{i1} = 0$ and from (15) $S_{i1} = 0$. By induction, we can easily show that if t_{i1}=0, then $S_{ij-1} = t_{ij} = T_{ij} = s_{ij} = S_{ij} = 0$. Now, let

$$F(t_{i1}) = \sum_{j=1}^{n}\{(t_{ij} - S_{ij-1}) + (T_{ij} - t_{ij}) + (s_{ij} - S_{ij}) + (S_{ij} - s_{ij})\} - H \qquad (36)$$

Then *F(0)* = -*H* < 0 and

$$F^{'}(t_{i1}) = \sum_{j=1}^{n}\{(t_{ij}^{'} - S_{ij-1}^{'}) + (T_{ij}^{'} - t_{ij}^{'}) + (s_{ij}^{'} - S_{ij}^{'}) + (S_{ij}^{'} - s_{ij}^{'})\}$$

Recalling relation (34), we have $F^{'}(t_{i1}) > 0$. That is $F(t_{i1})$ is an increasing function of t_{i1} and that it takes a negative value for t_{i1}=0. This implies that equation (35) has a unique solution which is a minimizing solution by Theorem 2 . Thus problem (P) has a unique and global optimal solution. This completes the proof of the theorem.

Next, we show that problem *(P)* has a unique and global optimal solution for any value of *n* . For more details, see Emet [13] . First, recall that the whole system depends on t_{i1} which is to be determined correctly for any value of *n*. The following result gives us an insight about such a determination. Before that, let us consider two different schedules with the same starting and finishing points , say :

Schedule 1 : ($S_{ij-1}, t_{ij}, T_{ij}, s_{ij}, S_{ij}), j = 1,2 \dots, n$ with $S_{i0} = 0$, and $S_{in} = H$

Schedule 2 : ($\overline{S_{ij-1}}, \overline{t_{ij}}, \overline{T_{ij}}, \overline{s_{ij}}, \overline{S_{ij}}), j = 1,2 \dots, n$ with $\overline{S_{i0}} = 0$, and $\overline{S_{in+1}} = H$.

Lemma 3: The turning points of schedule 2 lie between the turning points of schedule 1. That is

$$S_{ij-1} \le \overline{t_{ij}} \le t_{ij} \le \overline{T_{ij}} \le T_{ij} \le \overline{s_{ij}} \le s_{ij} \le \overline{S_{ij}} \le S_{ij}$$

with $\overline{S_{in-1}} \le \overline{t_{in}} \le \overline{T_{in}} \le \overline{s_{in}} \le \overline{S_{in}} = H$.

Proof : By corollary 1, if we reduce t_{ij} to $\overline{t_{ij}}$, then all other turning points are to be reduced. Now, suppose our conclusion fails for some value *k* of *j* and for one inequality, while all other inequalities of (37) hold for *j=k*. That is suppose we, for instance, have $\overline{S_{ik}} > S_{ik}$. Then, if we pass to the end points, we obtain

$\overline{S_{in}} > S_{in} = H$, which is a contradiction. If the conclusion fails for two inequalities, say $\overline{t_{ij}} > t_{ij}$ and $t_{ij} > \overline{T_{ij}}$, then $\overline{t_{ij}} > \overline{T_{ij}}$ which is also a contradiction with (37). Repeating the same arguments, we reach the desired result. This completes the proof of the Lemma.

As an important corollary of the previous lemma is the following :

Corollary 2 : If condition (31) holds then the quantity

$E = \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij}$

is a decreasing function of *n*.

Proof : Let *j*=1 in (9). The we have

$$\frac{\partial E}{\partial t_{i1}} = \frac{h_{i1}(P_{i1} - D_{i1})}{\theta_{i1}}(1 - e^{-\theta_{i1} t_{i1}}) +$$

$$\frac{h_{i1}D_{i1}}{\theta_{i1}}(1 - e^{\theta_{ij}(T_{i1} - t_{i1})}) + c_{i1}P_{i1} =$$

$c_{i1}P_{i1} > 0$

(Recall (14)). Hence E is an increasing function of t_{i1}.Now, consider the two schedules 1 and 2 as defined above. If we increase *n* by 1, then by Lemma 3, we have $\overline{t_{i1}} < t_{i1}$ which implies that $E(n+1, \overline{t_{i1}}) \leq E(n, t_{i1})$. Since E is an increasing function of t_{i1}, the last inequality means that E is a decreasing function of *n*. This completes the proof of the corollary.

Our last result in this paper is the following :

Theorem 4 : Under conditions (31), the underlying inventory system(Q) has a unique and global optimal solution.

Proof : As a direct consequence of all above results, we start with a suitable value of t_{i1} for n=1. If we increase n by 1, Then E would decrease. Such decrease of E shall stop after choosing new value of t_{i1} less than the previous ones. Continuing the procedure in this manner, we shall eventually reach to a value of n, say n^*, at which the function E starts to increase. Then the optimal value of n is $n^* -1$. This optimal value of n with the corresponding optimal values of S_{ij-1} , t_{ij}, T_{ij}, s_{ij}, and S_{ij} say S^*_{ij-1}, t^*_{ij}, T^*_{ij}, s^*_{ij} and S^*_{ij} are our unique and global optimal solution for problem (Q). This completes the proof of the theorem.

VI. CONCLUSION

In this paper , we have considered a general multi-item production lot size inventory problem for a given finite time horizon of H units long . The time horizon is divided into *n* different cycles in each of which a number of *m* items are produced. We have built an inventory model with the objective of minimizing the overall total related inventory cost. Then we have introduced a solution procedure by which we could determine the optimal stopping and restarting production times for each item in each cycle in the given time horizon when shortages are allowed but are completely backordered. Then, quite simple and feasible sufficient conditions that guarantee the uniqueness and global optimality of the obtained solution are established .Such optimal solutions can lead to optimal inventory policies for the different products. Further research may include the possibility of having some parameters of such systems including the cost parameters as known functions of time or as known probability distributions.

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