# A Simulation Study of Additive Outlier in ARMA (1, 1) Model

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*Abstract*— Abnormal observation due to an isolated incident such as a recording error is known as additive outlier and it is often found in time series. Since extreme value of additive outliers may contribute to the inaccuracy of model specification, proper detection procedure is significant to avoid such error. Equations that explain the nature of an additive outlier and the test statistics pertaining to it are discussed in this article. This is followed by two separate simulation studies that are conducted to investigate the sampling behavior and detection performance of the test statistics in ARMA (1, 1) models. Results for the first simulation study show that the test statistics is an increasing function of sample size. Whilst in the other simulation study we see that the performance of the test statistics improves as large magnitudes of outlier effect are used.

*Keywords*—Additive outlier, sampling behavior of test statistics, detection performance of test statistics, simulation

## I. INTRODUCTION

Abnormal observations in time series often signify important events such as an intervention or an unexpected incident like the outbreak of war, economic recession etc. These observations are known as outliers because they are aberrant from the rest of the observations. To identify an outlier based on the reasons associated with it, outliers are named based on their attributes like additive outlier (AO), innovational outlier (IO), temporary change (TC) and level shift (LS) each of which corresponds to a unique incident. The most common incident found in a time series is recording error and such event is noted by AO. Thus, this study aims to investigate the sampling behavior of the test statistics used to detect a single AO in ARMA (1, 1) models and its performance when selected criterion is applied.

An outlier-free time series $Z_t$ that follows an autoregressive moving average (ARMA) process is defined as

$$\Phi(B)Z_t = \Theta(B)a_t \qquad (1)$$

where $B$ is the backshift operator such that $BZ_t = Z_{t-1}$ and

$$\Phi(B) = 1 - \Phi_1 B - ... - \Phi_p B^p \; ; \; \Theta(B) = 1 - \Theta_1 B - ... - \Theta_q B^q$$

are polynomials in $B$, $\{a_t\}$ is a sequence of white noise random variables, identically and independently distributed as N(0, $\sigma_a^2$).

From (1), $Z_t$ can be defined as

$$Z_t = \frac{\Theta(B)}{\Phi(B)} a_t \qquad (2)$$

Fox (1972) is among the earliest to conduct a study on outliers in time series. He considered non-seasonal AR (p) process and two outliers which are AO and IO. In his work, Fox proposed a method which detects and removes the outlier effect [1]. Following that, many studies on outliers in ARMA (p,q) models were carried out like [2]-[4]. Making use of (1) and (2), the generating mechanism of an AO in ARMA process is described as

$$Y_t = \begin{cases} Z_t & t \neq T \\ \\ Z_t + \omega & t = T \end{cases}$$

(3)

$$= Z_t + \omega I_t^{(T)} \qquad (4)$$

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From equation (2)

$$= \frac{\theta(B)}{\phi(B)} a_t + \omega I_t^{(T)} \tag{5}$$

The observed outlier-free series of (1) and unobservable series are denoted as $Y_t$ and $Z_t$ respectively. Magnitude of AO is represented by $\omega$ and $I_t^{(T)}$ is a time indicator variable used to indicate the occurrence of an AO. Therefore, $I_t^{(T)} = 1$ when an AO is spotted and $I_t^{(T)} = 0$ otherwise. Following (4), AO is said to be deterministic in nature i.e. not affecting observations subsequent to it [1].

## II. TEST STATISTICS FOR AO DETECTION

Despite only affecting the observation at $t = T$, an AO is known to affect up to $p$ subsequent residuals following $t=T$ [6]. Therefore, residual estimates are used in the estimation of $\omega$ which later forms the basis of AO detection. In this section, discussions on the estimation of residuals and AO effects are shown in (6) and (7) respectively.

### A. Estimation of residuals

To facilitate understanding of how residual estimates are used in the AO detection procedure, consider a simple case when $T$ and all parameters in (1) are known [1]. Let

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = 1 - \pi_1 B^1 - \pi_2 B^2 \ldots \tag{6}$$

be the dynamic system where $\pi_j$ denote the weights for $j$ beyond a moderately large value $J$ that essentially equal to 0 when the roots of $\theta(B)$ lie outside of the unit circle. Estimated residuals can then be described as

$$\hat{e}_t = \pi(B) Y_t \tag{7}$$

Using (5), (6) and (7), estimated residuals of an AO contaminated series in an ARMA process can be written as

$$\begin{aligned}
\hat{e}_t &= \pi(B) Y_t \\
&= \pi(B) \left[ \frac{\theta(B)}{\phi(B)} a_t + \omega I_t^{(T)} \right] \\
&= \frac{\phi(B)}{\theta(B)} \left\{ \frac{\theta(B)}{\phi(B)} a_t + \omega I_t^{(T)} \right\} \\
&= a_t + \frac{\phi(B)}{\theta(B)} \omega I_t^{(T)} \\
&= a_t + \pi(B) \omega I_t^{(T)} \\
&= a_t + \omega \pi(B) I_t^{(T)}
\end{aligned} \tag{8}$$

Noting that $I_t^{(T)} = 1$ when $t = T$ and $I_t^{(T)} = 0$ otherwise; residuals for $t < T$ and $t = T$ are obtained as follows:

$$\hat{e}_t = \begin{cases} a_t & t < T \\ \\ a_t + \omega \pi(B) & t = T \end{cases} \tag{9}$$

On the contrary, residual estimates for $t = T + j (j = 1, 2 \ldots n - j)$ are not as straight forward as (9). To obtain the respected residual estimates, we expand equation (8).

$$\begin{aligned}
\hat{e}_t &= a_t + \omega \pi(B) I_t^{(T)} \\
&= a_t + \omega \left( 1 - \pi_1 B^1 - \pi_2 B^2 - \pi_3 B^3 \ldots \right) I_t^{(T)} \\
&= a_t + \omega \left( I_t^{(T)} - \pi_1 I_{t-1}^{(T)} - \pi_2 I_{t-2}^{(T)} - \pi_3 I_{t-3}^{(T)} \ldots \right)
\end{aligned}$$

Hence, for $t = T + j (j = 1, 2, \ldots n - T)$

$$\begin{aligned}
\hat{e}_{T+j} &= a_{T+j} + \omega \left( I_{T+j}^{(T)} - \pi_1 I_{T+j-1}^{(T)} - \pi_2 I_{T+j-2}^{(T)} \ldots - \pi_j I_{T+j-j}^{(T)} - \ldots \right) \\
&= a_{T+j} + \omega \left( I_{T+j}^{(T)} - \pi_1 I_{T+j-1}^{(T)} - \pi_2 I_{T+j-2}^{(T)} \ldots - \pi_j I_T^{(T)} \ldots \right) \\
&= a_{T+j} + \omega \left[ 0 - \pi_1 (0) - \pi_2 (0) \ldots - \pi_j (1) \ldots \right] \\
&= a_{T+j} + \omega \left( - \pi_j \right)
\end{aligned} \tag{10}$$

For $n$ number of observations, equations (8)-(10) can be summarized as the following [5]:

$$\begin{bmatrix} \hat{e}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{e}_{T-1} \\ \hat{e}_T \\ \hat{e}_{T+1} \\ \hat{e}_{T+2} \\ \cdot \\ \cdot \\ \cdot \\ \hat{e}_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{T-1} \\ a_T \\ a_{T+1} \\ a_{T+2} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} + \omega \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ -\pi_1 \\ -\pi_2 \\ \cdot \\ \cdot \\ \cdot \\ -\pi_{n-T} \end{bmatrix} \tag{11}$$

### B. Estimation of AO effect

Let $\hat{\omega}$ be the estimator of $\omega$ in (3), $\hat{\omega}$ is known as the least squares estimate of AO effect because $\{a_t\}$ are obtained from the least squares theory [5]. From (8), let $\pi(B) I_t^{(T)}$ be represented by $x_t$, we have that

$$\hat{\omega}_t = \frac{\sum_{t=T}^{n} \hat{e} \, x_t}{\sum_{t=T}^{n} x_t^2} \qquad (12)$$

and variance of the estimator given as

$$Var(\hat{\omega}) = \frac{\hat{\sigma}_a^2}{\sum_{t=T}^{n} x_t^2} \qquad (13)$$

Thus, the standardized version of $\hat{\omega}$ is

$$\tau_t = \frac{\hat{\omega}_t}{\sqrt{Var(\hat{\omega}_{AO})}}$$

$$= \frac{1}{\hat{\sigma}_a} \left[ \frac{\sum_{t=T}^{n} \hat{e}_t x_t}{\sum_{t=T}^{n} x_t^2} \right] \sqrt{\sum_{t=T}^{n} x_t^2} \qquad (14)$$

Having stated equations (6) to (14), the test statistics of interest denoted by $\eta_t$ is the absolute maxima of (14) as described in (15)

$$\eta_t = \max_{t=1,2,\ldots,n} \left\{ |\tau_t| \right\} \qquad (15)$$

## III. ILLUSTRATIONS

For the illustration of the sampling behavior and detection performance of the test statistics $\eta_t$, we consider a simple case when $T$ and all parameters of ARMA (1, 1) are known. To allow a more comprehensive analysis to be conducted, the parameters are carefully chosen to form unique combinations of ARMA (1, 1) models as shown in Table 1. Assuming the residuals, $e_t$ follow a normal distribution of $N$ (0, 1), outlier free time series for each of the model in Table 1 are generated using the *arima.sim* procedure in *R* package.

In section 3.1, outlier free time series of size $n$ are used to study the sampling behavior of $\eta_t$. Then, in section 3.2, AO contaminated series are generated by creating an AO at $T=n/2$ in the outlier free time series. These contaminated series are used to test the performance of $\eta_t$ in the detection of the artificial simulated AO.

### A. Sampling Behavior

In this section, we investigate the sampling properties of $\eta_t$ in relation to
(i) Sample size $n$ 60, 100 and 200

(ii) Coefficients chosen for ARMA (1, 1) in Table 1.

TABLE I
ARMA (1, 1) MODELS

| Model | AR | MA |
|---|---|---|
| 1 | 0.3 | 0.7 |
| 2 | -0.3 | -0.7 |
| 3 | -0.3 | 0.7 |
| 4 | 0.3 | -0.7 |

To achieve this, outlier-free time series of size 60,100 and 200 for each of the model in Table 1 are generated 500 times. For instance, given $n = 100$ and model =1, 500 test statistics for the additive outlier $\eta_t$ pertaining to the respected criterion are attained. Next, the upper percentiles of $\eta_t$ at 1%, 5% and 10% level are obtained for comparison.

Repeating the same procedure for all possible combinations of (i) and (ii), the results are then plotted in Figure 1-3.The plots show similar patterns as estimates of $\eta_t$ at given percentiles are increasing functions of sample size $n$. However magnitude for the increase varies for each model. Take model 3 as an example, at 5% upper percentiles, estimates of $\eta_t$ corresponding with sample size 60,100 and 200 are 2.55, 2.70 and 2.76 respectively as compared to the estimates of Model 2 which are 3.47, 3.71 and 4.03.
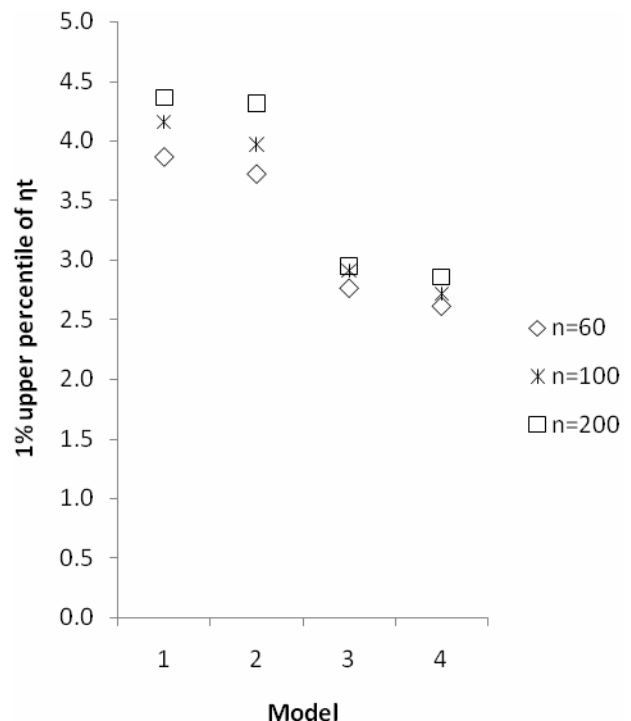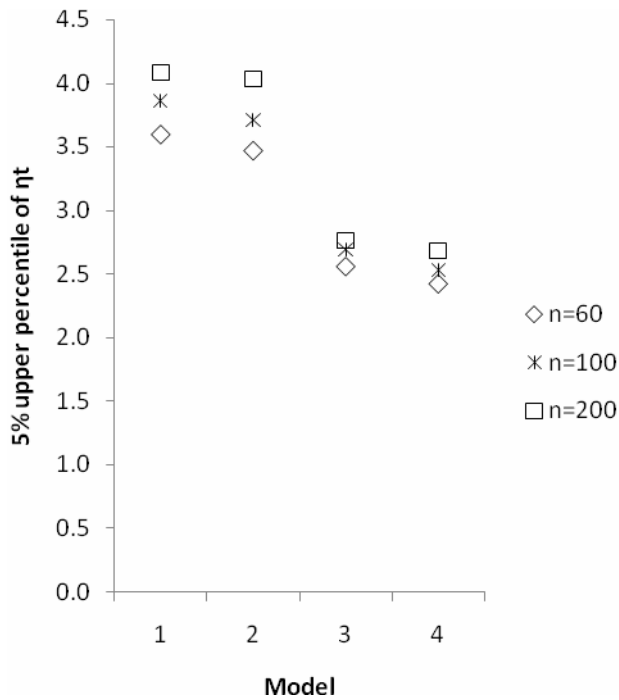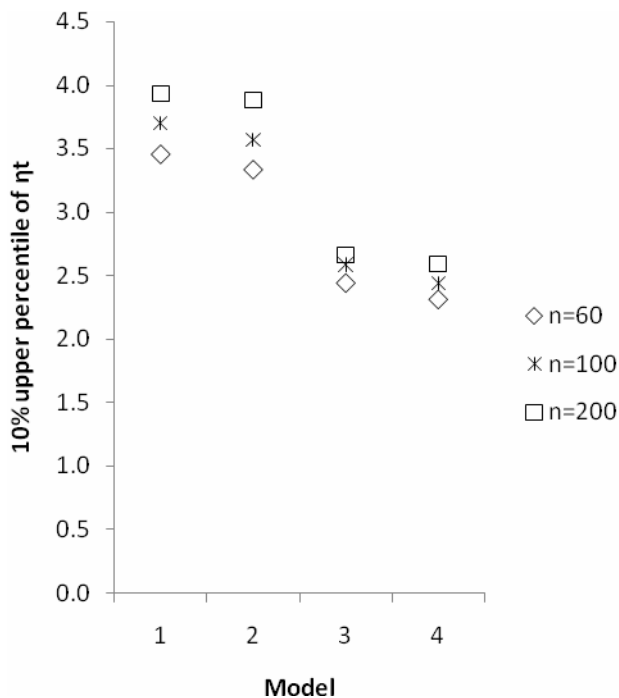
Fig.1    1% upper percentiles of $\eta_t$



Next, we examine the detection performance of $\eta_t$ associated with

(i) Sample size $n$ of 60, 100 and 200
(ii) Coefficients chosen for ARMA (1, 1) in Table 1
(iii) AO effect $\omega$ of magnitudes 5, 10 and 15

For each possible combination of (i),(ii) and (iii), 500 AO contaminated series are generated by allocating an AO of the respected $\omega$ at $T = n/2$ in each of the series as suggested in (3). For example, given $n = 100$ ,model $=1$ and $\omega =5$, we acquire the proportions of the AOs correctly detected at 1% significance level from the 500 AO contaminated series in relation to the respected criterion.

Repeating the same procedure for all possible combinations of (i), (ii) and (iii), the results are then plotted in Figure 4-15. Overall the plots suggest that the detection performance of $\eta_t$ improves when large $\omega$ are used, this is especially evident in model 3 as exhibited in Figure 10-12.

Fig.2    5% upper percentiles of $\eta_t$
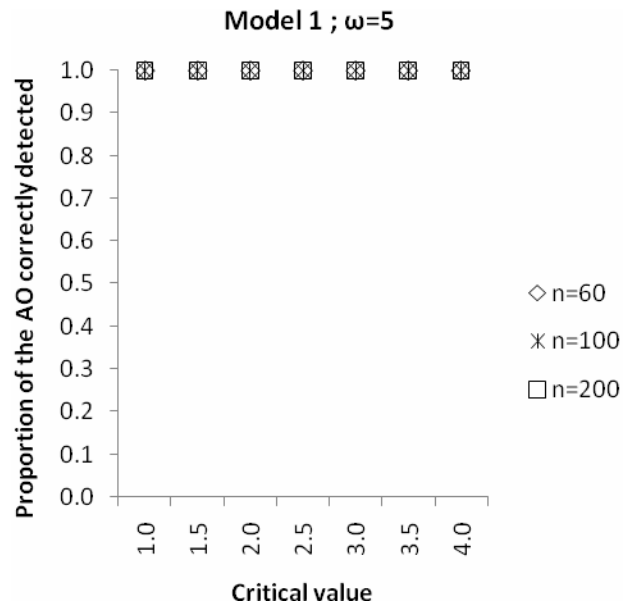




Fig. 4    Proportion of the AO correctly detected in Model 1 when $\omega = 5$

Fig.3    10% upper percentiles of $\eta_t$

*B.  Detection Performance*
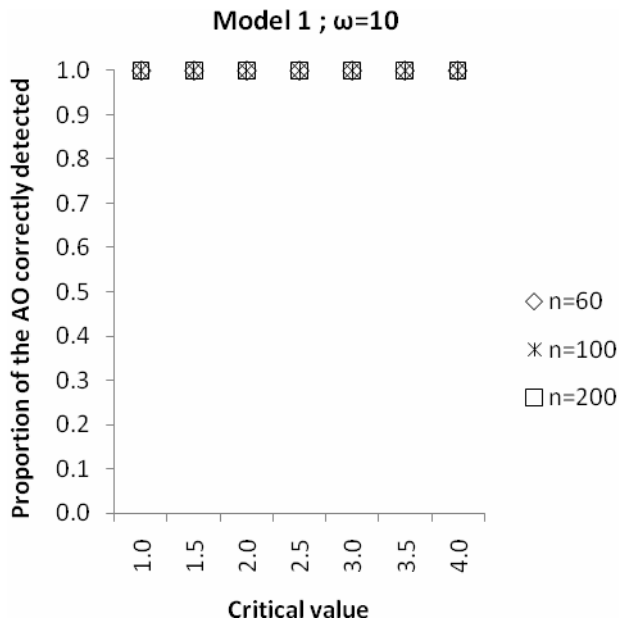
Fig. 5    Proportion of the AO correctly detected in Model 1 when $\omega = 10$
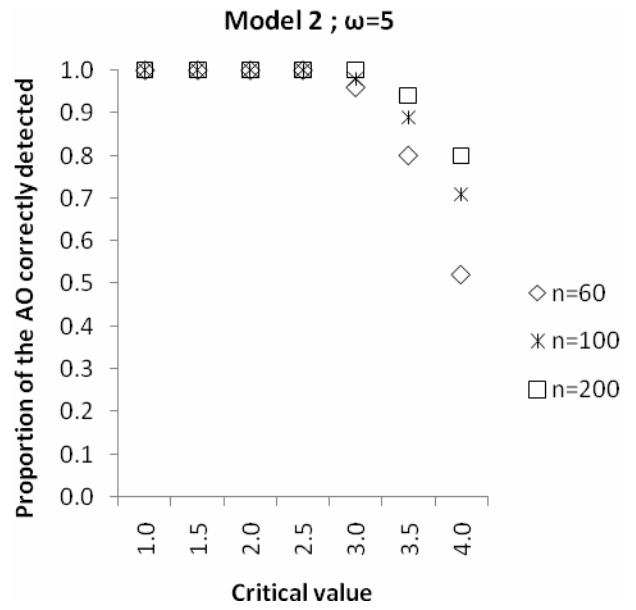


Fig. 7    Proportion of the AO correctly detected in Model 2 when $\omega = 5$
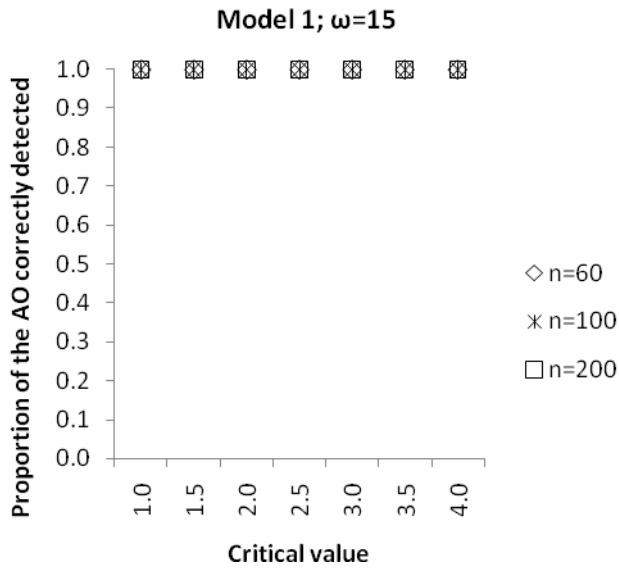


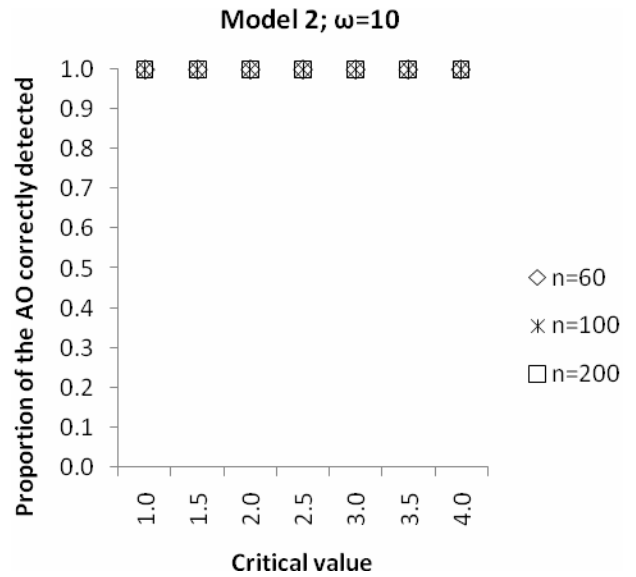Fig. 6    Proportion of the AO correctly detected in Model 1 when $\omega = 15$



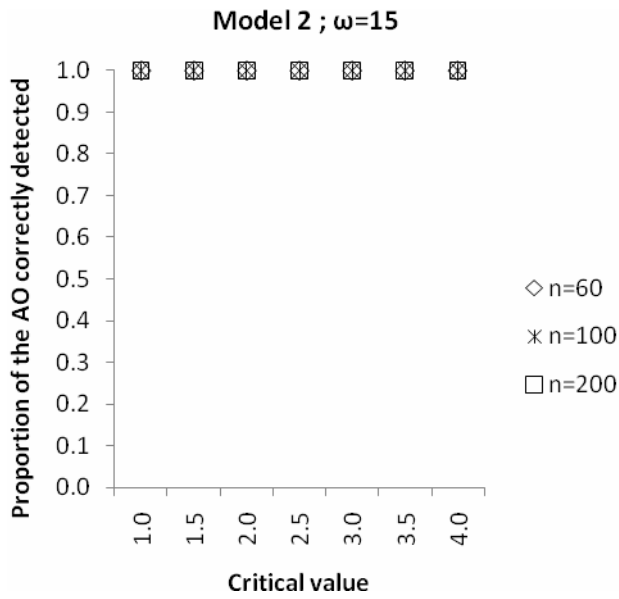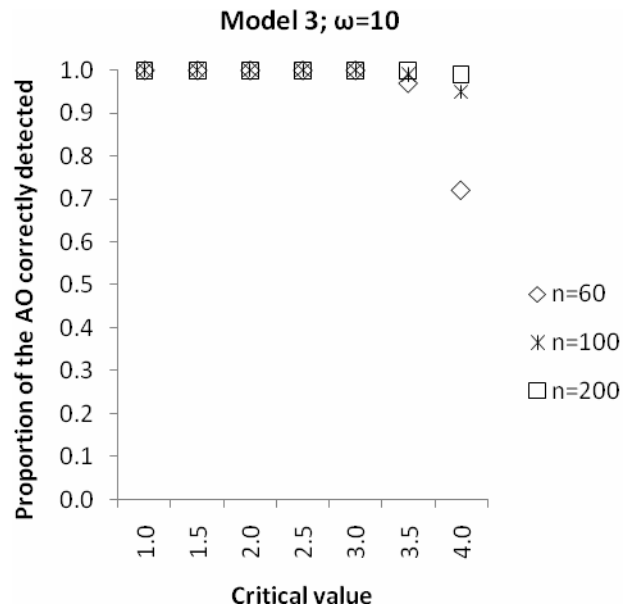Fig. 8    Proportion of the AO correctly detected in Model 2 when $\omega = 10$

Fig. 9  Proportion of the AO correctly detected in Model 2 when $\omega = 15$



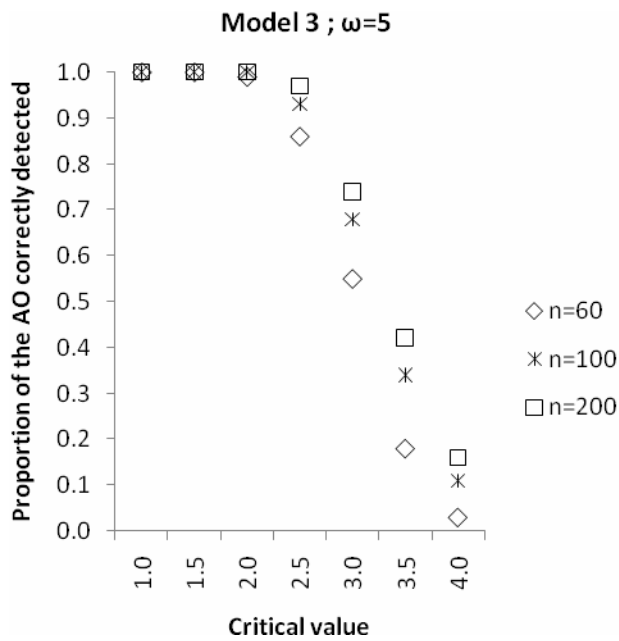Fig. 11  Proportion of the AO correctly detected in Model 3 when $\omega = 10$



Fig. 10  Proportion of the AO correctly detected in Model 3 when $\omega = 5$
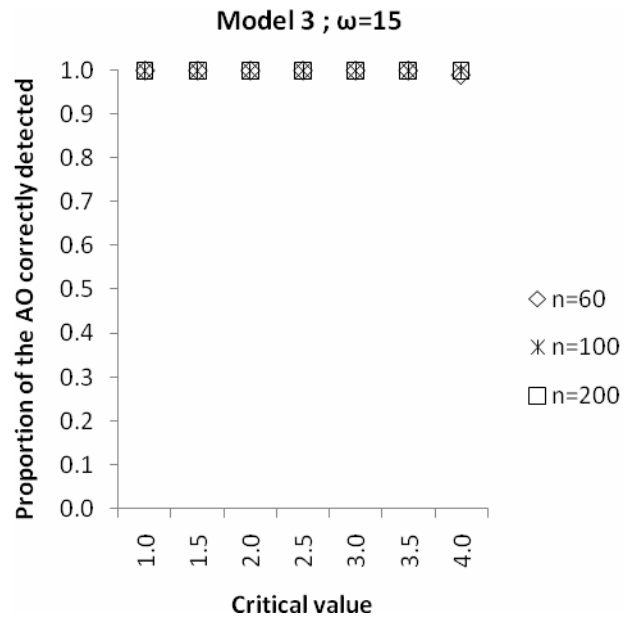


Fig. 12  Proportion of the AO correctly detected in Model 3 when $\omega = 15$
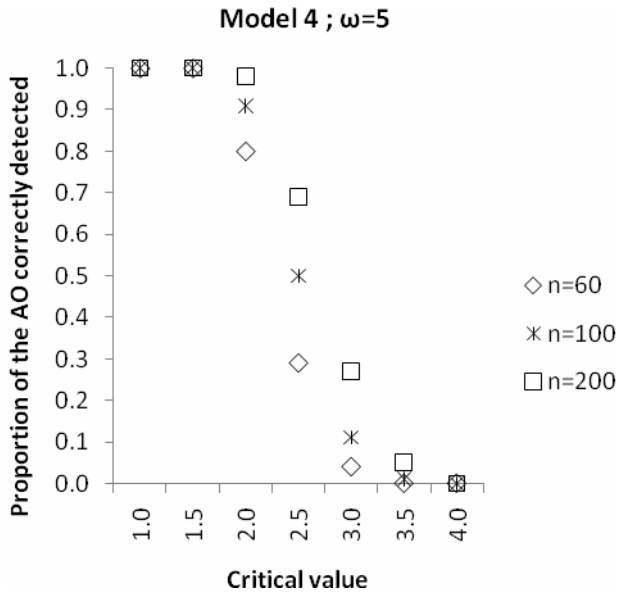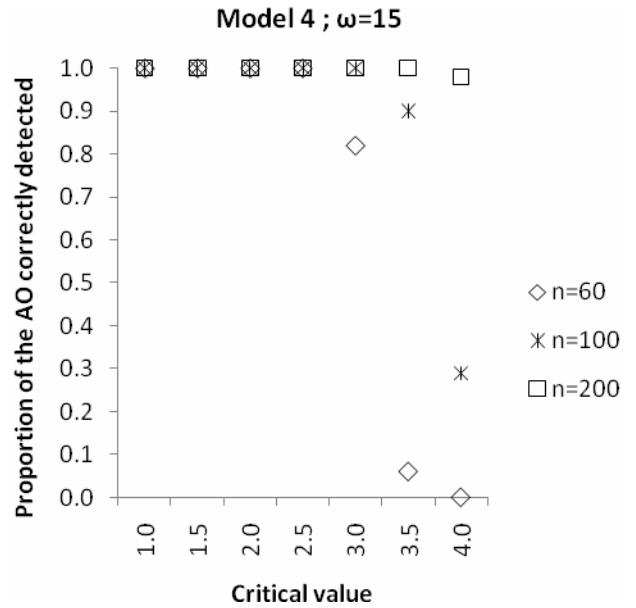
### Model 4 ; ω=5

Fig. 13   Proportion of the AO correctly detected in
Model 4 when $\omega = 5$

### Model 4 ; ω=15

Fig. 15   Proportion of the AO correctly detected in
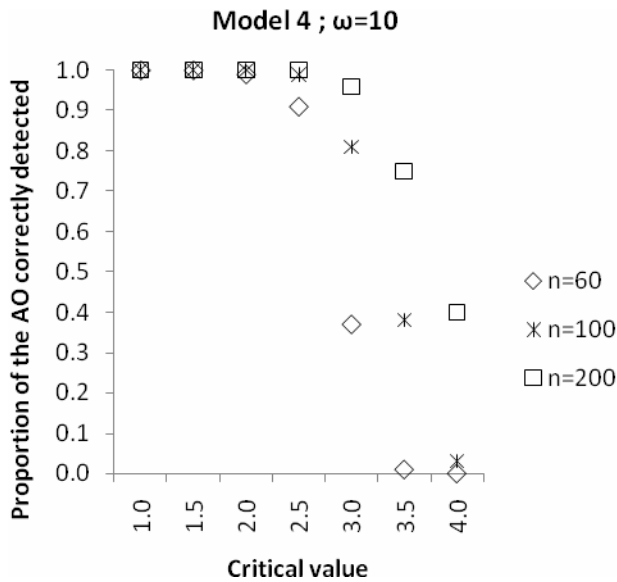Model 4 when $\omega = 15$

### Model 4 ; ω=10

Fig. 14   Proportion of the AO correctly detected in
Model 4 when $\omega = 10$

## IV.   CONCLUSION

An AO is often associated with isolated mistake such as a recording error. Therefore, it has a deterministic nature because the AO effect $\omega$ does not affect subsequent observations as described in equation (4). However, according to equations (8) - (10), residuals that come after an AO may substantially be affected by $\omega$. In section 3.1, simulation study on the sampling behavior suggests that estimates of the test statistics $\eta_t$ are increasing functions of $n$. On the other hand, the simulation study in section 3.2 show that the detection performance of $\eta_t$ improves when large magnitudes of $\omega$ are used. In this study, neither the sampling behavior nor the detection performance of $\eta_t$ indicates any obvious relationship with the coefficients chosen for ARMA (1, 1).

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