On some interpolation problems in polynomial spaces with generalized degree

Dana Simian, Corina Simian

Abstract— **The aim of this paper is to study many interpolation problems in the space of polynomials of w-degree n. In order to do this, some new results concerning the polynomial spaces of w-degree are given. In this article, we consider only the case of functions in two variables. More details are obtained for the weight $w = (1, w_2)$. We found a set of conditions for which, $\Pi_{n,w}$, the space of polynomials of w-degree n is an interpolation space.**
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I. Introduction

Multivariate polynomial interpolation of functions is usually used in practical problems in which is required the approximation of an unknown multivariate function with a polynomial, matching the initial function on a set of functionals, which represent the interpolation conditions. Let be $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset R^d$ a set of arbitrary linear functionals and $\mathcal{F} \supset \Pi^d$ a space of functions which includes polynomials. The polynomial interpolation problem is to find a polynomial subspace $\mathcal{P}$ such that for an arbitrary function $f \in \mathcal{F}$ there exists an unique polynomial $p \in \mathcal{P}$ such that $\lambda_i(f) = \lambda_i(p), \ \forall\ i \in \{1, \ldots, n\}$. In this case the pair $(\Lambda, \mathcal{P})$ is called correct.

Various interpolation schemes were studied, connected to various modelling problems:

1) Lagrange interpolation, with the set of conditions
$$\Lambda = \{\delta_\theta(f) = f(\theta) | \theta \in \Theta\}.$$
(see [1], [2], [3], [4], [5], etc).
2) Hermite interpolation, defined in many ways and involving certain derivatives of the unknown function. A general way of describing the Hermite conditions is given in [**?**]:
$$\Lambda = \left\{\lambda_{q,\theta} \,|\lambda_{q,\theta}(p) = \left(q(D)p\right)(\theta)\right\},$$
$q \in \mathcal{P}_\theta;\ \theta \in \Theta;\ \mathcal{P}_\theta \subset \Pi.$
Other definition can be found in [7] and uses chains of derivatives, organized in a tree.
3) An interpolation scheme, for a set of general functionals, Λ, named least interpolation scheme, is given in [3]. A minimal interpolation space for Λ is
$$H_\Lambda{\downarrow} = span\{g{\downarrow}; g \in H_\Lambda\} \tag{1}$$
with
$$H_\Lambda = span\{\lambda^\nu;\ \lambda \in \Lambda\} \tag{2}$$
λ^ν being the generating function of the functional λ and $g{\downarrow}$ the least term of g (the homogeneous polynomial of minimal degree from Taylor series of g).
4) Ideal interpolation schemes - are the interpolation schemes having the property that $ker(\Lambda)$ is an ideal of polynomials. Ideal interpolation schemes represent in fact Hermite type interpolation schemes (see [9]). Multivariate ideal interpolation schemes are deeply connected with H-bases. Any ideal interpolation space with respect to a set of conditions Λ, can be obtained like a space of reduced polynomials modulo a H-basis of the ideal $ker(\Lambda)$ (see [8]).
We remaind that a finite set of polynomials,
$$\mathcal{H} = \{h_1, \ldots, h_s\} \subset \Pi \setminus \{0\},$$
is a H-basis for the ideal $I =< \mathcal{H} >$ if and only if $\{p{\uparrow} \mid p \in I\} =< p{\uparrow} \mid p \in \mathcal{H} >$, with $p{\uparrow}$ the upper term of the polynomial p, that is the maximum degree homogeneous part of p.

D. Simian is with the Department of Computer Science, Faculty of Science, University Lucian Blaga of Sibiu, Romania e-mail: dana.simian@ulbsibiu.ro
C. Simian is with the Faculty of Science, University Lucian Blaga of Sibiu, Romania e-mail: corinafirst@yahoo.com

Both in the least interpolation scheme and in ideal interpolation schemes, the definition of the degree notion is determinant.

In 1994, T. Sauer proposed, in [8], a generalization of the degree, using a weight $w \in N^d$.

Definition 1: (T. Sauer, [8]) The w-degree of the monomial x^α is
$$\delta_w(x^\alpha) = w \cdot \alpha = \sum_{i=1}^{d} w_i \cdot \alpha_i.$$
$\forall\ \alpha \in N^d,\ w = (w_1, \ldots, w_d) \in N^d,\ x \in R^d.$

We will use the notations:
$$A^0_{n,w} = \{\alpha \in N^d \mid w \cdot \alpha = n\},\ w \in (N^*)^d,\ n \in N \tag{3}$$
$$r_w(n) = \#(A^0_{n,w}) \tag{4}$$
$$N_A = \{n \in N \mid \exists \alpha \in A^0_{n,w}\} \tag{5}$$
A Γ-grading is defined as follows. Let $(\Gamma, +)$ denotes an orderer monoid, with respect to the total ordering $\prec$, such that: $\alpha \prec \beta \Rightarrow \gamma + \alpha \prec \gamma + \beta,\ \forall\ \alpha, \beta, \gamma \in \Gamma$.

Definition 2: (T. Sauer, [10]) A direct sum $\Pi = \bigoplus_{\gamma \in \Gamma} \mathcal{P}^{(\Gamma)}_\gamma$ is called a grading induced by Γ, or a Γ-grading, if $\forall\ \alpha,\ \beta \in \Gamma$
$$f \in \mathcal{P}^{(\Gamma)}_\alpha,\ g \in \mathcal{P}^{(\Gamma)}_\beta \Rightarrow f \cdot g \in \mathcal{P}^{(\Gamma)}_{\alpha+\beta} \tag{6}$$

The total ordering induced by Γ gives the notion of degree for the components in $\mathcal{P}_\gamma^{(\Gamma)}$. Each polynomial $f \neq 0$, has a unique representation

$$f = \sum_{i=1}^{s} f_{\gamma_i}, \ f_{\gamma_i} \in \mathcal{P}_{\gamma_i}^{(\Gamma)}; \quad f_{\gamma_i} \neq 0 \tag{7}$$

The terms f_{γ_i} represent the Γ- homogeneous terms of degree γ_i.

Assuming that $\gamma_1 \prec \ldots \prec \gamma_s$, the Γ- homogeneous term f_{γ_s} is called the leading term or the maximal part of f, denoted by $f^{(\Gamma)}\uparrow$.

The w - degree induces on the space of polynomials in d variable a N_A-grading, in the sense given before, that is:

$$\Pi = \bigoplus_{\gamma \in N_A} \mathcal{P}_\gamma^{(N_A)} \tag{8}$$

Let be

$$\Pi_{n,w} = \left\{ \sum_{w\cdot\alpha \leq n} c_\alpha x^\alpha \mid c_\alpha \in R, \ \alpha \in N^d \right\} \tag{9}$$

The polynomial homogeneous subspace of w- degree n can be rewritten as:

$$\Pi_{n,w}^0 = \left\{ \sum_{\alpha \in A_{n,w}^0} c_\alpha x^\alpha \mid c_\alpha \in R, \ \alpha \in N^d \right\}$$

We observe that $r_w(n) = d_{n,w}^0$ is the dimension of the w-homogeneous subspace $\Pi_{n,w}^0$.
Will we denote by

$$f^{[j]_w}(x) = \sum_{\alpha\cdot w = j} \frac{(D^\alpha f)(0) x^\alpha}{\alpha!},$$

the w-homogeneous part of f and by $f\downarrow_w$, the w-least term of f, that is the term with the lest w-degree in Taylor series of f.

These type of conditions appear in practical applications in which we have spatial and temporal interpolation conditions.

The paper is organize as follows. In section II we prove three new results concerning the bivariate polynomial spaces of w-degree. In section III we introduce and analyze many interpolation schemes defined using the w - degree of polynomials.

II. Some results concerning the space of polynomials of w - degree

The dimensions of the homogeneous w- space, $r_w(n)$ and the set $A_{n,w}^0$ depend of the weight $w = (w_1, w_2) \subset Z_+^2$. In [12], we found a general expression for $r_w(n)$ and $A_{n,w}^0$, for arbitrary $w_1, \ w_2$ and implemented two variants of algorithms based on this expression. These results are given in theorem 1.

Theorem 1: ([12]) Let $w = (w_1, w_2) \in (N^*)^2$ and let consider the functions:

$$r : \{0, \ldots, w_1 - 1\} \to \{0, \ldots, w_1 - 1\}; \quad r(i) = (iw_2) \, mod \, w_1$$

$$\tilde{r} : \{0, \ldots, w_2 - 1\} \to \{0, \ldots, w_2 - 1\}; \quad \tilde{r}(i') = (i'w_1) \, mod \, w_2$$

Then

1) $r_w(0) = 1$ and $(0,0) \in A_{0,w}^0$.
2) If $j = cw_1$, then $r_w(j) = \left[\frac{c}{w_2}\right] + 1$, and $(\alpha_1, \alpha_2) \in A_{j,w}^0$ are given by
$\alpha_2 = kw_1, \text{ with } 0 \leq k \leq \left[\frac{c}{w_2}\right]; \ \alpha_1 = \frac{j - w_2\alpha_2}{w_1}$.
3) If $j = cw_2$, then $r_w(j) = \left[\frac{c}{w_1}\right] + 1$, and $(\alpha_1, \alpha_2) \in A_{j,w}^0$ are given by
$\alpha_1 = kw_2, \text{ with } 0 \leq k \leq \left[\frac{c}{w_1}\right]; \ \alpha_2 = \frac{j - w_1\alpha_1}{w_2}$.
4) For any j, $0 < j < \min(w_1, w_2)$, $r_w(j) = 0$.
5) If $j \not\vdots w_1$ and $j \not\vdots w_2$, with $j \geq \min(w_1, w_2)$, then $r_w(j) = \#(M_1)$ with $M_1 = \left\{ \left[0, \left[\frac{j - iw_2}{w_1 w_2}\right]\right] \cap N \right\}$, if $\frac{j - iw_2}{w_1 w_2} \geq 0$, where $[\cdot]$ is the integer part function, $i = r^{-1}(s)$, with $s = j \ mod \ w_1$, and $(\alpha_1, \alpha_2) \in A_{j,w}^0$ are given by $\alpha_1 = \frac{j - (q_1 w_1 + i) w_2}{w_1}$, with $q_1 \in M_1$ and $\alpha_2 = \frac{i - w_1\alpha_1}{w_2}$.
6) If $j \not\vdots w_1$ and $j \not\vdots w_2$, with $j \geq \min(w_1, w_2)$, then $r_w(j) = \#(M_2)$ with $M_2 = \left\{ \left[0, \left[\frac{j - i'w_1}{w_1 w_2}\right]\right] \cap N \right\}$, if $\frac{j - i'w_1}{w_1 w_2} \geq 0$, $i' = \tilde{r}^{-1}(p)$, with $p = j \ mod \ w_2$, and $(\alpha_1, \alpha_2) \in A_{j,w}^0$ are given by $\alpha_2 = \frac{j - (q_2 w_2 + i') w_1}{w_2}$, with $q_2 \in M_2$ and $\alpha_1 = \frac{i - w_2\alpha_2}{w_1}$.
7) If $j \not\vdots w_1$ and $j \not\vdots w_2$, with $j \geq \min(w_1, w_2)$ and $\min\{j - iw_1, j - i'w_2\} < 0$, i, i' defined in the previous statements of theorem, then $r_w(j) = 0$.

The following theorem proves that for obtaining the values of $r_w(n)$ it is sufficient to apply theorem 1 only for the case $n < w_1 w_2$ and then a recursive calculus can be performed.

Theorem 2: With the notations from theorem 1, if $n \geq w_1 w_2$, then

$$r_w(n) = r_w(n \ mod \ w_1 w_2) + n \ div \ w_1 w_2, \tag{10}$$

with $n \ div \ w_1 w_2 = \left[\frac{n}{w_1 w_2}\right]$ and $[\cdot]$ the integer part function.

Proof: Let be $n = w_1 \cdot w_2 \cdot q + p$, that is $q = n \ div \ w_1 \cdot w_2$ and $p = n \ mod \ w_1 \cdot w_2$.

1) If $n = c \cdot w_1$, then $p = w_1 \cdot c_1$, that is $c = w_1(w_2 \cdot q + c_1)$. From theorem 1 we obtain :
$$r_w(n) = q + \left(1 + \left[\frac{c_1}{w_2}\right]\right) = q + r_w(p)$$
2) If $n = c \cdot w_2$, then $p = w_2 \cdot c_2$, that is $c = w_2(w_1 \cdot q + c_2)$. From theorem 1 we obtain :
$$r_w(n) = q + \left(1 + \left[\frac{c_2}{w_1}\right]\right) = q + r_w(p)$$
3) If $n \not\vdots w_1$ and $n \not\vdots w_2$, than let be $n = jw_1 w_2 + i$, $i < w_1 w_2$. and let be $m = (j+1) w_1 w_2 + i$. We will use induction on n and the following result proved in [12]:
$$r_w(m) = r_w(n) + 1 \tag{11}$$

In practical problems, we are first interested in the case $w_1 = 1$ or $w_2 = 1$. Some results related to this case are presented next.

Theorem 3: If $w = (1, w_2) \in Z_+^2$, $w_2 > 1$, then the dimension of the w - homogeneous polynomial space of degree n and the exponents of the monomials which generate this space, are given by

$$r_w(n) = 1 + q \tag{12}$$

$$A_{n,w}^0 = \{(j, 0), (j - w_2, 1), \ldots, (j - qw_2, q)\}, \tag{13}$$

with $q = \left[\frac{n}{w_2}\right]$.
If $w = (w_1, 1) \in Z_+^2$, $w_1 > 1$, then

$$r_w(n) = 1 + q$$

$$A_{n,w}^0 = \{(0, j), (1, j - w_1), \ldots, (q, j - qw_1)\}. \tag{14}$$

Proof: Let $w = (w_1, w_2)$, with $w_1 = 1$. Then $n = n \cdot w_1$, and according theorem 1, $r_w(n) = \left[\frac{n}{w_2}\right] + 1 = q + 1$, $\alpha_2 \in \{0, \ldots, q\}$, $\alpha_1 = n - w_2\alpha_2$.
A similar proof can be made for $w_2 = 1$.

Theorem 4: If $w = (1, w_2) \in Z_+^2$, $w_2 > 1$, then the dimension of the w - polynomial space of degree n is

$$d_{w,n} = dim(\Pi_{n,w}) = \frac{(q + 1)(w_2 q + 2r)}{2}, \tag{15}$$

with $q = \left[\frac{n}{w_2}\right]$ and $r = n \ mod \ w_2$.

Proof: $dim(\Pi_{n,w}) = \sum_{j=0}^{n} dim(\Pi_{n,w}^0)$. From (13) we observe that $dim(\Pi_{iw_2,w}^0) = dim(\Pi_{(i+1)w_2,w}^0) = \ldots = dim(\Pi_{(i+w_2-1)w_2,w}^0) = i + 1$, $\forall \ i \geq 0$. Therefore $d_w = 1 \cdot w_2 + \ldots + q \cdot w_2 + (q+1) \cdot r = w_2 \cdot \frac{q(q+1)}{2} + (q+1) \cdot r$.

III. The interpolation problems

We consider, in the space of polynomial of w-degree, many interpolation problems having the conditions of type

$$\lambda_{j,k} = f^{[j]_w}(\theta_{j,k}), \tag{16}$$

$\theta_{j,k} \in \Theta \subset R^2, j \in \{1, \ldots, n\}$.

First, we consider the interpolation problem with the conditions:

$$\Lambda_{o,w} = \{\lambda_j(f) = f^{[j]_w}(\theta_j), \theta_j \in \Theta \subset R^2\}, \tag{17}$$

$j \in \{0, \ldots, n\}$. We want to find an interpolation polynomial with minimum w- degree, for these conditions. In order to do this we generalize least interpolation (see [3]) for the space of polynomials with w-degree.
We introduce the following notation:

$$< f, p >= (p(D)f)(0) = \sum_{\alpha \in N^2} \frac{D^\alpha p(0) D^\alpha f(0)}{\alpha!}. \tag{18}$$

The generating function of the functional λ_j is :

$$\lambda_j^\nu(z) =< \lambda_j, e_z > \tag{19}$$

The spaces from (1)-(2) become

$$H_{\Lambda_{o,w}} = span\left\{p_j(x) = \sum_{\alpha \cdot w = j} \frac{\theta_j^\alpha \cdot x^\alpha}{\alpha!}\right\}; \tag{20}$$

$$H_{\Lambda_{o,w}}\downarrow_w = span\{p_j\downarrow_w |p_j(x) \in H_{\Lambda_{o,w}}\} \tag{21}$$

$j \in \{0, \ldots, n\}$. Obviously $H_{\Lambda_{o,w}} = H_{\Lambda_{o,w}}\downarrow_w$, be cause it is generated by w- homogeneous polynomials.

For any $f \in \mathcal{A}_0$, we have,

$$\lambda_j(f) = f^{[j]_w}(\theta_j) =< f, p_j >= (p_j(D)f)(0) \tag{22}$$

Let $L_{\Lambda_{o,w}}$ be the interpolation operator for the conditions (17).

Theorem 5: The operator $L_{\Lambda_{o,w}}$ has the expression:

$$L_{\Lambda_{o,w}} = \sum_{j=0}^{n} p_j \cdot \frac{< p_j, f >}{< p_j, p_j >}, \tag{23}$$

with p_j given in (20).

Proof: We have that $< p_j, p_i > \neq 0 \iff j = i$. The following equality holds, for all $j \in \{0, \ldots, n\}$ and $f \in \mathcal{A}_0$:

$$< L_{\Lambda_{o,w}}, p_i >=< f, p_i > \tag{24}$$

By a simple computation, we obtain:
$\lambda_j(L_{\Lambda_{o,w}}(f)) =< L_{\Lambda_{o,w}}, p_j >=< f, p_i >= f^{[j]}(\theta_j)$.

Theorem 6: The fundamental interpolation polynomials, φ_i, $i \in \{0, \ldots, n\}$, for the interpolation scheme $(\Lambda_{o,w}, H_{\Lambda_{o,w}})$ are given by

$$\varphi_i = \frac{p_i}{< p_i, p_i >} \tag{25}$$

Proof: $\lambda_j(\varphi_i) = \left\langle p_j, \frac{p_i}{< p_i, p_i >}\right\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol.
Next, we will considerate the case $w_1 = 1$ or $w_2 = 1$.

Proposition 1: If in the interpolation problem with conditions (17) $w_1 = 1$ or $w_2 = 1$, then the maximum degree of the interpolation polynomial $(L_{\Lambda_{o,w}})(f)$ is n.

Proof: The degree of the interpolation polynomial is given by the maximum degree of p_n. Taking into account the theorem 3, p_n is a linear combination of the monomials which exponents are given in (13) or (14). The maximum degree of these monomials is n.

Let observe that $d_{n,w} = dim(\Pi_{n,w}) \neq \#(\Lambda)$. So the interpolation space falls to be $\Pi_{n,w}$.

We want to find a set of conditions for which $\Pi_{n,w}$ is an interpolation space. A necessary condition for this is that $d_{n,w} = \#(\Lambda)$. We consider the set of condition:

$$\Lambda_{w,\Theta} = \{\lambda_{j,k}(f) = f^{[j]_w}(\theta_{j,k})\}, \tag{26}$$

with $j \in \{0, \ldots, n = q \cdot w_2 + r\}$, $k \in \{1, \ldots d_{j,w}^0\}$, $\Theta = \{\theta_{j,k}\} \subset R^2$ a set of points having the following properties:

1) $\theta_{j,i} \neq \theta_{j,k}, \ \forall i \neq k$.
2)

$$\Delta_j = \left|\theta_{j,k}^\alpha\right| \neq 0, \tag{27}$$

$j \in \{0, \ldots, n\}$, $k \in \{1, \ldots d_{j,w}^0\}$,
$\alpha \in A_{j,w}^0 = \{(j - i \cdot w_2, i) | i = 0, \ldots, q\}$,

Theorem 7: The interpolation problem with conditions Λ_w, given in (26), has an unique solution in the space of polynomials of w-degree n, with $w = (1, w_2)$, $w_2 > 1$.

Proof: Let be $p = \sum_{\alpha \in A_{n,w}} c_\alpha x^\alpha$ the interpolation polynomial. We look for its coefficients, c_α. The interpolation conditions leads to $n+1$ Cramer systems, having $d^0_{j,w}$ equations and the determinant $\Delta_j \neq 0$, $j \in \{0, \ldots, n\}$. Hence, all of these systems have an unique solution.

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