# One-dimensional parabolic equation with a discontinuous nonlinearity and integral boundary conditions

Abdelkader Boucherif

**Abstract**— In this paper we are concerned with the existence of solutions of an initial-boundary value problem for a one-dimensional parabolic inclusion with nonlocal integral boundary conditions. Using the Green's function we transform the problem into an equivalent integral inclusion. Our technique is based on fixed point theorems for set-valued maps and the method of lower and upper solutions. We provide sufficient conditions that guarantee the existence of at least one solution.

**Keywords**—parabolic inclusion, integral boundary conditions, Green's function, set-valued maps, fixed point theorems, lower and upper solutions.

## I. INTRODUCTION

Consider the following one-dimensional parabolic inclusion subjected to integral boundary conditions.

$$u_t - u_{xx} \in F(x,t,u), 0 < x < \pi, 0 < t < 1, \quad (1)$$

$$u(x,0) = u_0(x), \qquad 0 \le x \le \pi, \quad (2)$$

$$u(0,t) = \int_0^\pi g(u(x,t))dx,\ 0 < t < 1, \quad (3)$$

$$u(\pi,t) = \int_0^\pi h(u(x,t))dx,\ \ 0 < t < 1, \quad (4)$$

where $u_0,\ g, h$ are given functions and $F$ is a multivalued map satisfying some conditions that will be specified later.

Manuscript received May 15, 07, Revised Dec.19, 07. Supported by a grant from the Arab Fund Fellowship Program, Kuwait. The author is with the Division of Applied Mathematics, Brown University, Providence, RI 02912 USA (phone: 401-863-7422; fax: 401-863-1355; e-mail: Abdelkader_Boucherif@brown.edu), on leave from the Department of Mathematics and Statistics, Box 5046, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia (e-mail: aboucher@kfupm.edu.sa).

Parabolic problems with discontinuous nonlinearities arise naturally in chemical reactor theory, porous medium combustion (see [11] and [12]); also in best response dynamics arising in game theory (see [15]), and the references therein. Parabolic equations with discontinuous nonlinearities generated by increasing functions of bounded variations have been investigated in [24], [22] and [4]. Parabolic problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [3], [16], [5], thermoelasticity [7]. Several papers have been devoted to the study of parabolic problems with integral conditions [6], [20], [25]. A good account on numerical treatment of parabolic problems with integral conditions can be found in [8].

In this paper we consider an intial boundary value problem for the one-dimensional heat equation with a convex multivalued right hand side and subjected to integral boundary conditions. We shall convert Problem (1), (2), (3), (4) to an integral inclusion using the properties of the Green's function corresponding to the linear problem. We, then, provide sufficient conditions on the data that will enable us to obtain a priori bounds on possible solutions of a one-parameter family of problems related to the original one. Our approach is based on fixed point theorems for suitable multivalued operators.

The outline of the paper is as follows. Section 2 is devoted to the study of the linear nonhomogeneous problem and the properties of the Green's function. In section 3, we shall recall the main properties of multivalued maps. We state and prove our main results in section 4.

## II. LINEAR NONHOMOGENEOUS PROBLEM

In this section we consider the linear nonhomogeneous problem

$$u_t - u_{xx} = f(x,t), 0 < x < \pi, 0 < t < 1, \quad (5)$$

$$u(x,0) = u_0(x), \qquad 0 \le x \le \pi, \quad (6)$$

$$u(0,t) = a(t), \qquad 0 < t < 1, \tag{7}$$

$$u(\pi,t) = b(t), \qquad 0 < t < 1, \tag{8}$$

We say that $u \in C^{2,1}(D)$ if $u$ has a continuous second order partial derivative with respect to $x$ and a continuous first order partial derivative with respect to $t$. Let $X = C(\overline{D})$ be the Banach space of real-valued continuous functions on $\overline{D}$, equipped with the norm $\|u\|_\infty = \max\{|u(x,t)|;\ (x,t) \in D\}$ for $u \in X$. A strong solution of the above problem is a function $u \in C^{2,1}(D) \cap C(\overline{D})$. The following result can be found in [13] and [21].

Assume that the functions $f,\ u_0$, are Hölder continuous, and the functions $a$ and $b$ are continuous. Then, Problem (5), (6), (7), (8) has a unique strong solution given by for each $(x,t) \in D = (0,\pi) \times (0,1)$,

$$\begin{aligned} u(x,t) &= \textstyle\int_0^t \int_0^\pi G(x,t;y,s)\ f(y,s)\, dyds \\ &+ \textstyle\int_0^\pi G(x,t;y,0)\ u_0(y)\, dy \\ &+ \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;0,s)\ a(s)ds \\ &- \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s)\ b(s)ds, \end{aligned} \tag{9}$$

where $G(x,t;y,s)$ is the Green's function corresponding to the linear homogeneous problem. This function satisfies the following

(i) $G_t - G_{xx} = \delta(t-s)\,\delta(x-y) \quad s < t,$ $0 < x, y < \pi$

(ii) $G(x,t;y,s) = 0 \quad s > t, 0 < x, y < \pi$

(iii) $G(0,t;y,s) = G(\pi,t;y,s) = 0 \quad s < t$

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in D$

(v) $G,\ G_t,\ G_x,\ G_{xx}$ are continuous functions of $(x,t),(y,s) \in D,\ t-s > 0.$

(vi) there exist $d_0 > 0$ and $\mu \in (0,1)$ such that

$$|G(x,t;y,s)| \le \frac{d_0\,|x-y|^{1-\mu}}{(t-s)^\mu},$$

and $\left|\dfrac{\partial G}{\partial y}(x,t;y,s)\right| \le \dfrac{d_0\,|x-y|^{\kappa-2+2\mu}}{(t-s)^\mu}$, with $1 - \frac{\kappa}{2} < \mu < 1.$

**Lemma 1** *Let $y_0$ be a fixed number in $[0,\pi]$.*

Then there exists a constant $\delta_0 > 0$ such that $\max_{(x,t)\in D} \int_0^t \left|\dfrac{\partial G}{\partial y}(x,t;y_0,s)\right| ds \le \delta_0.$

Proof. This follows from the estimate on $\left|\dfrac{\partial G}{\partial y}(x,t;y,s)\right|$.

We write (9) in the following convenient form, for each $(x,t) \in D$,

$$u(x,t) = G\,(f\ + u_0)\,(x,t) + \gamma\,(a,b)\,(x,t) \tag{10}$$

where

$$\begin{aligned} &G\,(f\ + u_0)\,(x,t) = \\ &\textstyle\int_0^t \int_0^\pi G(x,t;y,s)\ f(y,s)\, dyds \\ &+ \textstyle\int_0^\pi G(x,t;y,0)\ u_0(y)\, dy, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\gamma\,(a,b)\,(x,t) = \\ &\textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;0,s)\ a(s)ds \\ &- \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s)\ b(s)ds. \end{aligned} \tag{12}$$

The operators $G,\ \gamma$ map $C(\overline{D})$ into $C^{2,1}(D)$. Moreover, $v = G\,(f\ + u_0)$ solves the problem $v_t - v_{xx} = f,\ v(x,0) = u_0(x)$ and $w = \gamma\,(a,b)$ solves the problem $w_t - w_{xx} = 0,\ w(x,0) = 0,\ w(0,t) = a(t),\ w(\pi,t) = b(t).$

## III. MULTIVALUED FUNCTIONS

We, now, introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books [1], [2] and [9].

Let $(Y, |\cdot|)$ be a normed space. We shall denote the set of all subsets of $Y$ having property $\ell$ by $P_\ell(Y)$. For instance, $U \in P_{cl}(Y)$ means $U$ closed in $Y$; when $\ell = b$ we have the bounded subsets of $Y$, $\ell = cv$ for convex subsets, $\ell = cp$ for compact subsets and $\ell = cp, cv$ for compact and convex subsets. A multivalued map $R : Y \to 2^Y$ is convex (closed) valued if $R(z)$ is convex (closed) for each $z \in Y$. $R$ is bounded on bounded sets if $R(B) = \cup_{z\in B} R(z)$ is bounded in $Y$ for all $B \in P_b(Y)$ (i.e. $\sup_{z\in B}\{\sup\{|y|\,; y \in R(z)\}\} < \infty$). The multivalued map $R$ is called upper semicontinuous (usc) on $Y$ if for each $z \in Y$ the set $R(z) \in P_{cl}(Y)$ and is nonempty, and for each open subset $\Lambda$ of $Y$ containing $R(z)$, there exists an open neighborhood $\Pi$ of $z$ such that $R(\Pi) \subset \Lambda$. The set-valued map $R$ is called completely continuous if

$R(B)$ is relatively compact for every $B \in P_b(Y)$. If $R$ is completely continuous with nonempty compact values, then $R$ is usc if and only if $R$ has a closed graph (i.e. $z_n \to z,\ w_n \to w$, $w_n \in R(z_n) \Rightarrow w \in R(z)$). $R$ has a fixed point if there exists $z \in Y$ such $z \in R(z)$. A multivalued map $R : \overline{D} \to P_{cl}(\mathbb{R})$ is called measurable if for every $\theta \in \mathbb{R}$, the function $v \longmapsto dist(\theta, R(v)) = \inf\{|\theta - z|\,; z \in R(v)\}$ is measurable.

**Definition 2** $F : D \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ *is called*

an $L^2-$Carathéodory multifunction if
(i) $F(.,.,u) : D \to 2^{\mathbb{R}}$ is measurable for all $u \in \mathbb{R}$,
(ii) $F(x,t,.) \to \mathbb{R} \to 2^{\mathbb{R}}$ is usc for almost all $(x,t) \in D$
(iii) for each $\varrho > 0$ there exists $\omega_\varrho \in L^2(D)$ such that $|u| \leq \varrho$ implies
$|F(x,t,u)| := \{|w|\,; w \in F(x,t,u)\} \leq \omega(x,t)$ for a.e.$(x,t) \in D$.

**Definition 3** *Let $u \in X$. Then $S_{F,u}$ denotes the set of $L^2-$selections of the set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}}$, and is the set*

$\{w \in L^2(D);\ w(x,t) \in F(x,t,u(x,t))\,,\ \forall (x,t) \in D\}$.
The fact that this set is not empty follows from Lemma 3 in [18].

**Definition 4** *Let $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ have nonempty compact values. The Nemitsky operator $\mathcal{F}$ of $F$ is the set-valued operator defined by*

$\mathcal{F} : C(D) \to L^2(D),\ \mathcal{F}(u)$ is the set of all

$w : D \to \mathbb{R}$ measurable such that
$w(x,t) \in F(x,t,u(x,t))\,, \forall (x,t) \in D$.
It can be shown (see [14, page 40], [23]) that if $F$ is usc with convex bounded values then the operator $\mathcal{F}$ is well defined, usc, bounded on bounded sets in $C(D)$, and has convex values.

**Definition 5** *$u$ is a strong solution of (1), (2), (3), (4) if there exists a Lipschitz selection $f \in S_{F,u}$ and $u$ has the integral representation (9).*

**Remark**. If $F$ is a Lipschitz multifunction then it admits a Lipschitz selection. See [17].

**Definition 6** *Let $(Z,d)$ be a metric space and let $A, B$ be two nonempty subsets of $Z$. The Hausdorf distance between $A$ and $B$ is defined by*

$$d_H(A,B) = \max\{\sup_{a \in A} d(a,B),\ \sup_{b \in B} d(A,b)\}.$$

Here $d(a,B) = \inf\{d(a,b); b \in B\}$.Then $(P_{cl,b}(Z),\ d_H)$ is a metric space.

**Definition 7** *A multivalued operator $\mathcal{L}$: $Z \to P_{cl}(Z)$ is called*

(i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H(\mathcal{L}(u), \mathcal{L}(v)) \leq \delta\ d(u,v)$ for all $u, v \in Z$

(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

The following theorems play an important role in our existence results.

**Theorem 8** *[19] Let $E$ be a Banach space and $\mathcal{L} : E \to P_{cp,cv}(E)$ a condensing map. If the set $S := \{z \in E;\ \lambda z \in \mathcal{L}(z)$ for some $\lambda > 1\}$ is bounded, then $\mathcal{L}$ has a fixed point.*

We remark that a compact map is the simplest example of condensing maps.

**Theorem 9** *[10]Let $B_r(0)$ and $\overline{B_r(0)}$ denote respectively the open and closed balls in a Banach space $(E, \|\cdot\|)$ centered at $0$ and having radius $r$. Let $\mathcal{L}_1 : \overline{B_r(0)} \to P_{cl,cv,b}(E)$ and $\mathcal{L}_2 : \overline{B_r(0)} \to P_{cp,cv}(E)$ be two multivalued operators satisfying*

(i) $\mathcal{L}_1$ is a contraction,
(ii) $\mathcal{L}_2$ is compact and usc.
Then either
(j) the operator inclusion $u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ has a solution in $\overline{B_r(0)}$, or
(jj) there exists $u \in E$ with $\|u\| = r$ such that $\lambda u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ for some $\lambda > 1$.

**Theorem 10** *[14, page 11]Let $E$ be a normed linear space, $C$ convex subset in $E$ and $U$ open in $C$ with $0 \in U$. Let $\Lambda : U \to 2^C$ be an usc, compact multivalued operator with closed and convex values. Then either*

(a) $\Lambda$ has a fixed point, or
(b) there exists $z \in \partial U$ such that $z \in \lambda \Lambda z$ for some $\lambda \in (0,1)$.

## IV. MAIN RESULTS

In this section, we shall state and prove our main results.We shall assume throughout the remainder of the paper that the following conditions hold.

(H0) The multifunction $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ is nonempty, usc and has compact and convex values. Moreover, there exists a Lipschitz selection $f \in S_{F,u}$ for each $u \in X$.

(H1) $u_0 \in C([0,\pi])$.

**Theorem 11** *Suppose that, in addition to (H0) and (H1) the following assumptions are satisfied,*

(H2) $g,\ h : C(D) \to \mathbb{R}$ are continuous and bounded,

(H3) $F$ maps bounded sets into relatively compact sets, and there are positive constants $c_1,\ c_2$ such that $|F(x,t,u)| \le c_1 + c_2 |u|$.

Then Problem (1), (2), (3), (4) has at least one strong solution.

**Proof.** It follows from (9), (10), and (11) that $u$ is a solution of (1), (2), (3), (4) if and only if $u$ is a fixed pont of the multivalued operator $\mathcal{L}$, defined by

$$\mathcal{L}u = G\mathcal{F}(u) + \gamma(u), \tag{13}$$

where $\mathcal{F}$ is the Nemitski operator of $F$.

In fact, we have

$$\mathcal{L}u(x,t) = \int_0^t \int_0^\pi G(x,t;y,s)F(y,s,u(y,s))dyds + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds,$$

where

$$\int_0^t \int_0^\pi G(x,t;y,s)\,(\mathcal{F}u)(y,s)\,dyds$$

is the Aumann integral of $\mathcal{F}$. We see that $\mathcal{L}$ is the sum of a multivalued operator $G\mathcal{F}$ and a single valued operator $\gamma(\cdot)$. We apply Theorem 6 to the operator $\mathcal{L}$.

Let $u \in X$. We show that $\mathcal{L}u \in P_{cp,cv}(X)$.

(a) $\mathcal{L}u$ is a convex subset of $X$ for each $u \in X$. Let $v_1, v_2 \in \mathcal{L}u$. Then there exists $w_1, w_2 \in S_{F,u}$ such that for each $(x,t) \in D$ we have for $i = 1,2$

$$\begin{aligned} v_i(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w_i(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

Since $S_{F,u}$ is convex, it is clear from the above relation that any convex combination of $v_1, v_2$ is an element of $\mathcal{L}u$.

(b) $\mathcal{L}u$ is a compact subset of $X$ for each $u \in X$. Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $S_{F,u}$. By (H0) and (H3) the Nemitski operator $\mathcal{F}$ of $F$ is well defined, usc and maps bounded sets into relatively compact sets. The sequence $(v_n)_{n\in\mathbb{N}}$ given by, for each $n \in \mathbb{N}$

$$\begin{aligned} v_n(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w_n(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

is relatively compact in $\mathcal{L}u$. This implies that $\mathcal{L}u$ is a compact subset of $X$.

(c) We show that $\mathcal{L} = G\mathcal{F} + \gamma(\cdot)$ is a compact operator. To achieve this, we show that $\mathcal{L}$ is uniformly bounded and maps bounded sets into equicontinuous sets.

Let $B$ be a bounded subset of $X$, and let $u \in B$. Then there is $M > 0$ such that $\|u\|_\infty \le M$.

Now, for each $v \in \mathcal{L}u$ there exists $w \in S_{F,u}$ such that

$$\begin{aligned} v(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

Hence, if $m_g$ and $m_h$ denote the bounds on $g$ and $h$ respectively,

$$|v(x,t)| \le \int_0^t \int_0^\pi G(x,t;y,s)[c_1 + c_2 |u(y,s)|]dyds + \int_0^\pi G(x,t;y,0)\,|u_0(y)|\,dy + \pi m_g \max_D \int_0^1 \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| ds + \pi m_h \max_D \int_0^1 \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| ds.$$

It follows from Lemma 1

$$|v(x,t)| \le \pi\,(c_1 + \|u_0\|_\infty)\,\|G\|_\infty$$

$$+(\pi m_g + \pi m_h)\delta_0$$
$$+c_2 \int_0^1 \int_0^\pi G(x,t;y,s) \left| u(y,s) \right| dyds$$
$$\leq M_0 + c_2 \int_0^1 \int_0^\pi G(x,t;y,s) \left| u(y,s) \right| dyds$$

where
$M_0 = \pi (c_1 + \|u_0\|_\infty) \|G\|_\infty + (\pi m_g + \pi m_h)\delta_0$.
Then $|v(x,t)|$
$\leq M_0 + \pi c_2 \|G\|_\infty \|u\|_\infty . \leq M_0 + \pi c_2 \|G\|_\infty M$.
This shows that $\mathcal{L}u$ is uniformly bounded.

Next, let $(x,t)$ , $(\xi,\tau) \in D$. Then

$$|v(x,t) - v(\xi,\tau)| \leq$$
$$(c_1 + c_2 M) \int_0^1 \int_0^\pi |G(x,t;y,s) - G(\xi,\tau;y,s)| \, dyds$$

$$+ \|u_0\|_\infty \int_0^\pi G(x,t;y,0) - G(\xi,\tau;y,0) dy$$

$$+\pi m_g \int_0^1 \left| \frac{\partial G}{\partial y}(x,t;0,s) - \frac{\partial G}{\partial y}(\xi,\tau;0,s) \right| ds$$

$$+\pi m_h \int_0^1 \left| \frac{\partial G}{\partial y}(x,t;\pi,s) - \frac{\partial G}{\partial y}(\xi,\tau;\pi,s) \right| ds$$

It follows from the properties of the Green's function that, as $|x-\xi| + |t-\tau| \to 0$, the right hand of the last inequality tends to zero. This shows that $\mathcal{L}u$ is equicontinuous.

(d) Now, consider the set $S = \{u \in X; \lambda u \in \mathcal{L}u$, for some $\lambda > 1\}$. We show that this set is bounded.
We proceed as before to obtain

$$|\lambda u(x,t)| \leq$$
$$M_0 + c_2 \int_0^1 \int_0^\pi G(x,t;y,s) \left| u(y,s) \right| dyds$$

Since $\lambda > 1$ it follows from the above inequalities that,

$$|u(x,t)| \leq$$
$$M_0 + c_2 \int_0^1 \int_0^\pi G(x,t;y,s) \left| u(y,s) \right| dyds.$$

Gronwall's inequality implies

$$\|u\|_\infty \leq M_0 \exp\left(\pi c_2 \|G\|_\infty\right).$$

Therefore the set $S$ is bounded and consequently, $\mathcal{L}$ has a fixed point in $X$. This fixed point is the solution to our original problem.□

For our second result, we shall assume, in addition to (H0) and (H1), that the following conditions are satisfied.

(H4) $g$ and $h$ are Lipschitz continuous, with Lipschitz constants $k_g$ and $k_h$ respectively, with

$$\Delta := (k_g + k_h)\delta_0 < 1$$

and further $g(0) = h(0) = 0$.

(H5) $F$ has compact, convex values and there exists $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|F(x,t,u)| \leq \Psi(|u|)$

(H6) $\sup\limits_{\rho \in (0,\infty)} \dfrac{\rho(1-\pi\Delta)}{\pi \|G\|_\infty (\|u_0\|_\infty + \pi\Psi(\rho))} > 1$

**Theorem 12** *If the conditions (H0), (H1), (H4), (H5), and (H6) are satisfied. Then Problem (1), (2), (3), (4) has at least one solution.*

**Proof.** Condition (H6) implies that there exists $r > 0$ such that

$$\frac{r(1-\pi\Delta)}{\pi \|G\|_\infty (\|u_0\|_\infty + \pi\Psi(r))} > 1. \tag{14}$$

Consider the closed ball $\overline{B_r(0)}$ in the Banach space $X$. Let $u \in \overline{B_r(0)}$. Write $\mathcal{L}u$ as $\mathcal{L}_1 u + \mathcal{L}_2 u$,
with

$$\mathcal{L}_1 u(x,t) =$$

$$\int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s)) dyds \tag{15}$$

$$- \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s)) dyds,$$

and

$$\mathcal{L}_2 u(x,t) =$$

$$\int_0^t \int_0^\pi G(x,t;y,s) F(y,s,u(y,s)) dyds \tag{16}$$

$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy.$$

**Claim 1.** $\mathcal{L}_1 : \overline{B_r(0)} \to P_{cl,cv,b}(X)$ is a contraction.
Notice that $\mathcal{L}_1$ is a single valued operator. The continuity of the functions $g$ and $h$ implies that $\mathcal{L}_1 u \in P_{cl,cv,b}(X)$.
Now, let $u, v \in \overline{B_r(0)}$. Then

$$|\mathcal{L}_1 u(x,t) - \mathcal{L}_1 v(x,t)| \leq$$

$$\int_0^t \left| \frac{\partial G}{\partial y}(x,t;0,s) \right| \int_0^\pi |g(u(y,s)) - g(v(y,s))| \, dyds$$

$$+ \int_0^t \left| \frac{\partial G}{\partial y}(x,t;\pi,s) \right| \int_0^\pi |h(u(y,s)) - h(v(y,s))| \, dyds$$

Hence

$\|\mathcal{L}_1 u - \mathcal{L}_1 v\|_\infty \leq$

$k_g \int_0^t \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| \int_0^\pi |u(y,s) - v(y,s)|\, dyds$

$+k_h \int_0^t \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| \int_0^\pi |u(y,s) - v(y,s)|\, dyds.$

Condition (H4) implies that

$$d_H(\mathcal{L}_1 u, \mathcal{L}_1 v) = \|\mathcal{L}_1 u - \mathcal{L}_1 v\|_\infty \leq \Delta \|u - v\|_\infty .$$

Since $\Delta < 1$ it follows (see Definition 5) that $\mathcal{L}_1$ is a contraction.□

**Claim 2.** $\mathcal{L}_2 : \overline{B_r(0)} \to P_{cp,cv}(X)$ is compact and usc.

·Let $u \in \overline{B_r(0)}$. We proceed as in the proof of the previous theorem to show that $\mathcal{L}_2 u$ is a compact and convex subset of $X$.

·We show that $\mathcal{L}_2$ is a compact operator on $\overline{B_r(0)}$. For each $v \in \mathcal{L}_2 u$, there exists $w \in S_{F,u}$ such that for each $(x,t) \in D$ we have

$$v(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\, dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy.$$

Condition (H5) implies that

$$|v(x,t)| \leq \int_0^1 \int_0^\pi G(x,t;y,s)\, |w(y,s)|\, dyds$$
$$+ \int_0^\pi G(x,t;y,0)\ |u_0(y)|\, dy$$
$$\leq \int_0^1 \int_0^\pi G(x,t;y,s) \Psi(|u(y,s)|) dyds$$
$$+ \int_0^\pi G(x,t;y,0)\ |u_0(y)|\, dy$$
$$\leq \int_0^1 \int_0^\pi G(x,t;y,s) \Psi(\|u\|_\infty) dyds$$
$$+\pi \|G\|_\infty \|u_0\|_\infty$$

Thus,

$$\|v\|_\infty \leq \pi \|G\|_\infty [\Psi(r) + \|u_0\|_\infty].$$

·Next, we show that $\mathcal{L}_2$ maps bounded sets into equicontinuous subsets of $X$.

Let $(x,t)$ and $(\xi,\tau) \in D$. For each $v \in \mathcal{L}_2 u$ there is $w \in S_{F,u}$ such that

$$|w(y,s)| \leq \Psi(|u(y,s)|).$$

Thus,

$$|v(x,t) - v(\xi,\tau)| \leq$$
$$\int_0^1 \int_0^\pi |G(x,t;y,s) - G(\xi,\tau;y,s)|\, |w(y,s)|\, dyds$$
$$\leq \Psi(r) \int_0^1 \int_0^\pi |G(x,t;y,s) - G(\xi,\tau;y,s)|\, dyds.$$

The continuity of the Green's function implies that the right hand side of the above inequality tends to zero as $|x-\xi| + |t-\tau|$ tends to zero. By the Ascoli-Arzela theorem, we conclude that the operator $\mathcal{L}_2$ is compact.

·$\mathcal{L}_2$ has a closed graph. Let $(u_n, v_n) \in Gr(\mathcal{L}_2)$ converge to $(u,v)$ . We must show that $v \in \mathcal{L}_2 u$. We have $v_n \in \mathcal{L}_2 u_n$, and there exists $w_n \in S_{F,u_n}$ such that for each $(x,t) \in D$

$$\begin{aligned} v_n(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s) w_n(y,s)\, dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy. \end{aligned}$$

Obviously,

$$\|v_n - v\|_\infty \to 0 \text{ as } n \to \infty.$$

Consider the continuous operator $\Gamma : L^2(D) \to X$, defined by

$$\begin{aligned} (\Gamma w)(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\, dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy. \end{aligned}$$

Then $\Gamma \circ S_F$ has a closed graph (see [16, Theorem 2]). Also,

$$v_n \in \Gamma \circ S_{F,u_n}.$$

Since $u_n \to u$, uniformly, it follows that

$$v \in \Gamma \circ S_{F,u}.$$

Hence, there exists $w \in S_{F,u}$ such that

$$\begin{aligned} v(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\, dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\, u_0(y) . \end{aligned}$$

This shows that $v \in \mathcal{L}_2 u$, and hence $\mathcal{L}_2$ has a closed graph.

·Since $\mathcal{L}_2$ has compact values, it follows that $\mathcal{L}_2$ is usc.□

**Claim 3.** The second alternative in Theorem 7 does not hold.

Suppose, on the contrary, that there exists $u \in X$ with $\|u\|_\infty = r$ and $\lambda > 1$ such that $\lambda u \in \mathcal{L}_1 u + \mathcal{L}_2 u$. There exists $z \in S_{F,u}$ such that

$\lambda u(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) z(y,s)\, dyds$

$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy$

$+ \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds$

$- \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds.$

Then

$$\begin{aligned} & |u(x,t)| \\ \leq\ & \int_0^1 \int_0^\pi G(x,t;y,s)\, |z(y,s)|\, dyds \\ & + \int_0^\pi G(x,t;y,0)\ |u_0(y)|\, dy \\ & + \int_0^t \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| \int_0^\pi |g(u(y,s))|\, dyds \\ & + \int_0^t \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| \int_0^\pi |h(u(y,s))|\, dyds. \end{aligned}$$

Hence by (H5)

$$\begin{aligned} & |u(x,t)| \\ \leq\ & \int_0^1 \int_0^\pi G(x,t;y,s)\, |\Psi(u(y,s))|\, dyds \\ & +\pi \|G\|_\infty \|u_0\|_\infty \\ & +k_g \pi \int_0^1 \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| ds\, \|u\|_\infty \\ & +\pi k_h \int_0^1 \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| ds\, \|u\|_\infty . \end{aligned}$$

This last inequality implies that

$$\begin{aligned} & |u(x,t)| \\ \leq\ & \pi\Delta \|u\|_\infty + \pi \|G\|_\infty \|u_0\|_\infty \\ & + \int_0^1 \int_0^\pi G(x,t;y,s)\, |\Psi(\|u\|_\infty)|\, dyds \\ \leq\ & \pi\Delta r + \pi \|G\|_\infty \|u_0\|_\infty + \pi \|G\|_\infty \Psi(r). \end{aligned}$$

This last inequality infer that

$$r \leq \pi\Delta r + \pi \|G\|_\infty \|u_0\|_\infty + \pi \|G\|_\infty \Psi(r),$$

which, in turn, implies that

$$r(1-\pi\Delta) \leq \pi(\|G\|_\infty \|u_0\|_\infty + \|G\|_\infty \Psi(r)).$$

This contradicts the definition of $r$ (see(14)).
Therefore the first alternative holds, which means that $u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ has a solution in $\overline{B_r(0)}$. This proves that our problem has at least one solution.□

**Theorem 13** *Suppose that (H1) holds and $F$ is an $L^2-$Carathéodory multifunction satisfying (H0) and*

(H7) $|F(x,t,u| \leq H(x,t,|u|)$ for a.e $(x,t) \in D$, all $u \in \mathbb{R}$, where $H : D \times [0,\infty) \to [0,\infty)$ is an $L^2-$Carathéodory function, nondecreasing with respect to its third argument and such that

$\limsup_{\varrho\to\infty} \dfrac{\|G\|_\infty}{\varrho} \int_0^T \int_\Omega H(x,t,\varrho)dxdt < 1.$

(H8) $g, h$ are continuous, nondecreasing and

$\limsup_{u\to\infty} \dfrac{g(u)}{u} = 0 = \limsup_{u\to\infty} \dfrac{h(u)}{u}.$

Then problem (1), (2), (3), (4) has at least one solution.

**Proof**. For $\lambda \in [0,1]$, consider the following one-parameter family of problems

$u_t - u_{xx} \in \lambda F(x,t,u),\ 0<x<\pi,\ 0<t<1,$

$u(x,0) = \lambda u_0(x) \qquad 0 \leq x \leq \pi,$
$u(0,t) = \lambda \int_0^\pi g(u(x,t))dx,\ 0<t<1,$

$u(\pi,t) = \lambda \int_0^\pi h(u(x,t))dx,\ 0<t<1.$

Notice that this problem has only the trivial solution for $\lambda = 0$, while its solutions of are fixed points of the multivalued operator $\mathcal{L}_\lambda := \lambda G\mathcal{F} + \gamma$, where $\mathcal{L}_1 = \mathcal{L}$ is given by (13).
We have

$u(x,t) \in \lambda \int_0^t \int_0^\pi G(x,t;y,s)F(y,s,u(y,s))dyds$

$+\lambda \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy$

$+\lambda \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \int_0^\pi g(u(y,s))dyds$

$-\lambda \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \int_0^\pi h(u(y,s))dyds,$

so that $|u(x,t)|$

$\leq \int_0^t \int_0^\pi G(x,t;y,s)H(y,s,|u(y,s)|)dyds$

$+ \int_0^\pi G(x,t;y,0)\, |\, u_0(y)|\, dy$

$+ \int_0^t \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| \int_0^\pi |g(u(y,s))|\, dyds$

$+ \int_0^t \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| \int_0^\pi |h(u(y,s))|\, dyds.$

Then

$R_0 := \|u\|_\infty \leq$

$\|G\|_\infty \int_0^T \int_0^\pi H(y,s,R_0)dyds + \pi \|G\|_\infty \|u_0\|_\infty$
$+\delta_0 \pi\, (g(R_0) + h(R_0)).$

Thus

$$\begin{cases} 1 \leq \frac{\|G\|_\infty}{R_0} \int_0^T \int_0^\pi H(y,s,R_0) dyds \\ +\frac{\pi \|G\|_\infty \|u_0\|_\infty}{R_0} + \frac{\delta_0 \pi \left(g(R_0)+h(R_0)\right)}{R_0} \end{cases}$$

On the other hand, it follows from the conditions on the functions $H, g, h$ that there exists $R^* > 0$ such that for all $\varrho > R^*$ we have

$$\frac{\|G\|_\infty}{\varrho} \int_0^T \int_0^\pi H(y,s,\varrho) dyds + \frac{\pi \|G\|_\infty \|u_0\|_\infty}{\varrho}$$

$$+\frac{\delta_0 \pi \left(g(\varrho)+h(\varrho)\right)}{\varrho} < 1.$$

Comparing the last two inequalities we see that $R_0 \leq R^*$.
Hence, all possible solutions of $(17.\lambda)$ are a priori bounded, independently of $\lambda$.
Let $U := \{u \in X; \|u\|_\infty < R^* + 1\}$. Then $U$ is open in $X$ with $0 \in U$.
Assume that there exists $z \in \partial U$ such that $z \in \mathcal{L}_\lambda z$ for some $\lambda \in (0,1)$. This implies that $z$ is a solution of $(17.\lambda)$ with $\|z\|_\infty = R^* + 1$, which is not possible. This implies that the first alternative in Theorem 8 holds. Consequently, $\mathcal{L}_1 = \mathcal{L}$ has a fixed point $z_0$, which is a solution of the above family of problems for $\lambda = 1$, which is exactly our original problem. This completes the proof.

## V. LOWER AND UPPER SOLUTIONS

In this section we study a general case of problem (1), (2), (3), (4) by the method of lower and upper solutions. More specifically, we shall consider the case where the multifunction $F(x,t,u)$ has the form $[\varphi(x,t,u), \psi(x,t,u)]$, where $\varphi, \psi : D \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions
(j) $\varphi(.,.,u), \psi(.,.,u) : D \to \mathbb{R}$ are measurable,
(jj) $\varphi(x,t,.) : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous,
(jjj) $\psi(x,t,.) : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous,
(jv) $\varphi(x,t,u) \leq \psi(x,t,u)$
Then $F = [\varphi, \psi]$ is the general upper semicontinuous multifunction with compact, convex values (see [9, page 5]).
We will refer to the original problem, in this case, by problem $(P)$.

**Definition 14** *A solution of our problem is a function $u \in C(D)$ such that the exists $f \in L^2(D)$, $\varphi(x,t,u) \leq f(x,t) \leq \psi(x,t,u)$ and $u$ satisfies $u_t - u_{xx} = f$ for a.e. $(x,t)$, (2), (3), (4).*

**Definition 15** *$\theta \in C(D)$ is a lower solution of $(P)$ if it satisfies*

(l.1) $\theta_t - \theta_{xx} \leq \varphi(x,t,\theta)$
(l.2) $\theta(x,0) \leq u_0(x)$
(l.3) $\theta(0,t) \leq \int_0^\pi g(\theta(x,t)) dx$
(l.4) $\theta(\pi,t) \leq \int_0^\pi h(\theta(x,t)) dx$

**Definition 16** *$\Theta \in C(D)$ is an upper solution of $(P)$ if the above inequalities are reversed when we substitute $\Theta$ for $\theta$.*

**Definition 17** *Let $\theta \leq \Theta$ be as above. Then $[\theta, \Theta]$ denotes the set of all $u \in C(D)$ such that $\theta(x,t) \leq u(x,t) \leq \Theta(x,t)$ for all $(x,t) \in D$.*

**Theorem 18** *Assume that*

(1) there exists $\beta \in C(D; \mathbb{R}_+)$ such that $\max(|\varphi(x,t,u)|, |\psi(x,t,u)|) \leq \beta(x,t)$
(2) $(P)$ has a lower solution $\theta$ and an upper solution $\Theta$ such that $\theta \leq \Theta$,
(3) the functions $g$ and $h$ are continuous and bounded.
Then $(P)$ has at least one solution $u \in [\theta, \Theta]$.

**Proof**. Define a truncation operator $T : C(D) \to [\theta, \Theta]$ by $T(u) = \max\{\theta, \min(u, \Theta)\}$. Then, it can be shown that $T$ is continuous and bounded. Consider the modified problem
$u_t - u_{xx} \in F(x,t,T(u)),\ (x,t) \in D$
$u(x,0) = u_0(x)$
$u(0,t) = \int_0^\pi g(T(u(x,t))) dx$
$u(\pi,t) = \int_0^\pi h(T(u(x,t))) dx.$
Notice that that the multifunction $F_1 : D \times \mathbb{R} \to 2^{\mathbb{R}}$, given by $F_1(x,t,u) = F(x,t,T(u))$ is nonempty $L^2-$Carathéodory multifunction with compact and convex values, and bounded. So, we can apply Theorem 8 to obtain a solution $u$ of $(P)$.
We show that $u \geq \theta$.
Suppose on the contrary that the set $\omega := \{(x,t) \in D;\ u(x,t) < \theta(x,t)\}$ has positive measure. Then for all $(x,t) \in \omega$ we have $T(u(x,t)) = \theta(x,t)$.
Hence $F_1(x,t,u(x,t)) = [\varphi(x,t,\theta(x,t)), \psi(x,t,\theta(x,t))]$.
Let $w(x,t) = u(x,t)$-$\theta(x,t)$ . Then (recall that $\theta$ is a lower solution),
· for all $(x,t) \in \omega$ we have $w(x,t) < 0$
$\cdot w_t - w_{xx} = (u_t - u_{xx}) - (\theta_t - \theta_{xx}) \geq$
$\varphi(x,t,\theta(x,t)) - \varphi(x,t,\theta(x,t)) = 0$
$\cdot w(x,0) \geq u_0(x) - u_0(x) = 0$
$\cdot w(0,t) \geq \int_0^\pi g(T(u(x,t))) dx - \int_0^\pi g(\theta(x,t)) dx =$
$\int_0^\pi g(\theta(x,t)) dx - \int_0^\pi g(\theta(x,t)) dx = 0$

$\cdot w(\pi,t) \geq \int_0^\pi h\left(T(u(x,t))\right)dx - \int_0^\pi h\left(\theta(x,t)\right)dx =$

$$\int_0^\pi h\left(\theta(x,t)\right)dx - \int_0^\pi h\left(\theta(x,t)\right)dx = 0.$$

The maximum principle (see [13], [21]) implies that $w(x,t) \geq 0$ for all $(x,t) \in \omega$.
This is a contradiction. Hence the set $\omega$ has measure zero, and so $u(x,t) \geq \theta(x,t)$ for all $(x,t) \in D$.
Similarly, we can show that $u(x,t) \leq \Theta(x,t)$ for all $(x,t) \in D$.
Thus $T(u(x,t)) = u(x,t)$. We infer that $F_1(x,t,u(x,t)) = F(x,t,u(x,t))$.
Therefore problem $(P)$ has a solution $u$ in the order interval $[\theta, \Theta]$ .□

## VI. ACKNOWLEDGMENT

The author thanks the Division of Applied Mathematics, Brown University for the warm hospitality.

## REFERENCES

[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, Berlin, New york, 1984.

[2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[3] J. R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. 21 (1963), 155-160

[4] S. Carl, C. Grossmann and C. V. Pao, *Existence and monotone iterations for parabolic differential inclusions*, Comm. Appl. Nonl. Anal. 3 (1996), 1-24.

[5] R. Yu Chegis, *Numerical solution of a heat conduction problem with an integral boundary condition*, Litovsk. Mat. Sb. 24 (1984), 209-215

[6] D-Q. Dai and Yu Huang, *Remarks on a semilinear heat equation with integral boundary conditions*, Nonl. Anal. 67 (2007), 468-475

[7] W. A. Day, *A decreasing property of solutions of parabolic equations with applicatons to thermoelasticity*, Quart. Appl. Math. 41 (1983), 468-475

[8] M. Dehghan, *Efficient techniques for the second order parabolic equation subject to nonlocal specifications*, Appl. Numer. Math. 52 (2005), 39-62

[9] K. Deimling, *Multivalued Differential Equations*, W. de Gruyter, Berlin, 1992

[10] B. C. Dhage, *Multivalued mappings and fixed points II*, Tamkang J. Math. 37 (2006), 27-46

[11] E. Feireisl, *A note on uniqueness for parabolic problems with discontinuous nonlinearities*, Nonl. Anal. 16 (1991), 1053-1056

[12] E. Feireisl and J. Norbury, *Some existence, uniqueness and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities*, Proc. Royal Soc. Edinburgh 119 A (1991), 1-17

[13] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964

[14] M. Frigon, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations differentielles ordinaires*, Dissertationes Math. 296 (1990), 1-79

[15] J. Hofbauer and P. L. Simon, *An existence theorem for parabolic equations on $\mathbb{R}^N$ with discontinuous nonlinearity*, Electron. J. Qual. Theory Diff. Equ. No. 8 (2001), 1-9

[16] N. I. Ionkin, *Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions*, Differential Equations 13 (1977), 204-211

[17] I. Kupka, *Continuous selections for Lipschitz multifunctions*, Acta Math. Univ. Comeniane LXXIV,1 (2005), 133-141

[18] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965), 781-786

[19] M. Martelli, *A Rothe type theorem for noncompact acyclic-valued maps*, Boll. Un. Mat. Ital. 4 (3) (1975), 70-76

[20] W. E. Olmstead and C. A. Roberts, *The one dimensional heat equation with a nonlocal initial condition*, Appl. Math. Lett. 10 (1997), 89-94

[21] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992

[22] R. Pisani, *Boundary value problems for parabolic operators with discontinuous nonlinearities (in Italian)*, Rend. Ist. Mat. Univ. Trieste 14 (1982), 85-98

[23] T. Pruszko, *Topological degree methods in multivalued boundary value problems*, Non. Anal. 5 (1981), 953-973

[24] J. Rauch, *Discontinuous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. 64 (1977), 277-282

[25] H-M. Yin, *On a class of parabolic equations with nonlocal boundary conditions*, J. Math. Anal. Appl. 294 (2004), 712-728.

Abdelkader Boucherif holds a PhD in Applied Mathematics from Brown University, Providence, Rhode Island, USA.

He is currently interested in the study of parabolic problems with discontinuous nonlinearities and nonlocal conditions.

Prof. Boucherif is a regular member of the American Mathematical Society (AMS) and the International Federation of Nonlinear Analysts (IFNA).