On Algebraic Structures of Dynamical Systems

STANISŁAW P. KASPERCZUK
Institute of Physics
University of Zielona Góra
Podgórna 50, 65 246 Zielona Góra
POLAND
S.Kasperczuk@if.uz.zgora.pl

Abstract: The paper discusses the close relation between the existence of first integrals of vector fields $X : M \to TM$ and Poisson structures of a manifold M.

Key–Words: Poisson structures, Hamiltonian systems, Integrability

1 Introduction

The fundamental problem in mathematical physics is to discover new integrable systems. These systems always have algebraic structures that are responsible for their integrability. Therefore the most interesting problem in the study of dynamical systems is to give such general algebraic structures which provide a hidden treasure.

The present paper is devoted to the study of Poisson structures for three dimensional dynamical systems. Poisson structures play an important role in dynamical systems, fluid dynamics, magnetohydrodynamics, superconduity, chromodynamics, etc. Dimension three corresponds to the first nontrivial case where the Poisson structure does not imply the symplectic structure. On the other hand, there are surprisingly many natural phenomena modelled by the three dimensional vector fields:

$$X : \mathbb{R}^3 \to T\mathbb{R}^3, \qquad X = \sum_{i=1}^{3} X_i(x) \frac{\partial}{\partial x_i}, \tag{1}$$

such as the time evolution in chemistry and biology, economy, in laser physics, plasma physics, optics, dynamo theory, fields theory, etc. As is known these systems show a very reach behavior, from complete integrability to chaos and strange attractors (at least numerically). So, one of the most fundamental problems which arise in these systems is integrability. This requires the existence $n-1$ functionally independent integrals of motion for an n-dimensional dynamical system. The trajectories of integrable systems may be obtained as intersections of level surfaces of first integrals. We define *partially integrable systems* as systems with a number of first integrals smaller than $n-1$.

2 Preliminaries

Let M be a real smooth manifold of dimension n, TM and T^*M its tangent and cotangent bundles, respectively. For each $k \in N$, we denote by $\mathcal{X}^k(M)$ the space of smooth sections of $\bigwedge^k TM$, and by $\mathcal{V}^k(M)$ the smooth sections of $\bigwedge^k T^*M$. By convention, for $k = 0$, we set $\mathcal{X}^0(M) = \mathcal{V}^0(M) = \mathcal{F}(M)$, where $\mathcal{F}(M)$ is a space of real-valued smooth functions defined on M.

A *Poisson structure* on M is a skew-symmetric bilinear map [1,2]:

$$\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M),$$

called the Poisson bracket, which satisfies the following conditions:

(i) $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$
(ii) $\{f,gh\} = \{f,g\}h + \{f,h\}g$
(iii) $\{f,g\} = -\{g,f\}$

for all $f,g,h \in \mathcal{F}(M)$. The pair $(M, \{\ ,\ \})$ is called a *Poisson manifold*, and conditions (i) - (iii) make $(\mathcal{F}(\mathcal{M}), \{\ ,\ \})$ into a Lie algebra $\mathcal{A}$.

We define an *almost Lie algebra* [3] to be the same as a Lie algebra $\mathcal{A}$ except that the bracket operation does not satisfy the Jacobi identity (i). A pair $(M, \{\ ,\ \}')$ is called an *almost Poisson manifold* when $\{\ ,\ \}'$ is an almost Lie algebra structure.

The local expression for the bracket $\{\ ,\ \}$ is

$$\{f,g\} = \sum_{i,j} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}. \tag{2}$$

The bracket $\{f,g\}$ defines a twice covariant skew-symmetric tensor $\pi \in \mathcal{X}^2(M)$, such that

$$\{f,g\} = \pi(dg, df) \tag{3}$$

 Manuscript received Sept. 30, 2007; Revised received Jan. 19, 2008

then

$$\pi = \sum_{ij} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{4}$$

Since π is skew,

$$\pi_{ij} = -\pi_{ji} \tag{5}$$

and the Jacobi identity (i) is equivalent to

$$\sum_l \left(\pi_{li} \frac{\partial \pi_{jk}}{\partial x_l} + \pi_{lj} \frac{\partial \pi_{ki}}{\partial x_l} + \pi_{lk} \frac{\partial \pi_{ij}}{\partial x_l} \right) \equiv 0. \tag{6}$$

Let $(M, \{\ ,\ \})$ be a Poisson manifold. Given $h \in \mathcal{F}(M)$, defines the linear map $X_h : \mathcal{F}(M) \to \mathcal{F}(M)$ by $X_h(f) = \{f, h\}$. The correspondence $h \mapsto X_h$ defines a vector field, called the *Hamiltonian vector field*, and h plays the role of a Hamiltonian function. A *bi-Hamiltonian manifold* M is a manifold endowed with two Poisson tensors π and π^1 such that $\pi_\lambda = \pi + \lambda\pi^1$ is a Poisson tensor for any $\lambda \in \mathbb{R}$. A *Casimir function* on a Poisson manifold $(M, \{\ ,\ \})$ is a smooth function ψ such that $\{\psi, f\} = 0$ for all $f \in \mathcal{F}(M)$. A vector field X on a Poisson manifold (M, π) is called a *Poisson vector field* if it is an infinitesimal automorphism of the Poisson structure, i.e. the Lie derivative of π with respect to X vanishes

$$\mathcal{L}_X \pi = 0. \tag{7}$$

Another equivalent condition for X to be a Poisson vector field can be written in the following form

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\};\ \forall f, g \in \mathcal{F}(M). \tag{8}$$

If $X = \pi dh$, (8) is the Jacobi identity.

For a given volume element Ω on M we consider its induced isomorphism $\Phi : \mathcal{X}^k \to \mathcal{V}^{n-k} : u \mapsto i(u)\Omega$, where $i(u)$ is the contraction by the k-vector $u \in \mathcal{X}^k(M)$. Let

$$D_\Omega = \Phi^{-1} \circ d \circ \Phi : \mathcal{X}^k(M) \to \mathcal{X}^{k-1}(M) \tag{9}$$

be the pull-back operator under the isomorphism Φ. A k-vector $u \in \mathcal{X}^k$ is said to be exact if $D_\Omega(u) = 0$. In local coordinates, and taking $\Omega = \theta(x) dx_1 \wedge \cdots \wedge dx_n$, we have

$$D_\Omega(\pi) = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial(\theta \pi_{ij})}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \tag{10}$$

The vector field $D_\Omega(\pi)$ is known as *the modular vector field* of π with respect to Ω [4]. We say that a Poisson structure π on a manifold M is *unimodular* [4] if its modular vector field, with respect to the volume element Ω, is identically zero.

The Jacobi identity for a Poisson bracket is equivalent to the following condition on the Poisson bivector π:

$$[\pi, \pi]_S = 0, \tag{11}$$

where $[\ ,\ ]_S$ denotes the *Schouten bracket*. The Schouten bracket

$$[\ ,\ ]_S : \mathcal{X}^k(M) \times \mathcal{X}^m(M) \to \mathcal{X}^{k+m-1}(M) \tag{12}$$

is the extension of the Lie bracket of vector fields and the action of vector fields on smooth functions (cf. [2]). For $k = m = 1$, this bracket is precisely the commutator. The Schouten bracket satisfies the following properties:

$[P, Q]_S = (-1)^{pq}[Q, P]_S$,

$[P, Q \wedge R]_S = [P, Q]_S \wedge R + (-1)^{pq+q} Q \wedge [P, R]_S$,

$(-1)^{p(r-1)}[P, [Q, R]]_S + (-1)^{q(p-1)}[q, [R, P]]_S + (-1)^{r(q-1)}[R, [P, Q]]_S = 0$.

A Poisson structure π on a manifold M gives rise to a differential operator $\sigma_\pi : \mathcal{X}^k(M) \to \mathcal{X}^{k+1}(M)$, this operator is given by Schouten bracket with π as follows

$$\sigma_\pi Q = [\pi, Q]_S, \quad Q \in \mathcal{X}^k(M). \tag{13}$$

The cohomology

$$H^k_\pi(M) = \frac{Ker(\sigma_\pi : \mathcal{X}^k(M) \to \mathcal{X}^{k+1}(M))}{Im(\sigma_\pi : \mathcal{X}^{k-1}(M) \to \mathcal{X}^k(M))} \tag{14}$$

is called the *Poisson cohomology* of a Poisson manifold. As was noticed by many authors, the computation of the Poisson cohomology is very difficult (see e.g. [5]). Exceptions are nondegenerate Poisson structures and linear Poisson structures. The Poisson cohomology in low degrees has the following interpretations [5]: the zeroth cohomology group $H^0_\pi(M)$ is group of functions $f \in \mathcal{F}(M)$ such that $X_f = [f, \pi]_S = 0$, that is Casimir functions; $H^1_\pi(M)$ is the quotient of the Poisson vector fields, i.e., vector fields such that $[X, \pi]_S = 0$, by the Hamiltonian vector fields, i.e. vector fields of the type $[f, \pi]_S = X_f$; $H^2_\pi(M)$ is the quotient of the space of 2-vectors $\Lambda \in \mathcal{X}^2(M)$ which satisfy the equation $[\pi, \Lambda]_S = 0$ by the space of 2-vector fields of the type $\Lambda = [\pi, Y]_S$.

3 Almost Poisson Structures

Assume $M = \mathbb{R}^3$, with coordinates (x_1, x_2, x_3). We will consider a dynamical system

$$\frac{dx_i}{dt} = X_i(x) \quad i = 1, 2, 3. \tag{15}$$

We denote by

$$X : \mathbb{R}^3 \to T\mathbb{R}^3, \quad X = \sum_{i=1}^{3} X_i(x)\frac{\partial}{\partial x_i}, \qquad (16)$$

the vector field associated to system (15) by the relations $dx_i/dt = X(x_i), i = 1, 2, 3$. By integral of motion (or first integral) of (15) we mean a function $f \in \mathcal{F}(M)$ which satisfies

$$Xf = \sum_{i=1}^{3} X_i \frac{\partial f}{\partial x_i} = 0. \qquad (17)$$

Theorem 1 *Let the system (15) has a first integral $f \in \mathcal{F}(\mathbb{R}^3)$, then the vector field $X = \sum_{i=1}^{3} X_i(x)(\partial/\partial x_i)$ can be written in terms of an almost Poisson structure $X = \pi' df$.*

Proof If f is a first integral of X, then the vector field $Y : \mathbb{R}^3 \to T\mathbb{R}^3,\ Y = (Y_1(x), Y_2(x), Y_3(x))$, such that

$$X = (\nabla f \times Y) \cdot \nabla \qquad (18)$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, always exists. It is obvious that,

$$Xf = (\nabla f \times Y) \cdot \nabla f \equiv 0. \qquad (19)$$

To show that the vector field (16) has an almost Poisson structure we define the bivector

$$\pi' = Y_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + Y_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + Y_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}. \qquad (20)$$

Hence, the bivector π' is skew-symmetric: $\pi'_{ij} = -\pi'_{ji}$. Because of (18) and (20) we have

$$X = X_f = \pi' df. \qquad (21)$$

A skew-symmetric tensor field π' on a smooth manifold M defines a vector bundle map $\pi'^{\sharp} : T^*M \to TM$, such that for any $\alpha, \beta \in T^*M$

$$\pi'(\alpha, \beta) = i(\pi'^{\sharp}(\alpha))\beta.$$

For any pair of functions $f, g \in \mathcal{F}(\mathbb{R}^3)$, we set

$$\{f, g\}' = \pi'(df, dg) = i(\pi'^{\sharp}(df))dg = -i(\pi'^{\sharp}(dg))df. \qquad (22)$$

From (22) one can obtain:

$$\begin{aligned}\pi'(df, d(gh)) &= i(\pi'^{\sharp}(df))d(gh) \\ &= i(\pi'^{\sharp}(df))(hdg + gdh) \\ &= i(\pi'^{\sharp}(df))hdg + i(\pi'^{\sharp}(df))gdh \\ &= \pi'(df, dg)h + \pi'(df, dh)g.\end{aligned}$$

So, we have the Leibnitz rule:

$\{f, gh\}' = \pi'(df, d(gh)) = \pi'(df, gdh + hdg) = \pi'(df, dh)g + \pi'(df, dg)h = \{f, g\}'h + \{f, h\}'g.$

This completes the proof.

Let as previous, the vector Y defines the almost Poisson structure (20). We show that $\pi' = Y_1 (\partial/\partial x_2) \wedge (\partial/\partial x_3) + Y_2 (\partial/\partial x_3) \wedge (\partial/\partial x_1) + Y_3 (\partial/\partial x_1) \wedge (\partial/\partial x_2)$ is a Poisson tensor if the vector Y is orthogonal to the vector $curl\, Y$. Indeed, if

$$Y \cdot (\nabla \times Y) = 0,$$

then

$$Y_1 \frac{\partial Y_3}{\partial x_2} + Y_2 \frac{\partial Y_1}{\partial x_3} + Y_3 \frac{\partial Y_2}{\partial x_1} - Y_1 \frac{\partial Y_2}{\partial x_3} - Y_2 \frac{\partial Y_3}{\partial x_1} - Y_3 \frac{\partial Y_1}{\partial x_2} = 0, \qquad (23)$$

but the Jacobi identity (6) for bivector (20) is given by equation (23).

4 Poisson Vector Fields

Let $(\mathbb{R}^3, \pi)$ be a Poisson manifold and $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ be the volume form on $\mathbb{R}^3$. Assume that the modular vector field $D_\Omega(\pi)$ is identically zero, and the Poisson manifold is unimodular. Hence the 1-form $\omega^1 = i(\pi)\Omega$ is exact, that is,

$$i(\pi)\Omega = dg. \qquad (24)$$

In that case the Poisson tensor π_g reads

$$\pi_g = \left(\frac{\partial g}{\partial x_1}\right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \left(\frac{\partial g}{\partial x_2}\right) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + \left(\frac{\partial g}{\partial x_3}\right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}. \qquad (25)$$

Theorem 2 *Let f be a first integral of the system (15), the vector field $X = (\nabla f \times Y) \cdot \nabla$ is a Poisson vector field on the unimodular Poisson manifold $(\mathbb{R}^3, \pi_f)$, when $\nabla f \cdot (\nabla \times Y) = 0$.*

Proof X is a Poisson vector field if $\sigma_{\pi_f} X = \mathcal{L}_X \pi_f = 0$, then for

$$\pi_f = \left(\frac{\partial f}{\partial x_1}\right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \left(\frac{\partial f}{\partial x_2}\right) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}$$

$$+\left(\frac{\partial f}{\partial x_3}\right)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2} \qquad (26)$$

we have

$$\mathcal{L}_X\pi_f = [X,(\partial f/\partial x_1)(\partial/\partial x_2)\wedge(\partial/\partial x_3)]_S$$
$$+[X,(\partial f/\partial x_2)(\partial/\partial x_3)\wedge(\partial/\partial x_1)]_S \qquad (27)$$
$$+[X,(\partial f/\partial x_3)(\partial/\partial x_1)\wedge(\partial/\partial x_2)]_S.$$

Next, from (27) we obtain

$$\mathcal{L}_X\pi_f = \{\nabla X(\partial f/\partial x_1)-(\partial/\partial x_1)(X\nabla f)\}$$
$$(\partial/\partial x_2)\wedge(\partial/\partial x_3)+\{\nabla X(\partial f/\partial x_2)-(\partial/\partial x_2)$$
$$(X\nabla f)\}(\partial/\partial x_3)\wedge(\partial/\partial x_1)+\{\nabla X(\partial f/\partial x_3))$$
$$-(\partial/\partial x_3)(X\nabla f)\}(\partial/\partial x_1)\wedge(\partial/\partial x_2). \qquad (28)$$

Putting into (28) $X=(\nabla f\times Y)\cdot\nabla$ we get

$$\mathcal{L}_X\pi_f = -\nabla f\cdot(\nabla\times Y)\pi_f. \qquad (29)$$

Hence, $\mathcal{L}_X\pi_f=0$ when $\nabla f\cdot(\nabla\times Y)=0$. This ends the proof.

If $\sigma_{\pi_f}X=0$, then X fulfils the formula (8). The vector field Y satisfies the Jacobi identity for $\nabla\times Y=0$. Of course, a Hamiltonian vector field is a Poisson vector field, but the opposite is false. The first Poisson cohomology space $H^1_\pi(\mathbb{R}^3)$ is just the space of Poisson vector fields $X:\mathcal{L}_X\pi_f=0$, modulo the Hamiltonian vector fields $V_h=\pi_f dh$. Since the modular vector field $D_\Omega(\pi)$ preserves the Poisson structure $\pi:\mathcal{L}_{D_\Omega}(\pi)\pi=0$ (see [4]), it determines a class in the first Poisson cohomology , called the *modular class* of a Poisson manifold.

5 Hamiltonian Vector Fields

The possibility of describing a given vector field $X:\mathbb{R}^3\to T\mathbb{R}^3$ in terms of a Poisson structure is a question of fundamental importance.

Theorem 3 *Let $(\mathbb{R}^3,\pi,f)$ be a Hamiltonian system. Suppose, in addition, that there exists a second integral h, functionally independent of f, i.e. $dh\wedge df\neq 0$, and $\pi(dh,df)=0$. Then the Poisson tensor π satisfies the extra conditions*

$$\pi_{12}=\varrho(x)\frac{\partial h}{\partial x_3},\quad \pi_{31}=\varrho(x)\frac{\partial h}{\partial x_2}$$
$$\pi_{23}=\varrho(x)\frac{\partial h}{\partial x_1},\ \varrho(x)\in\mathcal{F}(\mathbb{R}^3). \qquad (30)$$

Proof If h is a first integral of X_f, then $X_f(h)=0$. From (18) we have

$$X_f(h)=(\nabla f\times Y)\cdot\nabla h=0. \qquad (31)$$

Since $df\wedge dh\neq 0$, from (20) it immediately follows that

$$\pi_{12}=\varrho(x)\frac{\partial h}{\partial x_3},\quad \pi_{31}=\varrho(x)\frac{\partial h}{\partial x_2}$$
$$\pi_{23}=\varrho(x)\frac{\partial h}{\partial x_1}. \qquad (32)$$

Denote by $\varrho(\partial h/\partial x_i)=u_i, i=1,2,3,$ the Jacobi identity (23) reads

$$u_1\partial u_3/\partial x_2+u_2\partial u_1/\partial x_3+u_3\partial u_2/\partial x_1$$
$$-u_1\partial u_2/\partial x_3-u_2\partial u_3/\partial x_1-u_3\partial u_1/\partial x_2=0 \quad (33)$$

But this equation is always satisfied. This ends the proof.

Theorem 4 *If the Hamiltonian system $(\mathbb{R}^3,\pi,h)$ possesses the second first integral $f\in\mathcal{F}(\mathbb{R}^3)$ functionally independent of Hamiltonian h, then the vector field X_h has a bi-Hamiltonian structure $\pi_\lambda=\pi+\lambda\pi^1,\lambda\in\mathbb{R}$, connected with functions $f,h\in\mathcal{F}(\mathbb{R}^3)$ respectively:*

$$\pi_{ij}=\varepsilon_{ijk}\varrho(x)\left(\frac{\partial f}{\partial x_k}\right)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \qquad (34)$$

and

$$\pi^1_{ij}=-\varepsilon_{ijk}\varrho(x)\left(\frac{\partial h}{\partial x_k}\right)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \qquad (35)$$

where ε_{ijk} is the Levi-Civita tensor.

Proof It can be easily seen that the vector field X_h has the form

$$X_h=\pi dh=\pi^1 df. \qquad (36)$$

We have to prove that $\pi_\lambda=\pi+\lambda\pi^1,\lambda\in\mathbb{R}$, is a Poisson tensor. Denoting by $\omega^1_f=i(\pi)\Omega$, $\omega^1_h=i(\pi^1)\Omega$, one can observe from (23) that the bivector π_λ defines the Poisson structure if

$$\omega^1_\lambda\wedge d\omega^1_\lambda=0. \qquad (37)$$

Therefore, if $\pi_\lambda=\pi+\lambda\pi^1$ is a Poisson tensor then

$$(\omega^1_f+\lambda\omega^1_h)\wedge d(\omega^1_f+\lambda\omega^1_h)=0. \qquad (38)$$

From (38) we obtain:

$(\omega^1_f+\lambda\omega^1_h)\wedge d(\omega^1_f+\lambda\omega^1_h)=\varrho(df-\lambda dh)\wedge d(\varrho(df-\lambda dh))=-\lambda\varrho(df\wedge d\varrho\wedge dh+dh\wedge d\varrho\wedge df)=0.$

This ends the proof.

Theorem 5 *Hamiltonian vector fields $X_h = \pi dh$ on a Poisson manifold $(\mathbb{R}^3, \pi)$ are completely integrable, thus have the bi-Hamiltonian structure.*

Proof The vector field $X_h = \pi dh$ is a Hamiltonian vector field if the bivector π is a Poisson tensor. Hence, the 1-form $\omega^1 = i(\pi)\Omega$ obeys the condition $\omega^1 \wedge d\omega^1 = 0$. More precisely, if ω^1 is not exact, by virtue of the Frobenius theorem there exist functions $f, g \in \mathcal{F}(\mathbb{R}^3)$, such that $\omega^1 = gdf$. The Poisson tensor is

$$\pi = \varepsilon_{ijk} g \left(\frac{\partial f}{\partial x_k}\right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{39}$$

and the Hamiltonian vector field becomes $X_h = g(\nabla f \times \nabla h) \cdot \nabla$. As the vector field X_h has two functionally independent first integrals f and h, by Theorem 4, it admits a bi-Hamiltonian form. This completes the proof.

6 Deformation of Poisson Structures

Let us consider a Hamiltonian vector field $X_h = \pi dh$, where $i(\pi)\Omega = \varrho(x)df$. There are infinitely many deformations of the bivector $\pi \mapsto \tilde{\pi}$, such that $\pi dh = \tilde{\pi} dh$. These deformations have the form: $\tilde{\pi} = \pi + \varepsilon \pi^1$, $i(\pi^1)\Omega = \varrho dh$.

Theorem 6 *The deformed bivector $\tilde{\pi}$ is a Poisson tensor if one of the following conditions holds:*
(i) $\varepsilon \in \mathbb{R}$,
(ii) $d\varepsilon \wedge df = 0$,
(iii) $d\varepsilon \wedge dh = 0$.

Proof If $\tilde{\pi}$ is a Poisson tensor, then $i(\tilde{\pi})\Omega \wedge d(i(\tilde{\pi})\Omega) = \varrho(df + \varepsilon dh) \wedge d(\varrho(df + \varepsilon dh)) = \varrho^2 df \wedge d\varepsilon \wedge dh = 0$. This ends the proof.

The modular vector field $D_\Omega(\tilde{\pi})$ of the deformed Poisson structure $\tilde{\pi}$ with respect to the volume element $\Omega = dx_1 \wedge dx_2 \wedge dx_3$, defined by (10), reads

$$D_\Omega(\tilde{\pi}) = \nabla \varrho \times (\nabla f + \varepsilon \nabla h) \cdot \nabla.$$

Assume $\varepsilon \in \mathbb{R}$, the condition for $\tilde{\pi} = \pi + \varepsilon \pi^1$ to be a Poisson tensor gives: $[\tilde{\pi}, \tilde{\pi}]_S = 2\varepsilon[\pi, \pi^1]_S + \varepsilon^2[\pi^1, \pi^1]_S = 0$. The second Poisson cohomology group $H^2_\pi(\mathbb{R}^3)$ is the quotient of the space of bivector fields μ such satisfy the equation $[\pi, \mu]_S = 0$ by the space of bivector fields of the type $\mu = [Z, \pi]_S$. The infinitesimal deformation $\mu = [Z, \pi]_S$ of Poisson tensor π always satisfies $[\pi, \mu]_S = 0$. If the cohomology class of π in $H^2_\pi(\mathbb{R}^3)$ vanishes, then π is called an *exact Poisson structure*. More precisely, an exact Poisson structure is a couple (π, Z) which satisfies the relation $[Z, \pi]_S = \pi$. In our situation, π^1 is a Poisson tensor compatible with π, but π^1 is not necessary obtained by an infinitesimal deformation.

7 Normal Form and Reduction for Hamiltonian Systems

Consider a Hamiltonian system on symplectic manifold $(\mathbb{R}^4, \omega^2 = \pi^{-1} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2, H)$. Recall that the Hamiltonian vector field X_H on $(\mathbb{R}^4, \omega^2)$ associated with H is defined by $i(X_H)\omega^2 = dH$, and assume that the Hamiltonian H is the Taylor series. Let the Taylor series starts with quadratic terms. In the study of the qualitative properties of the Hamiltonian system near an equilibrium point, which we assume to be 0, one of most powerful techniques available is to simplify H using nonlinear canonical transformations which leave the origin fixed. Let $L = \sum_{n \geq 2} L_n$ be a positive graded Lie algebra [6]. We subscript $H_m \in L_m$ and assume that the Lie bracket $[\,,]$ in L satisfies $[H_k, H_m] \in L_{k+m-2}$. We will say the element H in L is in *normal form* through terms of order $n \geq 2$ with respect to H_2 if $H_k \in Ker(ad_{H_2}|H_k)$ for $2 \leq k \leq n$ (cf.[7]). We define the adjoint representation of L as $ad_G(H) = [G, H]$. Consider the Hamiltonian system with $H \in L$ and $H_2 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$. Note that for $H, F \in L$: $[G, F] = \{F, G\} = \omega^2(X_F, X_G) = X_G(F)$. Since the vector field X_{H_2} has purely imaginary eigenvalues $\pm i$ we say X_{H_2} is in $1:1$ resonance. If we introduce complex conjugate variables $z_j = q_j + ip_j$, $w_j = q_j - ip_j$, $j = 1, 2$, the linear operator ad_{H_2} reads

$$ad_{H_2} = i\left(w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} - z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right).$$

The eigenvalues of ad_{H_2} are $i(m_1 + m_2 - k_1 - k_2)$ with corresponding eigenvectors $z_1^{k_1} z_2^{k_2} w_1^{m_1} w_2^{m_2}$. Thus among the generators for $Ker(ad_{H_2})$ are those given in the nonresonance case, which are called the Birkhoff generators. In addition to the Birkhoff generators we have the resonance generators given by $(k_1, k_2, m_1, m_2) = (1, 0, 0, 1)$ and $(k_1, k_2, m_1, m_2) = (0, 1, 1, 0)$. Therefore we may consider $Ker(ad_{H_2})$ as algebra of formal power series in the Hopf variables [7]

$$W_1 = \frac{1}{2}(z_1 w_2 + w_1 z_2), \quad W_2 = \frac{1}{2}(z_1 w_2 - w_1 z_2)$$

$$W_3 = \frac{1}{2}(z_1 w_1 - z_2 w_2), \quad W_4 \frac{1}{2}(z_1 w_1 + z_2 w_2),$$

with relation

$$W_1^2 + W_2^2 + W_3^2 = W_4^2. \tag{40}$$

The map $(p_1, p_2, q_1, q_2) \mapsto (W_1, W_2, W_3)$ is the standard Hopf map. Thus H being in normal form with respect to H_2 (i.e., $X_{H_2}(H) = \{H_2, H\} = 0$), means that can be written as a polynomial in the Hopf variables, and the flow of X_H leaves the 3-sphere $M_c = H_2^{-1}(c)$ invariant. The flow of X_{H_2} is periodic and defines free and proper S^1– action on M_c.

A reduced space for the flow on M_c is a symplectic manifold (N_R, ω_R) together with a surjective submersion $\phi : M_c \to N_R$ (cf. [8], and reference therein), such that $K \circ \phi = H \circ j$ and $\phi_* X(x) = X(\phi(x))$, $\forall x \in M_c$, where $j : M_c \to \mathbb{R}^4$ is inclusion, K is reduced Hamiltonian on N_R associated to H, and ϕ_* is the tangent map of ϕ. The related vector field on N_R is given by

$$X = (\nabla C \times \nabla K) \cdot \nabla \tag{41}$$

where $C = x_1^2 + x_2^2 + x_3^2$. The dynamics of such a system reduces to dynamics on $\mathbb{R}^3$, with respect Poisson structure. (In fact, such a structure is called the Lie-Poisson structure since the Casimir is a homogeneous quadratic polynomial.) The various reductions give different coordinate representations of the solutions. The dynamical system considered correspond to the bi-Hamiltonian dynamics with Poisson bracket. Each Poisson bracket $\{F, K\} = \nabla C \cdot (\nabla K \times \nabla F)$ is associated with the Casimir function C. When reduced to level sets of Casimir, the equations of motion take various symplectic forms. The various reductions give different coordinate representations of the solutions. These coordinate representations my be used to seek the simplest representation of the solutions.

8 Poisson Coalgebra and Integrability

First we recall some algebraic preliminaries. Detailed exposition of the theory can be found for example in [9]. A unital associative algebra over K is a linear space A together with two linear maps $m : A \bigotimes A \to A$ and $\eta : K \to A$ such that

$$m(m \otimes \mathbf{1}) = m(\mathbf{1} \otimes m),$$
$$m(\mathbf{1} \otimes \eta) = m(\eta \otimes \mathbf{1}) = id.$$

Here $A \bigotimes A$ is tensor product of two algebras, $\mathbf{1}$ is the unit element of A and id means the identity map. The usual notation for an algebra multiplication is simply $ab := m(a \otimes b)$, and we will use this notation. Let (A_1, m_1, η_1) and (A_2, m_2, η_2) be algebras, then the tensor product $A_1 \bigotimes A_2$ is naturally endowed with the structure of an algebra. The multiplication $m_{A_1 \bigotimes A_2}$ is defined as

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2).$$

A coalgebra is a triple $(A, \triangle, \epsilon)$ with a linear space A over K, and $\triangle : A \to A \bigotimes A$ is a linear map called comultiplication, $\epsilon : A \to K$ is a linear morphism called counit with the property

$$\triangle(ab) = \triangle(a)\triangle(b) \quad \forall a, b \in A$$
$$(\triangle \otimes id) \circ \triangle = (id \otimes) \circ \triangle$$
$$(id \otimes \epsilon) \circ \triangle = (\epsilon \otimes id) \circ \triangle = id$$

We note that $A \bigotimes A$ is both an algebra and coalgebra of A. One can define a tensor product of Poisson algebras. $\mathcal{F}(M) \bigotimes \mathcal{F}(M)$ is again a Poisson algebra with the vector product algebra structure and the tensor product coalgebra structure. We have to define a Poisson structure on $\mathcal{F}(M) \bigotimes \mathcal{F}(M)$ such that the axioms of Poisson algebra are satisfied. For our purpose the maps are defined as follows [10]. The multiplication $m_{\mathcal{F} \otimes \mathcal{F}}$

$$(f \otimes g)(h \otimes j) = (fh) \otimes (gj)$$

the coproduct on $\mathcal{F}(M)$

$$\triangle(x_i) = x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i$$

and the Poisson bracket on $\mathcal{F}(M) \bigotimes \mathcal{F}(M)$

$$\{f \otimes g, h \otimes j\}_{\mathcal{F} \otimes \mathcal{F}} = \{f, h\}_{\mathcal{F}} \otimes gj + fh \otimes \{g, j\}_{\mathcal{F}}.$$

This gives

$$\{\triangle(f), \triangle(g)\}_{\mathcal{F} \otimes \mathcal{F}} = \triangle(\{f, g\}_{\mathcal{F}}). \tag{42}$$

Assume that $\mathcal{C}$ is a Casimir for Poisson tensor π, then for any $h \in \mathcal{F}(\mathbb{R}^3)$

$$\triangle(\{\mathcal{C}, h\}_{\mathcal{F}}) = \{\triangle(\mathcal{C}), \triangle(h)\}_{\mathcal{F} \otimes \mathcal{F}} = 0. \tag{43}$$

So, if $\mathcal{C}$ is a Casimir function for π and h is an arbitrary smooth function on $\mathbb{R}^3$ then the Hamiltonian system defined by the Hamiltonian $H = \triangle(h(x_1, x_2, x_3))$ is completely integrable (see [10]).

9 Applications

9.1 Partially Integrable 3D Lotka-Volterra System

The 3D Lotka-Volterra system plays an important role in modelling many physical, chemical and biological processes. Let us consider the case (10) in Table I in Ref. [11]:

$$\begin{aligned} \dot{x}_1 &= x_1(x_3 + \lambda) \\ \dot{x}_2 &= x_2(x_1 + x_3 + \lambda) \\ \dot{x}_3 &= x_3(Bx_1 + x_2). \end{aligned} \tag{44}$$

The system (44) has the first integral [11]:

$$f = \frac{x_2}{x_1} + B(\ln x_2 - \ln x_1) - \ln x_3. \tag{45}$$

Due to Theorem 1, (44) can be rewritten as

$$X_f = (Y \times \nabla f) \cdot \nabla. \tag{46}$$

From (46) we obtain,

$$Y_1 = -x_1 x_2 x_3, \qquad Y_2 = 0$$

$$Y_3 = \frac{x_1^2 x_2 (x_3 + \lambda)}{B x_1 + x_2}. \tag{47}$$

Hence, an almost Poisson structure reads

$$\pi' = -x_1 x_2 x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$

$$+ \frac{x_1^2 x_2 (x_3 + \lambda)}{B x_1 + x_2} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}. \tag{48}$$

As it can be easily seen, the tensor (48) does not satisfy the extra conditions (30), and therefore the L-V system (44) is not completely integrable in this form. From Theorem 3 and Theorem 5 we can conclude that the system (44) is completely integrable if there exist functions $g, h, \varrho \in \mathcal{F}(\mathbb{R}^3)$ that

$$Y = \varrho \nabla h - g \nabla f. \tag{49}$$

So, it is very difficult to decide whether this system is completely integrable or not.

9.2 Integrable Lotka-Volterra System

Let us consider the case (4) in Table I in Ref. [11]:

$$\begin{aligned} \dot{x}_1 &= x_1(\lambda + C x_2 + x_3) \\ \dot{x}_2 &= x_2(\mu + x_1 + A x_3) \\ \dot{x}_3 &= x_3(\nu + B x_1 + x_2) \end{aligned} \tag{50}$$

where: $ABC + 1 = 0$, and $\nu = \mu B - \lambda AB$. This system has two functionally independent integrals [11]:

$$f = AB \ln x_1 - B \ln x_2 + \ln x_3, \tag{51}$$

$$h = ABx_1 + x_2 + \nu \ln x_2 - Ax_3 - \mu \ln x_3. \tag{52}$$

Hence, we can rewrite the system (50) as

$$X = \varrho(x)(\nabla f \times \nabla h) \cdot \nabla \tag{53}$$

where $\varrho(x) = -C x_1 x_2 x_3$. The modular vector field of the Poisson structure π_f with respect to the volume element $\Omega = dx_1 \wedge dx_2 \wedge dx_3$, reads $D_\Omega(\pi_f) = -C x_1 (\partial/\partial x_1) + (1 + C) x_2 (\partial/\partial x_2) - C(1 + B) x_3 (\partial/\partial x_3)$.

9.3 Hoyer System

The Hoyer system is defined by the three-dimensional dynamical system depending on nine parameters

$$\begin{aligned} \dot{x}_1 &= a_1 x_2 x_3 + b_1 x_3 x_1 + c_1 x_1 x_2 \\ \dot{x}_2 &= a_2 x_2 x_3 + b_2 x_3 x_1 + c_1 x_1 x_2 \\ \dot{x}_3 &= a_3 x_2 x_3 + b_3 x_3 x_1 + c_3 x_1 x_2 \end{aligned} \tag{54}$$

This system was introduced by P. Hoyer in 1879 in his PhD thesis. The existence of Poisson structures for the Hoyer system with quadratic Hamiltonians was studied in [12]. There exist four cases for which (54) has a Poisson structure with quadratic Hamiltonians

(i) $a_1, b_2, c_3 \neq 0$, $a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$
(ii) $a_1, b_2, b_3, c_2, c_3 \neq 0$, $a_2 = a_3 = b_1 = c_1 = 0$
(iii) $a_1, b_1, b_2, c_3 \neq 0$, $a_2 = a_3 = b_3 = c_1 = c_2 = 0$.

Evidently the authors were unconscious of finding completely integrable cases of the system (54). Indeed, consider case (i) for which the authors obtained two Poisson structures ((26) and (27) in Ref. [12])

$$\pi_1 = -\frac{\beta\alpha + h}{\alpha\beta c_3} x_3\, \partial x_1 \wedge \partial x_2 - \frac{\alpha - \gamma + h}{\alpha\gamma b_2} x_2\, \partial x_3 \wedge \partial x_1$$

$$+ \frac{\beta - \gamma + h}{\alpha\gamma b_2} x_2 \partial x_2 \wedge \partial x_3$$

and

$$\pi_2 = \left(\frac{a_1 + b_2}{c_3} + b_2 h\right) x_3 \partial x_1 \wedge \partial x_2 -$$

$$(1 + c_3 h) x_2 \partial x_3 \wedge \partial x_1 - \left(\frac{a_1 + b_2}{c_3} + b_2 h\right) x_3 \partial x_2 \wedge \partial x_3.$$

where α, β, γ are nonzero parameter, such that $\alpha + \beta + \gamma = 0$, and h is arbitrary.

It can be easily shown that in this case the Hoyer system has the bi-Hamiltonian form

$$X = (\nabla f \times \nabla g) \cdot \nabla \tag{55}$$

where $f = 1/2\left(x_1^2 - a_1 x_2^2 / b_2\right)$ and $g = 1/2\left(c_3 x_2^2 - b_2 x_3^2\right)$. For cases (ii) and (iii) in virtue of Theorem 6, the system (54) is completely integrable.

9.4 Henon-Heiles System

Consider the Henon-Heiles system given by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{3} a q_1^3 - q_1 q_2^2. \tag{56}$$

The normal form of (56) with respect to H_2 of order four is [8]

$$K = a_1 x_1^2 + a_2 x_2^2 + a_5 x_3 + a_5. \qquad (57)$$

From (41) we have

$$\frac{dx}{dt} = \nabla C \times \nabla K. \qquad (58)$$

Since the level sets of Casimir C are spheres, we choose a new basis of coordinates as follows: $x_1 = r\cos u\, sin\, v,\ x_2 = r\, sin\, u\, cos\, v,\ x_3 = r\, cos\, v$. In terms of these coordinates Casimir and Hamiltonian become

$$C = r^2$$
$$K = a_1 C cos^2 u\, sin^2 v + a_2 C \sin^2 u\, cos^2 v$$
$$+ a_5 C^{1/2} cos\, v.$$

The basis of the tangent space $T_x \mathbb{R}^3$ read

$$\frac{\partial}{\partial x_1} = cos\, u\, sin\, v \frac{\partial}{\partial r} + \frac{1}{r} cos\, u \cos\, v \frac{\partial}{\partial v} - \frac{sin\, u}{r\, sin\, v}\frac{\partial}{\partial u}$$
$$\frac{\partial}{\partial x_2} = sin\, u\, sin\, v \frac{\partial}{\partial r} + \frac{1}{r} sin\, u\, cos\, v \frac{\partial}{\partial v} + \frac{cos\, u}{r\, sin\, v}\frac{\partial}{\partial u}$$
$$\frac{\partial}{\partial x_3} = cos\, v \frac{\partial}{\partial r} - \frac{1}{r} sin\, v \frac{\partial}{\partial v}$$

The bi-Hamiltonian vector field (58) then reduces to

$$X_K = 2(\sqrt{C} sin\, v)^{-1}\left[\frac{\partial K}{\partial v}\frac{\partial}{\partial u} - \frac{\partial K}{\partial u}\frac{\partial}{\partial v}\right] \qquad (59)$$

and the equations of motion have the following form

$$\frac{du}{dt} = 4\sqrt{C}\,(a_1 cos^2 u + a_2 sin^2 u)\, cos\, v - 2a_5$$
$$\frac{dv}{dt} = 2\sqrt{C}\,(a_2 - a_1)\, sin\, 2u\, sin\, v. \qquad (60)$$

The system (60) is integrable, nevertheless obtaining the flow of X_K for this case is nontrivial task. A small calculations shows that the vector field X_K may be written as

$$X_K = (\nabla F \times \nabla G)\cdot \nabla \qquad (61)$$

where

$$F = 2(a_2 - a_1)x_1^2 + 2a_2 x_3^2 - 2a_4 x_3$$
$$G = x_2^2 - \frac{a_1}{2(a_2 - a_1)}\, x_3^2 - \frac{a_5}{2(a_2 - a_1)}\, x_3$$

We introduce a new basis of coordinates:
$y_1 = \sqrt{2(a_2 - a_1)}\, x_1, \quad y_2 = x_2, \quad y_3 = \sqrt{2a_2} x_3$
$-a_5/\sqrt{2a_2}$. In these coordinates we obtain

$$F = y_1^2 + y_3^2$$

$$G = y_2^2 \frac{a_1}{4a_2(a_2 - a_1)}\, y_3^2 + \frac{(2a_2 - a_1)a_5)}{2(a_2 - a_1)(2a_2)^{3/2}}$$
$$+ \frac{a_5^2(2a_2 - a_1)}{8a_2^2(a_2 - a_1)}.$$

Since the level sets of F are circular cylinders, we choose the cylindrical coordinates: $y_1 = r\, cos\, u,\ y_2 = v,\ y_3 = r\, sin\, u$. In these coordinates F and G take the form

$$F = r^2$$
$$G = v^2 - \frac{a_1 F}{4(a_2 - a_1)a_2}\, \sin^2 u$$
$$+ \frac{a_5(2a_2 - a_1)\sqrt{F}}{2(a_2 - a_1)(2a_2)^{3/2}}\, \sin u + \frac{a_5^2(2a_2 - a_1)}{8a_2^2(a_2 - a_1)}.$$

The basis of $T_y \mathbb{R}^3$ is $\frac{\partial}{\partial y_1} = \cos u \frac{\partial}{\partial r} - \frac{1}{r}\sin u \frac{\partial}{\partial u},\ \frac{\partial}{\partial y_2} = \frac{\partial}{\partial v},\ \frac{\partial}{\partial y_3} =$
$\sin u \frac{\partial}{\partial r} + \frac{1}{r}\cos u \frac{\partial}{\partial u}$. The reduced Hamiltonian vector fieldX_G reads

$$X_G = 2\left(\frac{\partial G}{\partial u}\frac{\partial}{\partial v} - \frac{\partial G}{\partial v}\frac{\partial}{\partial u}\right) \qquad (62)$$

In this form the flow of X_G can be easily obtained as (cf. [8])

$$r = const.$$
$$v = \chi[(1 + sn(\tau,k))^2 + (\beta + sn(\tau,k))^2]^{1/2} \times$$
$$cn(\tau,k)dn(\tau,k)$$
$$u = \arcsin\left(-\frac{\beta + sn(\tau,k)}{1 + \beta sn(\tau,k}\right)$$

where $\chi^{-2} = \frac{1}{8}[9\beta + s_1)(\beta + s_2)(\beta^2 - 1)]$.

9.5 Calogero System

In this subsection we consider a Hamiltonian system defined by the Hamiltonian [10]

$$H^{(n)} = \lambda\, n \sum_{i=1}^{n} p_i + \mu \sum_{i,k=1}^{n} \sqrt{p_i p_k}\cos\nu(q_1 - q_k) \qquad (63)$$

where λ, μ, ν are constants. We have showed [10] that this Hamiltonian system can be solved by extension of the Poisson structure

$$\pi = \nu\, x_3 \partial_1 \wedge \partial_2 + \nu\, x_2 \partial_3 \wedge \partial_1 - \frac{\nu}{2}\, \partial_2 \wedge \partial_3 \qquad (64)$$

where $\nu = const.$ and $\partial_i = \partial/\partial x_i$. Letting $\nabla = \nabla_1$ we define [9]:

$$\nabla_{n+1} = (\nabla \otimes id^n) \circ \nabla_n$$

that is, diagonalizing on the first factor after applying ∇_n. Hence for $n = 1$ and $a, b \in A$, we have

$$\triangle_2(a) = a \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes a \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes a$$

and

$$\{\triangle_2(a), \triangle_2(b)\}_{A\otimes A\otimes A} = \triangle_2(\{a, b\}_A)$$
$$= \{a, b\}_A \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \{a, b\}_A \otimes \mathbf{1}$$
$$+\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \{a, b\}_A.$$

Next, putting

$$a_1 = a \otimes \mathbf{1} \otimes \mathbf{1}, \quad a_2 = \mathbf{1} \otimes a \otimes \mathbf{1}, \quad a_3 = \mathbf{1} \otimes \mathbf{1} \otimes a$$

we get

$$\triangle_2(a) = \sum_{i=1}^{3} a_i$$

and

$$\{\triangle_2(a), \triangle_2(b)\}_{A\otimes A\otimes A} = \sum_{i=1}^{3} \{a, b\}_i.$$

Thus for arbitrary $n \geq 2$ we have

$$\triangle_{n-1}(a) = \sum_{i=1}^{n} a_1$$

and

$$\{\triangle_{n-1}(a), \triangle_{n-1}(b)\}_{A^{\otimes n}} = \sum_{i=1}^{n} \{a, b\}_i.$$

One can easily check that the Casimir for the Poisson structure (63) is

$$C = x_2^2 + x_3^2 - x_1 \tag{65}$$

The Poisson brackets are fulfilled by the following functions [10]

$$x_1 = p, \quad x_2 = \sqrt{p} \sin \nu q, \quad x_3 = \sqrt{p} \cos \nu q.$$

Consider the simple Hamiltonian

$$H = \alpha_1 x_1 + \alpha_2 (x_2^2 + x_3^2). \tag{66}$$

The relations (65) and (66) give

$$\triangle_{n-1}(H) = \alpha_1 \sum_{i=1}^{n} p_i + \alpha_2 \sum_{ij=1}^{n} \sqrt{p_i p_j} \cos \nu (q_i - q_j) \tag{67}$$

and

$$\triangle_{n-1}(C) = 2 \sum_{1=i<j}^{n} \sqrt{p_i p_j} \cos \nu (q_i - q_j). \tag{68}$$

The Hamiltonian (67) and the Calogero Hamiltonian (63) are equivalent for $\alpha_1 = \lambda\nu$ and $\alpha_2 = \mu$. The integrals of the Calogero system are given by the co-products of the Casimir (65) and read [10]

$$C_k = \triangle_{k-1}(C) = 2 \sum_{1=i<j}^{j=k} \sqrt{p_i p_j} \cos \nu (q_i - q_j), \tag{69}$$

$k = 2, \cdots, n$. Thus, the Calogero system (63) is completely integrable because the first integrals $H_n = \triangle_{n-1}(H)$ and $C_1, \cdots, C_{n-1}$ are functionally independent (cf. [10]).

References:

[1] A. Weinstein, Poisson Geometry, *Diff. Geom. Appl.* 9, 1998, pp. 213–238.

[2] I. Vaisman, *Lectures on the Geometry of Poisson Manifold*, Brikhäuser Basel, Berlin 1994.

[3] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncomutaive Algebras*, AMS, Providence, RI 1999.

[4] A. Weinstein, The modular automorphism group of a Poisson manifold , *J. Geom. Phys.* 23, 1997, pp. 379 – 394.

[5] J. P. Dufour and N.-T. Zung, *Poisson Structures and their Normal Forms*, Brikhäuser Basel, Berlin 2005.

[6] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, vol. 3, Academic Press, New York 1976.

[7] J. C. Van der Meer, *The Hamiltonian Hopf Bifurcations*, Lectures Notes in Math. Vol. 1160, Spronger, Berlin 1985.

[8] S. Kasperczuk, Normal Form, Lie-Poisson Structure and Reduction for the Henon-Heiles System, *Celest. Mech. and Dynam. Astr.* 63, 1996, pp. 245–253.

[9] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York 1969.

[10] S. P. Kasperczuk, Poisson structures and integrable systems, *Physica A* 284, 2000, pp. 113 – 123.

[11] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J.-M. Strelcyn and S. Wojciechowski, Integrals of quadratic ordinary differential equations in $\mathbb{R}^3$: the Lotka-Volterra system, *Physica A* 163, 1980, pp. 683–722.

[12] A. J. Maciejewski and M. Przybylska, Studies of the Hoyer System, *Rep. Math. Phys.* 48, 2001 pp. 131 – 138.